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On some Aspects of my Work with Gabriel Dirac

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Dedicated to the memory of my friend and coworker G. A. Dirac

Several results and problems from areas of discrete mathematics of joint interest to G. A. Dirac and the author are discussed.

I must have known Dirac when he was a child in the 1930's, but I really became aware of his existence when I visited England for the first time after the war in February and March 1949. We met in London and he told me of his work on chromatic graphs. Dirac defined a k-chromatic graph to be vertex critical if the omission of any vertex decreases the chromatic number and edge critical if the removal of any edge decreases the chromatic number. I immediately liked these concepts very much and in fact felt somewhat foolish that I did not think of these natural and obviously fruitful concepts before.

At that time I was already very interested in extremal problems and asked Gabriel to prove that for every k an edge critical k-chromatic graph must have $o(n^2)$ edges. More precisely define $f_k^{(e)}(n)$ to be the largest integer for which there is a $G(n; f_k^{(e)}(n))$ (i.e. a graph on n vertices and $f_k^{(e)}(n)$ edges) which is k-chromatic and edge critical. Estimate or determine $f_k^{(e)}(n)$ as accurately as possible.

Trivially $f_3^{(e)}(2n+1) = f_3^{(e)}(2n+2) = 2n+1$ and I expected that for $k > 3$, $f_k^{(e)}(n) = o(n^2)$. To my great surprise very soon Dirac showed [3]:

$$f_6^{(e)}(4n+2) \geq (2n+1)^2 + 4n + 2. \tag{1}$$

After I recovered from my surprise I immediately asked: is (1) best possible? This question is still open. Toft [16] proved that

$$f_6^{(v)}(n) \geq \frac{3}{10}n^2 \tag{2}$$

where $f_k^{(v)}(n)$ is the largest integer for which there is a graph $G(n; f_k^{(v)}(n))$ which is k-chromatic and vertex critical. It seems quite likely that for every $k \geq 4$

$$\lim_{n\to\infty} f_k^{(v)}(n)/n^2 > \lim_{n\to\infty} f_k^{(e)}(n)/n^2 \tag{3}$$

but as far as I know (3) is still open, and it has not even been proved that the limits exist.

I also asked what about $f_5^{(e)}(n)$ and $f_4^{(e)}(n)$. I still hoped that perhaps $f_4^{(e)}(n) = o(n^2)$. In 1970 Toft [14] proved that

$$f_4^{(e)}(n) > \frac{n^2}{16}. \tag{4}$$

Toft's proof is based on the idea of Dirac for the case $k \geq 6$. Simonovits and I proved that $f_4^{(e)}(n) \leq \frac{n^2}{4}+n$ which was later improved to $f_4^{(e)}(n) \leq \frac{n^2}{4}$. It would be very desirable to improve (4) or my result with Simonovits and to determine $\lim_{n\to\infty} f_4^{(e)}(n)/n^2$.

By the way, the graph of Toft has many vertices of bounded degree. This lead me to ask: Is there an edge critical four chromatic graph of n vertices each vertex of which has degree $> c_1 n$? I conjecture with some trepidation that such a graph does not exist. The strongest known result is due to Simonovits [10] and to Toft [15]. They proved that there is a $G(n)$ which is four chromatic and edge critical, each vertex of which has degree $> cn^{1/3}$. Dirac's six chromatic edge critical graph is regular of degree $\frac{n}{2}+2$. It is not impossible that there is a four critical regular graph of n vertices and degree $c_1 n$, but this seems unlikely to me. As far as I know there is no example of a regular edge critical four chromatic graph of degree ℓ for $\ell \geq 6$, but I expect that such graphs exist for every ℓ.

By the way all of the early examples of these critical graphs contained abnormally large bipartite graphs and also small odd circuits. At first I thought that this was not an accident and must be so, but I learned that Rödl found counterexamples some time ago. He showed that there is a constant c_r so that there is a four chromatic edge critical graph on n vertices and $c_r n^2$ edges which contains no odd circuit of size $\leq 2r+1$. Exact results are not known, of course, though no doubt it can be proved that c_r tends to 0 as r tends to infinity, but probably showing $c_r < c_{r-1}$ will not be easy.

I recently heard from Toft the following conjecture of Dirac: Is it true that for every $k \geq 4$ there is a k-chromatic vertex critical graph which remains k-chromatic if anyone of its edges is omitted. If the answer as expected is yes then one could ask whether it is true that for every $k \geq 4$ and r there is a vertex critical k-chromatic graph which remains k-chromatic if any r of its edges are omitted. Perhaps there is an $f(n)$ so that for every $k \geq 4$ there is a k-chromatic vertex critical graph on n vertices which remains k-chromatic if any $f(n)$ of its edges are omitted. If so one could try to determine the largest such $f(n)$.

Recently Toft asked: Is there a four chromatic edge critical graph with n vertices and $c_1 n^2$ edges which can be made bipartite only by the omission of $c_2 n^2$ edges? The original example of Toft could be made bipartite by the omission of cn edges. Rödl and Stiebitz [11] constructed such a graph but it is quite possible that the largest such c_1 for which this is possible will be less than $\frac{1}{16}$, and it might be of some interest to determine the dependence of c_2 from c_1.

Dirac [5] and I independently noticed the following strengthening of Turán's classical theorem. Denote by $f(n;k(r))$ the smallest integer for which every $G(n;f(n;k(r)))$ contains a complete graph $k(r)$ on r vertices. We noticed that then our $G(n;f(n;k(r)))$ already contains a $k(r+1)$ from which at most one edge is missing. In fact we showed that there is a $k(r-1)$ and $c_r n$ further vertices $y_1, \dots y_t$, $t = c_r n$, for which each of the y's is joined to every vertex of our $k(r-1)$. The exact value of c_r is known only for $r = 3$. Bollobás and I conjectured and Edwards proved that every $G(n;[n^2/4]+1)$ contains an edge (x_1, x_2) and $\frac{n}{6}$ y's, each of them joined to both x_1 and x_2, and $\frac{n}{6}$ is best possible.

It would be very nice if one could prove some analogue to our result for hypergraphs. The simplest problem would be: denote by $f(n;k^{(3)}(4))$ the smallest integer for which every 3-uniform hypergraph $G^{(3)}(n;f(n;k^{(3)}(4)))$ on n vertices and $f(n;k^{(3)}(4))$ triples contains all four triples of some set of 4 elements. The determination of $f(n;k^{(3)}(n))$ is a classic unsolved problem of Turán. Is it true that our $G^{(3)}(n;f(n;k^{(3)}(n)))$ contains 5 points x_1, x_2, x_3, y_1, y_2 and all the triples of the quadruples $\{x_1, x_2, x_3, y_1\}$ and $\{x_1, x_2, x_3, y_2\}$? Unfortunately nothing is known about this. Our result led me to the following theorem. By the theorem of Kövári, V.T. Sós and Turán [9] every $G(n;c_1 n^{2-1/r})$ contains a complete bipartite graph $k(r,r)$. I showed that every $G(n;c_2 n^{2-1/r})$ contains a $k(r+1,r+1)$ with at most one edge missing [8].

Simonovits and I have the following problem: Let $f(n;H)$ be the smallest integer for which every $G(n;f(n;H))$ contains a subgraph isomorphic to the graph H. Is it then true that every $G(n; f(n; H))$ contains at least two subgraphs isomorphic to H? We could not decide this question for $H = C_4$. In fact we expect that every $G(n; f(n; C_4))$ contains $[\varepsilon\sqrt{n}]$ C_4's.

Nearly thirty years ago I conjectured that every $G(n; 3n - 5)$ contains a topological complete pentagon, i.e. 5 vertices every two of which are joined by paths no two of which have any interior point in common. It is easy to see that if true this conjecture is best possible. I soon found out that Dirac [6] anticipated me; he made the same conjecture before me. The conjecture is still open. As far as I know the best result is due to Thomassen [13] who proved that every $G(n; 4n - 10)$ contains a topological complete pentagon.

Despite our many contacts Dirac and I only had one joint paper [7]. This paper was perhaps undeservedly neglected; one reason was that we have few easily quotable theorems there, and do not state any unsolved problems. We prove there, among other results, that if $G(n; n + 3)$ is planar then it contains two edge disjoint circuits.

Finally, let me remind the reader of a nice conjecture of Dirac [2]. This was conjectured also independently and simultaneously by Motzkin. Let there be given n points $x_1, \ldots, x_n$ in the plane not all on a line. Then for at least one x_i there are $\frac{n}{2} - c$ distinct lines among the (x_i, x_j). In a weaker form this conjecture has recently been proved by J. Beck [1] and independently by Szemerédi and Trotter [12].

Dirac gave several beautiful consequences of his theory of critical graphs. For example, he proved in [4] that any graph G on the orientable surface S_γ, $\gamma \geq 1$, for which equality holds in Heawood's inequality

$$\chi(G) \leq \lfloor \frac{1}{2}(7 + \sqrt{1 + 48\gamma}) \rfloor = H(\gamma)$$

must contain a complete graph on $H(\gamma)$ vertices. He also obtained the similar result for non-orientable surfaces. This was one of the most significant contributions to map-colour-theory since Heawood's pioneering paper in 1890. (The case of the torus was first obtained by P. Ungár).

Dirac made many deep and significant contributions to several other parts of graph theory than those mentioned above, among them paths and circuits, Menger's theorem and connectivity, extremal results for contractions and subdivisions, and infinite graph theory. He seemed much influenced by the work of König. His own influence is now present everywhere in graph theory.

Finally, I want to thank B. Toft for his help in writing this paper and for supplying some references.

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