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ON CERTAIN SATURATION PROBLEMS

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1. Introduction

1.1. In his paper [1] G. Somorjai proved a saturation theorem for the positive interpolatory linear operators

$$(1.1)\qquad L_n(f,x) := \frac{\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left|x-\frac{k}{n}\right|^{-r}}{\sum_{k=0}^{n}\left|x-\frac{k}{n}\right|^{-r}}, \quad 0 \leqq x \leqq 1, \quad n = 1, 2, \ldots,$$

where $r>2$ is a fixed real number, $f\in C[0,1]$. He considered the expressions

$$(1.2)\qquad M_f(x) := \overline{\lim_{y\to x}} \frac{|f(y)-f(x)|}{|y-x|}, \quad M_f := \sup_{0\leqq x\leqq 1} M_f(x)$$

for arbitrary $f\in C[0,1]$ and proved that

$$(1.3)\qquad \begin{cases} M_f(x) = 0 \quad \textit{for each} \quad x\in[0,1] \quad \textit{iff} \quad f(x) = \textit{const}, \\ M_f < +\infty \quad \textit{iff} \quad f\in \mathrm{Lip}\, 1. \end{cases}$$

(Here and later "const", $c_1, c_2, \ldots$ denote absolute positive real numbers.)

By (1.2) and using that every number $x_0\in[0,1]$ can be approximated with certain positive fractions $\{k_t/n_t\}$ (k_t, n_t are integers) such that

$$(1.4)\qquad \left|x_0-\frac{k_t}{n_t}\right| \leqq \frac{c}{n_t^2}, \quad t = 1, 2, \ldots,$$

(Dirichlet-theorem), and finally, exploiting that the fractions $\{k_t/n_t\}$ are interpolatory nodes for the operator (1.1), *he succeeded in proving that*

$$(1.5)\qquad \begin{cases} \|L_n(f,x)-f(x)\| = o(1/n) \quad \textit{iff} \quad f = \text{const}, \\ \|L_n(f,x)-f(x)\| = O(1/n) \quad \textit{iff} \quad f\in \mathrm{Lip}\, 1 \end{cases}$$

where $\|\cdot\|$ stands for the supremum norm.

Later in T. Hermann, P. Vértesi [2] we tried to extend his argument for other interpolatory operators, but as F. Pintér very recently remarked, [2; Lemma 3.2] was incorrect.

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1.2. The aim of this paper is to give some useful generalizations of (1.3) and (1.4) which help to get some other saturation theorems. Especially, we can use them to correct the proof of [2; Theorem 2.4] (cf. [8]).

2. Results

2.1. Using the previous notations first we verify the following

STATEMENT 2.1. *Let* $f\in C[0,1]$ *and* $E\subset[0,1]$ *a set such that* $CE:=[0,1]\setminus E$ *is countable. Then, by* $M_f(E):=\sup_{x\in E} M_f(x)$, *we get*

$$M_f(E)=M_f. \tag{2.1}$$

2.2. A simple consequence of the previous theorem is that

$$\begin{cases} M_f(x)=0 \quad \textit{for each} \quad x\in E \quad \textit{iff} \quad f(x)=\textit{const}, \\ M_f(E)<\infty \quad \textit{iff} \quad f\in \operatorname{Lip} 1 \end{cases} \tag{2.2}$$

(see (1.3)).

2.3. Now we state a Dirichlet-type approximation relation. Namely

THEOREM 2.2. *Let* $x_0\in(0,1)$ *be a fixed irrational number,* $\{y_r\}_{r=1}^{\infty}$ *an arbitrary sequence with* $y_r\neq x_0$, $r=1,2,\dots$ *and* $\lim_{r\to\infty} y_r=x_0$, $0<\varrho\leqq 1/3$ *and* $0\leqq\gamma$, $\delta<1$ *fixed real numbers. Then there exists a sequence* $\{x_k\}\subset\{y_r\}$ *and positive integers* $\{l_k\}_{k=1}^{\infty}$ *and* $\{n_k\}_{k=1}^{\infty}$ *with*

$$1<n_1<n_2<\dots, \quad \text{i.e.} \quad \lim_{k\to\infty} n_k=\infty \tag{2.3}$$

such that for the fractions $\dfrac{l_k+\gamma}{n_k+\delta}$ *the relations*

$$\left|x_0-\frac{l_k+\gamma}{n_k+\delta}\right|=o\left(\frac{1}{n_k}\right), \quad k=1,2,\dots, \tag{2.4}$$

$$\frac{\varrho}{2n_k}\leqq\left|x_k-\frac{l_k+\gamma}{n_k+\delta}\right|\leqq\frac{4\varrho}{n_k} \tag{2.5}$$

hold true.

2.4. For our purposes Theorem 2.2 is quite satisfactory. On the other hand, using its proof the following slight generalization can be proved (see Section 3.4).

THEOREM 2.3. *Let* $x_0\in(0,1)$ *be a fixed irrational number,* $\{y_r\}_{r=1}^{\infty}$ *be an arbitrary sequence with* $y_r\neq x_0$, $r=1,2,\dots$, $\lim_{r\to\infty} y_r=x_0$. *Further, let* $0<\varrho\leqq 1/3$ *(real),* $p,q>0$, $(p,q)=1$ *(integers),* $0\leqq\gamma<p$, $0\leqq\delta<q$ *(reals), be fixed numbers. Then there exist a sequence* $\{x_k\}\subset\{y_r\}$ *and positive integers* $\{l_k\}_{k=1}^{\infty}$ *and* $\{n_k\}_{k=1}^{\infty}$ *with*

$$1<n_1<n_2<\dots, \quad \textit{i.e.} \quad \lim_{k\to\infty} n_k=\infty$$

such that relations

$$\left|x_0-\frac{pl_k+\gamma}{qn_k+\delta}\right| = o\left(\frac{1}{n_k}\right), \quad k=1,2,\ldots, \tag{2.6}$$

$$\frac{\varrho}{2n_k} \leqq \left|x_k-\frac{pl_k+\gamma}{qn_k+\delta}\right| \leqq \frac{(2p+2)\varrho}{n_k}, \quad k=1,2,\ldots \tag{2.7}$$

hold true.

2.5. As it was mentioned earlier, Theorems 2.1—2.2 are very useful in some saturation problems. They can be applied, e.g., to get the following statement.

Let

$$t_{kn}=\frac{2k\pi}{2n+1}, \quad k=0,\pm 1,\pm 2,\ldots, \tag{2.8}$$

and consider the trigonometric polynomials

$$p_n(f,t)=\sum_{k=-n}^{n} f(t_{kn})u_{kn}(t), \quad n=1,2,\ldots,$$

for the continuous 2π-periodic f (shortly $f\in\tilde{C}$). Here

$$u_{kn}(t)=4l_{kn}^3(t)-3l_{kn}^4(t), \quad k=0,\pm 1,\pm 2,\ldots,$$

where $l_{kn}(t)$ are the fundamental functions of trigonometric interpolation based on (2.8). One can prove that

$$\deg p_n \leqq 4n,$$

$$p_n(f,t_{kn})=f(t_{kn}), \quad k=0,\pm 1,\pm 2,\ldots,$$

$$p_n(g,t)\equiv 1 \quad \text{if} \quad g(t)\equiv 1,$$

$$|p_n(f,t)-f(t)| \leqq 7\omega\left(f,\frac{1}{n}\right)$$

$\big(\omega(f,\delta)$ is the modulus of continuity of f; cf. [6]$\big)$. *Then* $\big($see A. K. Varma, P. Vértesi [3]$\big)$

$$\begin{cases} \|p_n(f,t)-f(t)\| = o\left(\frac{1}{n}\right) & \textit{iff} \quad f=\textit{const}, \\ \|p_n(f,t)-f(t)\| = O\left(\frac{1}{n}\right) & \textit{iff} \quad f\in \mathrm{Lip}\,1. \end{cases} \tag{2.9}$$

2.6. Other applications can be find in [8] and [4].

3. Proofs

3.1. Proof of Statement 2.1.[1] As a simple application of [7; Ch. 6, § 7, Theorem 7.2.] we get that *if* $h \in C[0, 1]$ *further*

$$\varlimsup_{y \to x+0} \frac{h(y)-h(x)}{y-x} > 0 \quad \textit{for each} \quad x \in E \tag{3.1}$$

then h is strictly monotone increasing on $[0, 1]$.

Now let us suppose that

$$M_f(E) < c < M_f. \tag{3.2}$$

Then, for

$$g(x) := cx - f(x), \tag{3.3}$$

$g \in C[0, 1]$, further for any fixed $x \in E$

$$\frac{g(x)-g(y)}{x-y} = c - \frac{f(x)-f(y)}{x-y} > 0 \tag{3.4}$$

if y is close enough to x (see (3.2)). Especially,

$$\varlimsup_{y \to x+0} \frac{g(x)-g(y)}{x-y} > 0, \quad x \in E. \tag{3.5}$$

By (3.5) and the quoted theorem, g is strictly monotone increasing i.e. for *arbitrary* $x, y \in [0, 1]$, $x \neq y$ we have

$$\frac{f(x)-f(y)}{x-y} < c, \quad x, y \in [0, 1], \quad x \neq y. \tag{3.6}$$

Applying the same argument for the function

$$t(x) := cx + f(x) \tag{3.7}$$

we get

$$\frac{f(x)-f(y)}{x-y} > -c, \quad x, y \in [0, 1], \quad x \neq y. \tag{3.8}$$

By (3.6) and (3.8)

$$\left| \frac{f(x)-f(y)}{x-y} \right| < c, \quad x, y \in [0, 1], \quad x \neq y, \tag{3.9}$$

from where

$$M_f = \sup_{0 \leq x \leq 1} \varlimsup_{y \to x} \left| \frac{f(x)-f(y)}{x-y} \right| \leq c, \tag{3.10}$$

a contradiction.

[1] This argument, which is much simpler than our previous one, is due to G. Petruska.

3.2. Proof of Theorem 2.2. By [5; Theorem 7.11, p. 196], for any fixed irrational number $x_0 \in (0, 1)$ there exist integer numbers $\{u_s\}$, $\{v_s\}$ such that

$$(3.11)\quad \left|x_0 - \frac{u_s}{v_s}\right| < \frac{1}{v_s v_{s+1}}, \quad (u_s, v_s) = 1, \quad 1 < v_1 < v_2 < \ldots, \quad \text{i.e.} \quad \lim_{s\to\infty} v_s = \infty.$$

Then, by $y_l \to x_0$ and $\lim_{s\to\infty} v_s = \infty$, one can find infinitely many different $x_k \in \{y_l\}$ for which

$$(3.12)\qquad \frac{\varrho}{v_{k+1}} < |x_k - x_0| \leqq \frac{\varrho}{v_k}, \quad k = k_1, k_2, \ldots.$$

If

$$(3.13)\qquad M_k := \left[\frac{\varrho}{|x_k - x_0|}\right],$$

then, if k_1 is big enough (which will be supposed from now on), by (3.12) and (3.13)

$$(3.14)\qquad \frac{1}{v_{k+1}} < \frac{1}{M_k} < \frac{2}{v_k}, \quad k = k_1, k_2, \ldots.$$

Now let $A_k := v_k \gamma - u_k \delta$. For fixed u_k and v_k consider the expression

$$\left|\frac{u_k}{v_k} - \frac{l+\gamma}{n+\delta}\right| = \left|\frac{nu_k - lv_k - A_k}{v_k(n+\delta)}\right|.$$

3.3. Here, by $(u_k, v_k) = 1$, the Diophantine equation $nu_k - lv_k = 1$ is solvable; let (n^*, l^*) be a solution. Take $n_0 := n^*[A_k]$ and $l_0 := l^*[A_k]$. Then, by $n^* u_k - l^* v_k = 1$, we get as follows.

$$(3.15)\quad \left|\frac{u_k}{v_k} - \frac{l_0+\gamma}{n_0+\delta}\right| = \left|\frac{n_0 u_k - l_0 v_k - A_k}{v_k(n_0+\delta)}\right| = \left|\frac{[A_k](n^* u_k - l^* v_k) - A_k}{v_k(n_0+\delta)}\right| \leqq \frac{1}{|v_k(n_0+\delta)|}.$$

Now we state that by a proper shifting of n_0 and l_0 we get the solution (n_k, l_k) with the relations

$$(3.16)\qquad M_k \leqq n_k < 3M_k,$$

$$(3.17)\qquad \left|\frac{u_k}{v_k} - \frac{l_k+\gamma}{n_k+\delta}\right| \leqq \frac{1}{v_k(n_k+\delta)}.$$

a) Indeed, if $n_0 < M_k$, consider $n' := n_0 + v_k$ and $l' := l_0 + u_k$. By definition $n' u_k - l' v_k = n_0 u_k - l_0 v_k = [A_k]\ (n^* u_k - l^* v_k) = [A_k]$, i.e. we have (3.15), if we replace (n_0, l_0) by (n', l'). If $n' \geqq M_k$, then by (3.14)

$$n_0 < M_k \leqq n_0 + v_k (= n') < M_k + v_k \leqq M_k + 2M_k = 3M_k,$$

i.e. for $n_k := n'$ we have both (3.16) and (3.17). On the other hand if $n' < M_k$, with a proper integer t, $n_0 + (t-1)v_k < M_k \leqq n_0 + tv_k$. Then $n_k := n_0 + tv_k$ and $l_k := l_0 + tu_k$ will satisfy (3.16) and (3.17).

b) If $n_0 \in [M_k, 3M_k)$, by $n_k := n_0$ and $l_k := l_0$ we get (3.16) and (3.17). If $n_0 \geqq 3M_k$, with a proper integer $\tilde{q}$ let $n_k := n_0 - \tilde{q} v_k$, $l_k := l_0 - \tilde{q} u_k$ be chosen such that $n_k - v_k < M_k \leqq n_k$. Then, as above we can verify (3.16) and (3.17).

Now, by (3.11), (3.17), (3.14), (3.16) and $\lim_{k\to\infty} v_k = \infty$

$$(3.18) \qquad \left| x_0 - \frac{l_k+\gamma}{n_k+\delta} \right| \leqq \left| x_0 - \frac{u_k}{v_k} \right| + \left| \frac{u_k}{v_k} - \frac{l_k+\gamma}{n_k+\delta} \right| < \frac{1}{v_k v_{k+1}} +$$

$$+ \frac{2}{v_k n_k} < \frac{1}{v_k M_k} + \frac{2}{v_k n_k} \leqq \frac{3}{v_k n_k} + \frac{2}{v_k n_k} = o\left(\frac{1}{n_k}\right), \quad k = k_1, k_2, \ldots,$$

which actually gives (2.4). To get (2.5) we write for any fixed $k = k_1, k_2, \ldots, k_1 \geqq k_0$,

$$(3.19) \qquad \left| x_k - \frac{l_k+\gamma}{n_k+\delta} \right| \geqq \left| |x_k - x_0| - \left| x_0 - \frac{l_k+\gamma}{n_k+\delta} \right| \right| < \frac{0.9\varrho}{M_k} - \frac{\varrho^2}{n_k} > \frac{\varrho}{2n_k}$$

(see (3.13) and (3.18)), further

$$\left| x_k - \frac{l_k+\gamma}{n_k+\delta} \right| \leqq |x_k - x_0| + \left| x_0 - \frac{l_k+\gamma}{n_k+\delta} \right| \leqq \frac{1.1\varrho}{M_k} + \frac{\varrho^2}{n_k} < \frac{4\varrho}{n_k}$$

(see (3.13) and (3.18)) which complete the proof.

3.4. Proof of Theorem 2.3. Being very similar to the previous one, we only sketch it. The first difference is that instead of $nu_k - lv_k = 1$, we solve $qu_k n - pv_k l = B_k$, where $B_k := (qu_k, pv_k)$ (cf. 3.3.). By $(u_k, v_k) = (p, q) = 1$, it is easy to see that $B_k \leqq p \cdot q$. Then, if (n^*, l^*) are solutions, with

$$n_0 = n^* \left[\frac{A_k}{B_k} \right] \quad \text{and} \quad l_0 := l^* \left[\frac{A_k}{B_k} \right]$$

we get that

$$\left| \frac{u_k}{v_k} - \frac{pl_0+\gamma}{qn_0+\delta} \right| \leqq \frac{B_k}{v_k(qn_0+\delta)} \leqq \frac{pq}{v_k(qn_0+\delta)}$$

(cf. (3.15)). Now if $n_0 < M_k$, say, then we take $n' := n_0 + pv_k$ and $l' := l_0 + q\mu_k$ finally we get $M_k \leqq n_k < (2p+1) M_k$ (cf. (3.16)). We omit the further details.

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