Multipartite Graph-Tree Ramsey Numbers

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INTRODUCTION

Let $T = T_n$ be a tree of order n, and $F = K(m_0, m_1, \ldots, m_k)$ be the complete multipartite graph with parts of order $m_0 \le m_1 \le \cdots \le m_k$. For n sufficiently large and $m_1 = m_2 = 1$, it has been shown that the Ramsey number $r(F, T) = k(n-1) + 1$. This result is generalized by showing that with just $m_0 = 1$ and n sufficiently large,

$$k(n-1) + 1 \le r(F, T) \le k(r(K(1, m_1), T) - 1) + 1.$$

Since it is known that $r(K(1, m_1), T) = n$ for the large class of trees that have no vertices of large degree, the upper and lower bounds are frequently identical. In all cases, these bounds are shown to differ by at most k.

For simple graphs F and G, the *Ramsey number* $r(F, G)$ is the smallest integer p such that if the edges of the complete graph K_p are colored red and blue, either the red subgraph contains a copy of F or the blue subgraph contains a copy of G. If F is a graph with *chromatic number* $\chi(F)$, then the *chromatic surplus* $s(F)$ is the smallest number of vertices in a color class under any $\chi(F)$-coloring of the vertices of F.

For any connected graph G of order $n \ge s(F)$, the Ramsey number $r(F, G)$ satisfies the inequality

$$r(F, G) \ge (\chi(F) - 1)(n - 1) + s(F). \tag{1}$$

This inequality follows from coloring red or blue the edges of a complete graph on $(\chi(F) - 1)(n - 1) + s(F) - 1$ vertices such that the red subgraph is isomorphic to $(\chi(F) - 1)K_{n-1} \cup K_{s(F)-1}$ and the red subgraph is isomorphic to the complement. When equality occurs in (**1**) we say that G is F-*good*. The concept of F-goodness generalizes the classical simple result of Chvátal that $r(K_k, T_n) = (k-1)(n-1) + 1$

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[6], where K_k denotes the complete graph on k vertices and T_n denotes a tree on n vertices. The result of Chvátal has been generalized in many special cases by replacing the complete graph K_k by a graph F of chromatic number k, the tree T_n by a "sparse" graph G of order n, and the number 1 by $s(F)$ (i.e., G is F-good). Even in the case when G is "sparse" but not F-good, the lower bound given in inequality (1) is in most cases a good approximation to the Ramsey number $r(F, G)$.

Our purpose is to investigate the Ramsey number $r(F, T_n)$, when T_n is a large-order tree, and in particular to determine which large trees T_n are F-good. Since $\chi(F)$ is important in determining the value of $r(F, T_n)$, it is natural to carefully consider the case when $F = K(m_0, m_1, \ldots, m_k)$ is a complete multipartite graph.

Not all trees are $K(m_0, m_1, \ldots, m_k)$-good when each $m_i \geq 2$ or when $m_0 = 1$ and $m_i \geq 2$ for $1 \leq i \leq k$, [2, 4]. For example in [4] it was shown that

$$r(K(2, 2), K(1, n-1)) > n + n^{1/2} - 5n^{3/10}$$

for n large. However, the principal result of [5] is that each large-order tree T_N is $K(1, 1, m_2, m_3, \ldots, m_k)$-good. We will generalize this last result by proving the following theorem.

THEOREM: If n is sufficiently large, then

$$r(K(1, m_1, \ldots, m_k), T_n) \geq \max\{k(n-1), k(r(K(1, m_1), T_n) - 2)\} + 1,$$

and $r(K(1, m_1, \ldots, m_k), T_n) \leq k\{r(K(1, m_1), T_N) - 1\} + 1$.

The upper and lower bounds for $r(K(1, m_1, \ldots, m_k), T_n)$ differ by at most k. For "most" trees $r(K(1, m_1), T_N) = n$, so in this case the upper and lower bounds agree and the tree is good. Also, if $m_1 = 1$, then clearly $r(K(1, m_1), T_n) = n$ and the principal result of [5] follows as a corollary. There are other cases when we can prove a stronger lower bound that agrees with the upper bound of the theorem. These will be discussed in more detail later. One example that is already in the literature is the special case when T_n is the star $K(1, n-1)$, which was studied in [2]. There it was proved that

$$r(K(1, m_1, \ldots, m_k), K(1, n-1)) = k(r(K(1, m_1), K(1, n-1)) - 1) + 1.$$

In this special case the value of the Ramsey number agrees with the upper bound of the theorem.

KNOWN RESULTS

Several known results dealing with the Ramsey number $r(K(m_0, m_1, \ldots, m_k), T_n)$ when n is large will be needed in the proof of the main theorem, so we list these results with references. In all cases we assume that $m_0 \leq m_1 \leq \cdots \leq m_k$.

The following gives a general upper bound on such Ramsey numbers.

THEOREM A [8]: For n sufficiently large, there is a constant $\gamma = \gamma(k, m_k)$ with $0 < \gamma < 1$, such that

$$r(K(m_0, m_1, \ldots, m_k), T_n) \leq k(n-1) + n^{\gamma}.$$

If $m_0 = 1$, then a stronger result about the Ramsey number can be stated.

THEOREM B [5]: For n sufficiently large, there is a constant $C = C(k, m_k)$, such that

$$r(K(1, m_1, \ldots, m_k), T_n) \leq k(n - 1) + C.$$

If $m_0 = m_1 = 1$, then the precise value of the Ramsey number has been determined, and all trees are $K(1, 1, m_2, \ldots, m_k)$-good.

THEOREM C [5]: For n sufficiently large,

$$r(K(1, 1, m_2, \ldots, m_k), T_n) = k(n - 1) + 1.$$

The complete multipartite graphs in each of the previous three theorems can be replaced by an appropriate graph F with chromatic number $k + 1$. In Theorem A, any F will suffice; in Theorem B, the chromatic surplus of F must be 1; and in Theorem C, a $(k + 1)$-coloring of F in which two color classes have only one vertex will suffice.

The following is the special case when the tree is a star, and it proves that a graph F with chromatic surplus 1 is not sufficient for all large-order trees to be F-good.

THEOREM D [2]: For n sufficiently large,

$$r(K(1, m_1, \ldots, m_k), K(1, n)) = k(r(K(1, m_1), K(1, n)) - 1) + 1.$$

Since $r(K(1, m_1), K(1, n)) = m_1 + n - \delta$, where $\delta = 1$ if m_1 and n are both even, and $\delta = 0$ otherwise, Theorem D completely determines the value of the Ramsey number. This value is not, in general, $k(n - 1) + 1$, so the star $K(1, n)$ is not $K(1, m_1, \ldots, m_k)$-good.

Before we can state the next result, some additional notation must be given. The *independence number* of a graph G will be denoted by $\alpha(G)$. If T is a tree, then set

$$\alpha'(T) = \min\{\alpha(T - V(S)) : S \text{ is a star contained in } T\}.$$

Thus, $\alpha'(T)$ is a measure of how small the independence number of the nonneighborhood of a vertex of the tree can be. The parameter α' is related to the Ramsey number $r(K(1, m), T)$ of a star and a tree, as the following result indicates.

THEOREM E [7]: Let m be a positive integer and T_n a tree of order n with $n \geq 12m^3$. Then

$$\max\{n, n + m - 1 - \alpha' - \beta\} \leq r(K(1, m), T_n) \leq \max\{n, n + m - 1 - \alpha'\},$$

where $\alpha' = \alpha'(T_n)$, and $\beta = 0$ if $n + m - 2 - \alpha'$ is divisible by m, and $\beta = 1$ otherwise.

Therefore, in general, the upper and lower bounds for $r(K(1, m), T_n)$ differ by at most 1. There are several cases in which there is equality. In particular, if the tree T_n has no vertex of degree at least $n - 2m - 3$, then $\alpha' \geq m - 1$ and $r(K(1, m), T_n) = n$.

The examples that give the lower bound in Theorem E will be needed later, so we briefly describe these graphs and their properties now. A careful verification of their properties can be found in [7].

Clearly, the graph K_{n-1} does not contain a T_n and its complement does not contain any star. If $n + m - 1 - \alpha' - \beta > n$, then the tree T_n must have a vertex of large degree. Let G be the graph on $n + m - 2 - \alpha' - \beta \geq n$ vertices that is the disjoint union of complete graphs K_m if $\beta = 0$ and of complete graphs K_m and K_{m-1} if $\beta = 1$. If n is large, then this can always be achieved. Clearly, G contains no $K(1, m)$. Finally, using the fact that the tree contains a vertex of large degree, we show that the complement of G does not contain T_n. This follows from the fact that if G contains T_n, then the large degree vertex of T_n will be in an independent set with $m - \beta$ vertices, and only an additional α' vertex of T_n can be in this independent set. Thus, $m - \alpha' - 1 - \beta$ vertices of G will not be in T_n, which leaves at most $n - 1$ vertices of G in T_n. The graphs K_{n-1} and G imply colorings that give the lower bounds in Theorem E.

RESULTS AND PROOFS

The principal result is contained in the following two theorems.

THEOREM 1: For $1 \leq m_1 \leq m_2 \leq \cdots m_k$ and n sufficiently large,

$$r(K(1, m_1, m_2, \ldots, m_k), T_n) \leq k(r(K(1, m_1), T_n) - 1) + 1.$$

THEOREM 2: Let $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$ and let n be sufficiently large. Then, in general

$$r(K(1, m_1, m_2, \ldots, m_k), T_n) > \max\{k(n - 1), k(r(K(1, m_1), T_n) - 2)\}.$$

Also,

$$r(K(1, m_1, m_2, \ldots, m_k), T_n) > k(r(K(1, m_1), T_n) - 1)$$

for each of the following cases:

(i) m_1 divides $n + m_1 - \alpha' - 2$,
(ii) $n \geq n + m_1 - \alpha' - 1$, or
(iii) $r(K(1, m_1), T_n) = n + m_1 - \alpha' - 2$.

Theorems 1 and 2 give upper and lower bounds for the Ramsey number $r(K(1, m_1, m_2, \ldots, m_k), T_n)$ that differ by at most k. In fact, unless the three T_n has a vertex of degree at least $n - 2m_1 + 3$, the upper bound and lower bound agree and imply that the Ramsey number is precisely $k(n - 1) + 1$. Thus if $\Delta(T)$ represents the maximal degree of a vertex in T, we have the following corollary.

COROLLARY 1: If n is sufficiently large and $\Delta(T_n) \leq n - 2m_1 + 2$, then

$$r(K(1, m_1, m_2, \ldots, m_k), T_n) = k(n - 1) + 1.$$

In particular, if $m_1 = 1$, this is true.

The multipartite graph in the corollary need not be a complete multipartite graph, because inequality (**1**) gives a lower bound that depends only on the chromatic number of the graph. Therefore, if F is any subgraph of $K(1, m_1, m_2, \ldots, m_k)$ with $\chi(F) = k + 1$, and $\Delta(T_n) \leq n - 2m_1 + 2$, then for n sufficiently large,

$$r(F, T_n) = k(n - 1) + 1.$$

The upper bound given in Theorem 1 clearly does not require that the first graph be a complete multipartite graph. However, the lower bounds given in Theorem 2 do depend on the first graph being a complete multipartite graph. This will be clear from the proof of Theorem 2, which follows.

Proof of Theorem 2: We will assume the notation used in Theorem E and the examples associated with that result. Thus,

$$r(K(1, m_1), T_n) = n, \qquad n + m_1 - \alpha' - 2, \qquad \text{or} \qquad n + m_1 - \alpha' - 1.$$

Consider the graph H defined as follows.

(i) If $r(K(1, m_1), T_n) = n$, then $H = K_{n-1}$.

(ii) If $r(K(1, m_1), T_n) > n$, then H has order $n + m_1 - \alpha - 2 - \beta$, and is the complement of a vertex disjoint union of complete graphs each of order

(a) m_1 if $\beta = 0$ (i.e., m_1 divides $n + m_1 - \alpha - 2$)

(b) m_1 or $m_1 - 1$ if $\beta = 1$ (this can always be done for large values of n).

The graph H does not contain T_n and the complement $\overline{H}$ does not contain a $K(1, m_1)$. Therefore, the graph $k \cdot H$ (k disjoint copies of the graph H) clearly does not contain T_n, and the complement of this graph does not contain $K(1, m_1, m_2, \ldots, m_k)$. The last observation is a direct consequence of the fact that $\overline{H}$ does not contain $K(1, m_1)$ and $m_1 \leq m_2 \leq \cdots \leq m_k$. Thus, considering $k \cdot H$ as the blue graph and its complement as the red graph, we have a coloring that completes the proof of Theorem 2. □

Before giving the proof of Theorem 1, some additional special notation will be introduced and a lemma will be stated. Notation not specifically mentioned will follow that in [1]. For simplicity, the multipartite graph $K(m_1, m_2, \ldots, m_k)$ will be denoted by $K(m{:}k)$ when $m_1 = \cdots = m_k = m$. A *suspended* path in a graph is a path in which each vertex has degree 2 in the graph. *End-edges* are edges incident to vertices of degree 1 (i.e., incident to *end-vertices*). If a graph contains a vertex v adjacent to m end-vertices, we say that v is the center of a *talon* of degree m.

LEMMA 1 [3]: If T_n is a tree with n vertices that does not contain a suspended path with more than s vertices, then T_n has at least $n/(2s)$ end-vertices. If, in addition, there are no more than t independent end-edges, then T_n has a talon of degree at least $n/(2st)$.

Proof of Theorem 1: Let $M = k(r(K(1, m_1), T_n) - 1) + 1$, and assume that the edges of the complete graph K_M are colored red and blue so that there is no red $K(1, m_1, m_2, \ldots, m_k)$ or blue T_n. We will show that this leads to a contradiction.

To shorten the notation, we denote the graph $K(1, m_1, m_2, \ldots, m_k)$ by just F, the order of F by f, and the red and blue subgraphs of K_M by R and B, respectively. It is assumed that n is sufficiently large so that the results of Theorems A–E apply. The notation of those theorems is used without additional comment.

The proof is by induction on k. The result is trivial for $k = 1$, so we can assume that $k \geq 2$. Three cases that depend on the structure of the tree T_n are considered: (1) T_n has a long suspended path, (2) T_n has many independent end-edges, and (3) T_n has a vertex of large degree. Lemma 1 implies that these cases exhaust the possibilities.

(1) T_n has a suspended path with at least $C + f^2$ vertices.

Let T' be the tree obtained from T_n by shortening the suspended path by C vertices. By Theorem B there is a copy of T' in B. Let T be a tree in B that is obtained from T' by lengthening this suspended path as much as possible up to C vertices (in fact, $C - 1$, since $T_n \not\subseteq B$). Let P, which has at least f^2 vertices, denote this suspended path of T. By the maximality of T this suspended path cannot be lengthened in B by using a vertex not in T.

By the induction assumption there is in R a copy H of $K(1, m_1, m_2, \ldots, m_{k-1})$ that is vertex disjoint from T. The maximality of P implies that each vertex of H cannot be adjacent in B to two consecutive vertices of P, and if a vertex is adjacent in B to two vertices of P, then the successors along P of these vertices are adjacent in R. Thus, no vertex h of H can have as many as f adjacencies on the path in B, for otherwise, the successors along P (except possibly for an end-vertex of the path) of these vertices and h would form a red complete $K_f \supseteq F$. Therefore, there are at least $f^2 - f(f - m_k) \geq m_k$ vertices of P adjacent in R to every vertex of H. This implies there is a red F, a contradiction that completes the proof in this case.

(2) T_n has at least $C + f^2$ independent end-edges.

By Theorem A, there are f vertex disjoint copies of $K(1, m_1, m_2, \ldots, m_k)$ in R, which we will denote by $H_1, \ldots, H_f$. Each vertex not in an H_i must have a blue adjacency in H_i to avoid a red copy of F. Let T be the tree obtained from T_n by deleting the end-vertices of the $C + f^2$ independent end-edges of T_n, and let x be the $C + f^2$ vertices of T adjacent to these end-vertices. By Theorem B, there is in B a copy of T that is vertex disjoint from $H_1, \ldots, H_f$.

There is no matching between X and $V(K_M - T)$ in B that saturates X, for this would imply that B contains T_n. Also, each vertex of X is adjacent in B to at least f vertices not in T, because each vertex of X has a blue adjacency in each H_i. Hall's matching theorem [10] implies that there is a subset $X' \subseteq X$ with $|X'| \geq f \geq m_k$ whose blue neighborhood in $V(k_M - T)$ has order at most $|X'| - 1$. Hence, there is a subgraph Y disjoint from T and with at least $M - n + 1$ vertices that is adjacent in R to each vertex of X'. The subgraph Y contains a red $K(1, m_1, m_2, \ldots, m_{k-1})$ by the induction assumption. This implies the existence of a red F, a contradiction that completes the proof in this case.

Lemma 1 implies that the following case exhausts the possibilities for the tree T_n.

(3) T_n has a talon of degree at least $n/(2(C + f^2)^2)$.

Since $r(K(m_1, m_2, \ldots, m_k), T_n) \leq (k - 1)(n - 1) + n^\tau$, no vertex of K_M has red degree $(k - 1)(n - 1) + n^\tau$. Therefore, each vertex has blue degree at least $n - 1 - n^\tau$. Thus, any tree with no more than $n - n^\tau$ vertices can be embedded in B with an arbitrary vertex of the tree embedded at any vertex of B.

We first show that there is no vertex of K_M of blue degree at least $n - 1$. Assume y is such a vertex, and let x be the vertex of T_n that is the center of a maximal-degree talon. The degree of the talon is at least $n/(2(C + f^2)^2)$, which is a positive fraction of n, since C and f are constants independent of n. Let T be the tree obtained from T_n by deleting the end-vertices of the talon with center x. Thus, T can be embedded in B with x embedded at y. This embedding can be extended to T_n, since y has degree at least $n - 1$. In the remainder of the proof we assume that there is no vertex of blue degree at least $n - 1$.

Our objective at this point is to show that the restriction on the (red and blue) edge-coloring of K_M forces the blue graph B to be a disjoint union of k graphs that are nearly complete, and the red graph R to be approximately a complete k-partite graph. This will be used to obtain a contradiction.

By Theorem A, there is a red $K(l{:}k)$, where l is large in comparison of f, but small in comparison to n. Since $R \not\supseteq F$, no one of the k parts of the $K(l{:}k)$ can contain a red $K(1, m_1)$. Therefore by Chvátal's theorem [6], each part of the $K(l{:}k)$ contains a complete blue subgraph of order at least (l/m_1). Hence, by replacing the original l by lm_1, we can assume that the two-colored K_M contains a red $K(l{:}k)$ with each part inducing a blue K_l. Denote the vertices of the k parts by $P_1, \ldots, P_k$, let P be their union, and p the number of vertices in P. Hence $|P_i| = l$, $|P| = \kappa l$, each edge in a P_i is blue, and each edge between different P_i is red.

Consider the vertices not in P. For each i $(1 \le i \le k)$, let Q_i be those vertices not in P that are adjacent in B to at most $(l - m_k)(m_1 + 1)$ vertices of P_j for each $j \ne i$. Since $F \not\subseteq R$, the Q_i are pairwise disjoint. Note that any set of $m_1 + 1$ vertices of Q_i (in fact, $P_i \cup Q_i$) has a common red neighborhood of order at least m_k in each P_j for $j \ne i$. Consequently, there is no red $K(1, m_1)$ in the graph induced by the vertices $P_i \cup Q_i$, for this would imply $F \subseteq R$. Hence, if for each i, $q_i = |Q_i|$, then $q_i + l \le r(K(1, m_1), T_n) - 1$.

Let Q be the union of the Q_i $(1 \le i \le k)$, Q_{k+1} be the remaining vertices not in $P \cup Q$, and $q_{k+1} = |Q_{k+1}|$. We will now show that each q_i $(1 \le i \le k)$ is approximately n, and q_{k+1} is bounded as a function of n and l.

Let Q'_i $(1 \le i \ge k)$ be the vertices of Q_i that have at least one red adjacency in P_i, Q' the union of these sets, and q' the number of vertices in Q'. Since there is no red $K(1, m_1)$ in $P_i \cup Q_i$, there are no more than lm_1 vertices in Q'_i. Hence, $q' \le klm_1$. All of the vertices of $Q - Q'$ have at least l blue adjacencies in P, and the vertices of Q' have at least $l - m_1$ blue adjacencies in P. Any vertex in Q_{k+1} has $l - m_k$ blue adjacencies in some P_i and at least $(l - m_k)/(m_1 + 1)$ in some P_j for $j \ne i$. Thus, a vertex in Q_{k+1} has at least $l(m_1 + 2)/(m_1 + 1) - c_1$ blue adjacencies in P, where $c_1 = c_1(f, k)$ (a constant independent of n and l). No vertex has blue degree $n - 1$, so each vertex of P has at most $n - l$ blue adjacencies outside of P. Counting the number of blue edges between P and $Q \cup Q_{k+1}$, we get the following inequality:

$$p(n - l) \ge q_{k+1}((l(m_1 + 2)/(m_1 + 1) - c_1) + q'(l - m_1) + (M - p - q' - q_{k+1})l.$$

Substituting appropriate values for p, q, q_{k+1}, and M, and performing straightforward calculations will yield the following inequality:

$$q_{k+1} \le c_2 = c_2(f, k).$$

Since $q_i \le r(K(1, m_1), T_n) - l - 1$ for $(1 \le i \le k)$, and $\Sigma_{i=1}^{k} q_i = k(r(K(1, m_1), T_n) - l - 1) + 1 - q_{k+1}$, each

$$q_i \ge n - l - c_3, \qquad \text{where } c_3 = c_3(f, k).$$

Since both c_2 and c_3 are independent of n and l, this verifies our claim about the q_i $(1 \le i \le k + 1)$.

Since $P \cup Q$ has at most $k(r(K(1, m_1), T_n) - 1)$ vertices, there is a vertex in Q_{k+1}, which we will denote by x. Let Q_i'' be the vertices of Q_i adjacent in R to x, and Q'' the union of the Q_i'' $(1 \leq i \leq k)$. There is an appropriate $c_4 = c_4(f, k)$ such that

$$|Q_i''| \geq l/(m_1 + 1) - c_4.$$

If this were not true, then x would have blue degree at least

$$(n - l - c_3) - (l/(m_1 + 1) - c_4) + l(m_1 + 2)/(m_1 + 1) - c_1 \leq n - c_1 - c_3 + c_4 \leq n,$$

which gives a contradiction for an appropriate choice of c_4.

There is no red $K(1, m_1)$ in $P_i \cup Q_i$, so each vertex of Q_i'' has at least $n - c_3 - m_1$ blue adjacencies in $P_i \cup Q_i$. Thus, a vertex in Q_i'' has at most $c_3 + m_1$ blue adjacencies in each Q_j'' for $i \neq j$. Hence, there is clearly a red F in $Q'' \cup \{x\}$. This contradiction completes the proof of Theorem 1. □

QUESTIONS

There are two questions directly related to the results of this paper that would be nice to clear up. First, can the precise value of $r(K(1, m), T_n)$ be determined for all large-order trees? The examples used to verify the present lower bound do not in general give the correct value for $r(K(1, m), T_n)$. However, when they do give the correct value, the proof of Theorem 1 implies that the present upper bound of Theorem 1 is the precise value for the multipartite graph-tree Ramsey number. Second, when the canonical examples used to give the lower bound for $r(K(1, m), T_n)$ do not give the precise result, can the precise value of $r(K(1, m_1, m_2, \ldots, m_k), T_n)$ still be determined?

The chromatic number, the chromatic surplus of the multipartite graph, and the Ramsey number $r(K(1, m), T_n)$ are important parameters in expressing the upper bound for the Ramsey number $r(K(1, m_1, m_2, \ldots, m_k), T_n)$ given in Theorem 1. It is natural to ask if there is a corresponding result for an arbitrary complete multipartite graph-tree Ramsey number. In particular, is it true that for n sufficiently large,

$$r(K(m_1, m_2, \ldots, m_k), T_n) \leq (k - 1)(r(K(m_1, m_2), T_n) - 1) + m_1? \qquad (2)$$

Ramsey numbers $r(B, T_n)$ have been investigated for B a bipartite graph, and good upper bounds have been obtained in [9]. Thus, a proof of inequality (2) would improve the known bounds of the multipartite graph-tree Ramsey numbers.

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