Domination in Colored Complete Graphs

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ABSTRACT

We prove the following conjecture of Erdös and Hajnal: For any fixed positive integer t and for any 2-coloring of the edges of K_n, there exists $X \subset V(K_N)$ such that $|X| \leqq t$ and X monochromatically dominates all but at most $n/2^t$ vertices of K_n. In fact, X can be constructed by a fast greedy algorithm.

1. INTRODUCTION

A 2-colored graph G is a graph with edges colored red or blue. A set $X \subset V(G)$ r-*dominates*, (b-*dominates*) $Y \subset V(G)$ if $X \cap Y = \emptyset$ and for each $y \in Y$ there exists $x \in X$ such that the edge (x, y) is red (blue). The set $X \subset V(G)$ *dominates* $Y \subset V(G)$ if either X r-dominates Y or X b-dominates Y.

Note that in this definition of domination X does not dominate itself. In particular, a set A on t vertices is said to dominate all but at most k vertices of G if A dominates B and $|V(G) - A - B| \leq k$. The following conjecture is due to Erdös and Hajnal ([2]). For given positive integers n, t, any 2-colored K_n (complete graph on n vertices) has a set X_t of at most t vertices dominating all but at most $n/2^t$ vertices of K_n. The conjecture is trivial for $t = 1$, and the case $t = 2$ has been proved by Erdös and Hajnal. In this paper the general conjecture is proved. In fact, the proof method shows that one vertex of X_t can be chosen arbitrarily. The following "antisymmetric" or "off-diagonal" generalization of the conjecture is also proved: for any $\beta \in (0, 1)$, i.e., real β, $0 < \beta < 1$, a 2-

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colored K_n *either* contains a set X_t such that $|X_t| \leqq t$ and X_t r-dominates all but at most $\beta^t n$ vertices of K_n, *or* contains a set X_t such that $|X_t| \leq t$ and X_t b-dominates all but at most $(1 - \beta)^t n$ vertices of K_n.

The results mentioned so far are corollaries of the following theorem:

Theorem 1. Let $G = [X, Y]$ be a 2-colored complete bipartite graph, t be a nonnegative integer, and $\beta \in (0, 1)$. Then at least one of the following two statements is true:

1. Some subset of at most t vertices of X r-dominates all but at most $\beta^{t+1}(|X| + |Y|)$ vertices of Y.
2. Some subset of at most t vertices of Y b-dominates all but at most $(1 - \beta)^{t+1}(|X| + |Y|)$ vertices of X.

Corollary 1. Let K_n be 2-colored, p a vertex of K_n, and k a positive integer and $\beta \in (0, 1)$. Then there exists a set $A \subset V(K_n)$ such that $p \in A$, and $|A| \leqq k$ and either A r-dominates all but at most $(n - 1)\beta^k$ vertices of K_n or A b-dominates all but at most $(n - 1)(1 - \beta)^k$ vertices of K_n.

Proof. Let X denote the set of red adjacencies of p in K_n and let Y denote the set of blue adjacencies of p in K_n. Apply the theorem with $t = k - 1$.

Choosing $\beta = 1/2$ in Corollary 1 gives the following corollary:

Corollary 2. Let K_n be 2-colored, $p \in V(K_n)$ and k is a positive integer. There exists a set $A \subset V(K_n)$ such that $p \in A$, $|A| \leqq k$ and A dominates all but at most $(n - 1)/2^k$ vertices of K_n.

If $k = \lfloor \log(n - 1) \rfloor + 1$ (log is of base 2), then $(n - 1)/2^k < 1$, so the next corollary follows from Corollary 2.

Corollary 3. Let K_n be 2-colored, $p \in V(K_n)$. Then there exists a set $A \subset V(K_n)$ such that $|A| \leqq \lfloor \log(n - 1) \rfloor + 1$, $p \in A$, and A dominates all vertices of $K_n - A$.

The proof of Theorem 1 is given in the next section. The third section of the paper is a summary of remarks and related results.

2. PROOF OF THEOREM 1

The following proposition will be used in the proof of Theorem 1:

Proposition. Let $\gamma \in [0, 1]$ and let t be a nonnegative integer. If $[A, B]$ is a 2-colored complete bipartite graph such that the red degree of each vertex in A is at most $\gamma|B|$, then there exists a subset of at most t vertices of B that b-dominates all but at most $\gamma^t|A|$ vertices of A.

Proof. The proposition is trivial for $t = 0$. Assume that $t \geqq 1$ and let $y_1 \in B$ be a vertex of maximum blue degree. Since $[A, B]$ has at least $(1-\gamma)|A||B|$ blue edges, the blue degree of y_1 is at least $(1-\gamma)|A|$. Therefore, y_1 b-dominates all but at most $\gamma|A|$ vertices of A. Let A_1 denote the set of vertices in A not b-dominated by $\{y_1\}$, and repeat the process with $[A_1, B]$. Since the number of blue edges of $[A_1, B]$ is at least $(1-\gamma)|A_1||B|$, there exists $y_2 \in B$ with blue degree at least $(1-\gamma)|A_1|$ in $[A_1, B]$. Therefore y_2 b-dominates all but at most $\gamma|A_1|$ vertices of A_1, which implies that $\{y_1, y_2\}$ b-dominates all but at most $\gamma|A_1| \leqq \gamma^2|A|$ vertices of A. Note that $y_2 = y_1$ is possible. The proposition follows by repeating this argument.

The following inequality of Minkowski is needed (see [1], p. 26):

Lemma. If a_i, b_i are nonnegative real numbers for $i = 1, 2, \ldots, n$, then

$$\prod_{i=1}^{n} (a_i + b_i)^{1/n} \geq \left(\prod_{i=1}^{n} a_i\right)^{1/n} + \left(\prod_{i=1}^{n} b_i\right)^{1/n}$$

Proof of Theorem 1. The theorem is trivial for $t = 0$. Assume that $t \geqq 1$ and let x_1 be a vertex of X with largest red degree in $[X, Y]$. Set $Y_1 = \Gamma_{\text{red}}(x_1)$, where $\Gamma_{\text{red}}(x)$ denotes the set of red adjacencies of x. Let x_2 be a vertex of X with largest red degree in $[X, Y - Y_1]$, set $Y_2 = \Gamma_{\text{red}}(x_2) \cap (Y - Y_1)$. Continue this process until x_t is defined. In general, x_i is a vertex of X with largest red degree in the complete bipartite graph

$$\left[X, Y - \bigcup_{j=1}^{i-1} Y_j\right]$$

and

$$Y_i = \Gamma_{\text{red}}(x_i) \cap \left(Y - \bigcup_{j=1}^{i-1} Y_j\right).$$

Note that the vertices $x_1, x_2, \ldots, x_t$ are not necessarily distinct.
For $i = 1, 2, \ldots, t$ set

$$\alpha_i = \frac{|Y_i|}{\left|Y - \bigcup_{j=1}^{i-1} Y_j\right|}.$$

With this notation

$$\bigcup_{j=1}^{t} \{x_j\}$$

r-dominates all but at most $a = (1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_t)|Y|$ vertices of Y.

Choose l so that

$$\alpha_l = \min_{1 \leqq j \leqq t} \{\alpha_j\} .$$

From the definition of x_l, each vertex of X has red degree at most $|Y_l|$ in the complete bipartite graph

$$\left[X, Y - \bigcup_{j=1}^{l-1} Y_j\right] .$$

Since

$$|Y_l| = \alpha_l \left| Y - \bigcup_{j=1}^{l-1} Y_j \right|$$

one can apply the proposition with $\gamma = \alpha_l$ to the complete bipartite graph

$$\left[X, Y - \bigcup_{j=1}^{l-1} Y_j\right].$$

It follows from the proposition that some subset of at most t vertices of

$$Y - \bigcup_{j=1}^{l-1} Y_j$$

b-dominates all but at most $\alpha_l^t|X|$ vertices of X. The choice of α_l implies that $\alpha_l^t|X| \leqq \alpha_1\alpha_2 \ldots \alpha_t|X|$, so some subset of at most t vertices of Y b-dominates all but at most $b = \alpha_1\alpha_2 \ldots \alpha_t|X|$ vertices of X.

The proof is completed by showing that either $a \leqq \beta^{t+1}(|X| + |Y|)$ or $b \leqq (1 - \beta)^{t+1}(|X| + |Y|)$. Set $a_i = (1 - \alpha_i), b_i = \alpha_i$ for $i = 1, 2, \ldots, t$ and $a_{t+1} = |Y|/(|X| + |Y|)$, $b_{t+1} = |X|/(|X| + |Y|)$. Apply the lemma with $n = t + 1$ to obtain

$$1 \geqq \left(\frac{a}{|X| + |Y|}\right)^{1/(t+1)} + \left(\frac{b}{|X| + |Y|}\right)^{1/(t+1)}$$

Since $\beta + (1 - \beta) = 1$, either

$$\left(\frac{a}{|X| + |Y|}\right)^{1/(t+1)} \leqq \beta$$

or

$$\left(\frac{b}{|X| + |Y|}\right)^{1/(t+1)} \leqq (1 - \beta) ,$$

and the proof of Theorem 1 is complete.

3. REMARKS AND RELATED PROBLEMS

It is worth mentioning that the proof of Theorem 1 is constructive; in fact, it is a greedy-type low-order polynomial algorithm to find the required (red or blue) dominating set. The same remark is true for the corollaries of Theorem 1; in particular, a dominating set of at most log n vertices can be found in a 2-colored K_n by a fast greedy algorithm. One might expect that the reason for this algorithmically nice behavior is that the results are not sharp. However, this is not the case; the random 2-coloring of K_n shows that Corollaries 2 and 3 are reasonably sharp.

Theorem 2. For fixed $\epsilon > 0$ and t there exists $n_0 = n_0(\epsilon, t)$ and a 2-coloring of K_n for $n \geqq n_0$ such that each t-element subset fails to dominate at least $((1/2^t) - \epsilon)n$ vertices of K_n.

Proof. Let t be fixed, ϵ fixed, and set $p = ((1/2^t) - \epsilon)n$. Assume that the edges of K_n are colored red or blue with probability 1/2. The probability that a fixed t-element vertex set of K_n r-dominates all but exactly k vertices is

$$\binom{n-t}{k}\left(1-\frac{1}{2^t}\right)^{n-t-k}\left(\frac{1}{2^t}\right)^k.$$

Therefore, the probability that some t-element vertex set of K_n dominates all but at most p vertices is at most

$$2\binom{n}{t}\sum_{k=0}^{p}\binom{n-t}{k}\left(1-\frac{1}{2^t}\right)^{n-t-k}\left(\frac{1}{2^t}\right)^k = x. \tag{1}$$

If $x < 1$ then there exists a 2-coloring of K_n such that each subset of t vertices of K_n fails to dominate at least p vertices as required.

The condition for nondecreasing terms in the summation of (1) is that $n \geq (t + 2^t - 1)/2^t\epsilon$. So in case

$$n \geq \frac{t + 2^t - 1}{2^t\epsilon}, \tag{2}$$

the summation has the trivial upper bound $(p + 1)$ times the $(p + 1)th$ term. Thus $p \leq n$, $\binom{n-t}{p} < \binom{n}{p} < n^n/(p^p(n-p)^{n-p})$, $\binom{n}{t} < n^t/t!$, and $c_t = 2(1 - (1/2^t))^{-t}/t!$ gives

$$x < c_t n^{t+1}\frac{n^n}{p^p(n-p)^{n-p}}\left(1-\frac{1}{2^t}\right)^n\left(\frac{1}{2^t-1}\right)^p \tag{3}$$

Set $q = p/n = 1/2^t - \epsilon$, so that (3) can be written as

$$x < c_t n^{t+1}\left(\left(\frac{1}{2^tq}\right)^q\left(\frac{1-(1/2^t)}{1-q}\right)^{1-q}\right)^n = c_t n^{t+1}A^n \tag{4}$$

The following inequality is needed. For positive a, b, α, β such that $\alpha + \beta = 1$, $a^\alpha b^\beta \le \alpha a + \beta b$ with equality if and only if $a = b$ ([1], p. 15). With $\alpha = q$, $\beta = 1 - q, a = 1/(2^t q), b = (1 - (1/2^t))/(1 - q)$ this inequality gives that $A \le 1$ with equality if and only if $q = 1/2^t$. Since $q = 1/2^t - \epsilon$, equality cannot hold. Therefore $A < 1$. Since A depends only on ϵ and t, the right-hand side of (4) clearly tends to zero if ϵ, t are fixed and n tends to infinity. Therefore $x < 1$ holds for $n \ge n_0 = n_0(t, \epsilon)$.

Theorem 3. For given $\epsilon > 0$, there exists $n_0 = n_0(\epsilon)$ and a 2-coloring of K_n such that for $n \ge n_0$ each set of at most $(1 - \epsilon) \log n$ vertices fails to dominate some vertices of K_n.

Proof. The proof (and the theorem) is almost the same as the proof a result of Erdös about the $S(k)$ property of tournaments ([3] or [4], p. 40). If

$$2\binom{n}{k}\left(1 - \frac{1}{2^k}\right)^{n-k} < 1,$$

then there exists a 2-coloring of K_n where each set of k vertices fails to dominate some vertices of K_n. It is easy to check that this inequality is true if $k = (1 - \epsilon) \log n$ and n is large.

It is natural to ask analogous questions when the edges of K_n are colored with more than two colors.

If the edges of K_n are colored with r colors then for each t there exist some subset of at most t vertices of K_n that (monochromatically) dominates all but at most $((r - 1)/r)^t n$ vertices of K_n.

One can check that the statement is essentially true for $t = 2$ (the required color can be the one used most frequently on K_n) and it is also true if the majority color class induces a regular subgraph of K_n. However, as H. A. Kierstead observed ([5]), if $t \ge 3$ and $r \ge 3$, the statement is false. The simple example is a K_n whose vertices are partitioned into three sets, A_1, A_2, A_3. If $1 \le i \le j \le 3$ and $x \in A_i, y \in A_j$, then the edge xy is colored with color i. Clearly, any 3 vertices fail to dominate at least $n/3 (> n(\frac{2}{3})^3)$ vertices showing that the statement is false.

References

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