Reprinted from the AMERICAN MATHEMATICAL MONTHLY
Vol. 96, No. 7, August–September 1989

A Problem of Leo Moser About Repeated Distances on the Sphere

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Abstract. We disprove a conjecture of Leo Moser by showing that (i) for every natural number n and $0 < \alpha < 2$ there is a system of n points on the unit sphere S^2 such that the number of pairs at distance α from each other is at least const $\cdot\, n \log^* n$ (where $\log^*$ stands for the iterated logarithm function) (ii) for every n there is a system of n points on S^2 such that the number of pairs at distance $\sqrt{2}$ from each other is at least const $\cdot\, n^{4/3}$. We also construct a set of n points in the plane in general position (no 3 on a line, no 4 on a circle) such that they determine fewer than const $\cdot\, n^{\log 3/\log 2}$ distinct distances, which settles a problem of Erdös.

1. Points in the plane. In most extremal problems in combinatorial and discrete geometry the configurations, arrangements, packings, coverings, etc. which are expected or proved to be optimal, are symmetric in one sense or another. In fact, it is a major obstacle in the way of the research in this field that very few symmetric patterns using a large number of objects are known. Perhaps this is one of the reasons why "latticelike" configurations have attracted so much attention in recent years, and two leading geometers devoted the last couple of years to writing a monograph about "Tilings and Patterns" **[GSh]**. In spite of the fact that applications in crystallography and coding theory also inspired extensive computer searches for symmetric configurations, the lack of constructions still remains a general characteristic of the field.

Under these circumstances it is no surprise that so little is known about one of the oldest and most important unsolved problems in discrete geometry: Whether or not the regular lattice packing is the densest packing of equal balls in 3-dimensional Euclidean space. In one of his papers C. A. Rogers made the ironic remark that "many mathematicians believe, and all physicists know" that the answer to this question is affirmative [**R**]. The "knowledge" of the physicists originates in their belief (so often emphasized by Einstein) that the laws of Nature must be simple, and if there existed more economical configurations, then Nature would surely have "invented" them.

In this article we will be concerned with variants of the following questions:

1. What is the minimum number of distances which a set of n points can determine?
2. How many times can a given distance α occur among n points?

We first consider these questions for sets of points in the plane. Beliefs similar to those of the physicists mentioned above led the senior author, more than 40 years ago, to state the following conjectures [**E1**]:

(i) Every set of n points in the plane determines at least $c_1 n/\sqrt{\log n}$ distinct distances (for some constant $c_1 > 0$);

(ii) The number of times a given distance can occur among n points in the plane is at most $n^{1+c_2/\log\log n}$ (for some $c_2 > 0$).

Both bounds are attained for the point system

$$\{(x, y): 0 \leqslant x, y < \sqrt{n},\ x \text{ and } y \text{ are integers}\},$$

i.e., for a $\sqrt{n}$ by $\sqrt{n}$ piece of the integer grid, one of the few known truly symmetric configurations in the plane. In the past many serious attempts were made to attack these problems (see, e.g. [**M1**], [**Ch**], [**JSz**], [**BS**], [**SSzT**], [**ChSzT**], [**EGS**] or the surveys [**E2**], [**EP**], [**MP**]), but the gaps between the existing lower and upper bounds are still enormous. (Erdös offered 500 dollars for a proof or disproof of (i) or (ii) several times.)

Due to the small number of known instances of regular point systems in the plane, fighting against these problems is a little bit like shadow boxing: You do not know exactly where the enemy is. The known results in this field reflect the strength (and limits) of the weaponry of combinatorics rather than throw any light on the geometric structure behind. On the other hand, for similar reasons, we must admit that beyond the belief there is very little real evidence supporting the above conjectures.

Even less is known about question 1 under the restriction that the points are in *general position*, i.e. there are no 3 of them on a straight line and no 4 on a circle. We shall need some notation.

Given a set $P = \{p_1, p_2, \dots, p_n\}$ of n distinct points and a positive number α, let

$$\begin{aligned} f(P, \alpha) &= \#\text{ pairs } (p_i, p_j),\ i < j, \text{ at distance } \alpha \text{ from each other,} \\ g(P) &= \#\text{ distinct distances determined by pairs of points of } P. \end{aligned} \tag{1}$$

Using this notation, let $G(n) = \min g(P)$, where the minimum is taken over all n-element point sets P in the plane in general position. Erdös has asked many times (see, e.g. [**E3**]) the following questions: Is it true that

$$\lim_{n\to\infty} \frac{G(n)}{n} = \infty, \tag{a}$$

$$\lim_{n\to\infty} \frac{G(n)}{n^2} = 0? \tag{b}$$

(Our ignorance in this area is really shocking!) Szemerédi [**Sz**] observed that $G(n) \geqslant (n-1)/3$. (In fact, he conjectures $G(n) \geqslant (n-1)/2$, which would generalize a theorem of Altman [**A**]). Our next result answers question (b) in the affirmative.

THEOREM 1. *For every natural number n, $G(n) < (3/2)n^{\log 3/\log 2} < (3/2)n^{1.585}$*

Proof. First consider the case $n = 2^k$, and let P be the set of all vertices of the unit cube in $\mathbb{R}^k$, i.e. all $(0,1)$-sequences $x = (x_1, x_2, \ldots, x_k)$ of length k. Since $x - x'$ is always a $(0, +1, -1)$-sequence, the pairs of distinct points belonging to P determine $3^k - 1$ different vectors. These occur in $(3^k - 1)/2$ pairs of opposite vectors.

One can obviously choose a 2-dimensional plane $\Pi \subseteq \mathbb{R}^k$ such that the orthogonal projection of P onto Π is in general position. The projection set P' also determines at most $(3^k - 1)/2$ pairs of opposite vectors, and hence at most this many different distances. Thus $G(2^k) < 3^k/2$.

Now let n be arbitrary. Pick k so that $2^{k-1} < n \leqslant 2^k$. Since G is clearly nondecreasing, we have $G(n) \leqslant G(2^k) < 3^k/2$. But $k < 1 + \log n/\log 2$, so $G(n) < (3/2)3^{\log n/\log 2} = (3/2)n^{\log 3/\log 2}$

Note that the same construction was used in [**DG**] for different purposes. Since the points of P' determine a large number of parallelograms, one cannot resist asking the following question: Does there exist a set P of n points in the plane in general position, such that P does not contain all the vertices of a parallelogram, but $g(P)$, the number of distinct distances determined by P, is $o(n^2)$?

2. Points on the sphere. What happens if, instead of point systems in the plane, we consider point systems on the sphere? The situation here differs from that in the plane in two important respects. First, there is nothing analogous to the integer lattice, so there are no obvious candidates for the sets which answer questions 1 and 2. Second, the answer to question 2 will depend on the particular distance α.

Let S^{d-1} denote the surface of the d-dimensional unit ball, i.e.,

$$S^{d-1} = \{(x_1, \ldots, x_d) \colon x_1^2 + \cdots + x_d^2 = 1\}.$$

More than 20 years ago Leo Moser [**M2**] (see also [**Gu**], [**MP**]) conjectured that there exists a constant c such that among any n points on the unit sphere S^2 the same distance can occur at most cn times, i.e.,

$$f(P, \alpha) \leqslant cn \tag{2}$$

for any n-element set $P \subseteq S^2$ and for any $0 < \alpha \leqslant 2$. This conjecture was partly motivated by a well known result conjectured by Vázsonyi and proved independently by several authors ([**G**], [**H**], [**St**]), which states that if $P = \{p_1, \ldots, p_n\}$ is the vertex set of a 3-dimensional convex polytope and $\alpha = \max_{1 \leq i < j \leq n} |p_i - p_j|$, then $f(P, \alpha) \leq 2n - 2$.

However, our next theorem shows that Moser's conjecture is false. Let $\log^* n$ denote the minimum integer r such that, starting with n, one has to iterate the logarithm function r times to get a value smaller than or equal to 1.

THEOREM 2. *There exist* $c_1, c_2 > 0$ *such that*

(i) *for every natural number* n *and for every* $0 < \alpha < 2$ *one can find* n *points in* S^2 *with the property that each is at distance* α *from at least* $c_1 \log^* n$ *others;*

(ii) *for every natural number* n *one can find* n *points in* S^2 *with the property that each is at distance* $\sqrt{2}$ *from at least* $c_2 n^{1/3}$ *others.*

Proof. (i) Given any $\varepsilon \geq 0$, let

$$S_\varepsilon = \{(x, y, z) \colon x^2 + y^2 + z^2 = 1 \text{ and } |z| \leq \varepsilon\}.$$

S_0 is the *equator* of S^2, and S_ε is called a *strip* of radius ε around the equator.

Let $0 < \alpha < 2$ be fixed. We also fix a small positive ε such that

$$2\sqrt{1 - \varepsilon^2} > \alpha, \tag{3}$$

i.e., the diameter of the two circles bounding S_ε is larger than α.

For each $k \geq 1$ we shall construct a point set P on the sphere in which each point is at distance α from at least k others. Our construction will be recursive. For $k = 1$, let P consist of 2 points on the equator, at distance α from each other.

To motivate the recursive step, consider the analogous situation in the plane. Given a set P in the plane in which each point is at distance α from at least k others, let $P^* = P \cup \pi(P)$, where π is a translation by a vector of length α, chosen so that $P \cap \pi(P)$ is empty. Then P^* provides the desired set for $k + 1$. This doesn't work on the sphere because there is no isometry which moves every point the same distance. Instead, we will replace π by a set of rotations about a fixed axis, one for each point of P.

So suppose that for some k we have a set $P = \{p_1, p_2, \ldots, p_{n(k)}\}$ such that each p_i is at distance α from at least k others. Assume further that all points of P are in a narrow strip around the equator, i.e., $P \subseteq S_{\varepsilon(k)}$ for some $\varepsilon(k) < \varepsilon$. Let u and v be two antipodal points on the sphere such that u is at distance δ from the north pole $(0, 0, 1)$ for some small δ which will be specified later. We turn S^2 around the axis uv so as to bring p_1 into a new position p_1' such that $|p_1 - p_1'| = \alpha$. (Note that, in view of (3), this is possible if δ is sufficiently small.) The rotation of S^2 which takes p_1 into p_1' and keeps u and v fixed, is denoted by π_1.

Let $P^{(0)} = P$, $P^{(1)} = \pi_1 P^{(0)}$. If the sets $P^{(0)}, P^{(1)}, \ldots, P^{(i-1)}$ have already been determined for some $1 < i \leq n(k)$, then we define $P^{(i)}$ as follows. Let π_i denote a rotation of S^2 around the axis uv for which $|p_i - \pi_i(p_i)| = \alpha$. Set

$$P^{(i)} = \pi_i(P^{(0)} \cup P^{(1)} \cup \cdots \cup P^{(i-1)}).$$

Finally, let

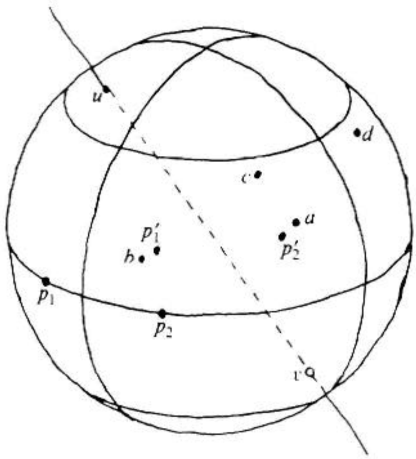
$$P^* = P^{(0)} \cup P^{(1)} \cup \cdots \cup P^{(n(k))}.$$

It is now clear that, for a proper choice of δ and the axis uv, (a) the above definitions are correct, i.e., all π_i's exist; (b) the sets $P^{(0)}, P^{(1)}, \ldots, P^{(n(k))}$ are pairwise disjoint; (c) there is an $\varepsilon(k+1) < \varepsilon$ such that $P^* \subseteq S_{\varepsilon(k+1)}$.

Set $n(k+1) = n(k)2^{n(k)}$. According to (b), we have $|P^{(i)}| = 2^{i-1}|P^{(0)}|$ for $1 \leqslant i \leqslant n(k)$, hence $|P^*| = n(k+1)$.

It is now a straightforward matter to show that every point of P^* is at distance α from at least $k+1$ others, which establishes Theorem 2(i) for the numbers $n(k)$. The result then follows easily for all n.

The figure below shows this construction for $k = 1$. Here $P^{(1)} = \{p_1', a\}$ is obtained by applying π_1 to $P^{(0)} = P = \{p_1, p_2\}$. Then $P^{(2)} = \{b, p_2', c, d\}$ is obtained by applying π_2 to $P^{(0)} \cup P^{(1)}$. The following pairs of points are at distance α: p_1p_2, p_1p_1', $p_1'a$, p_2p_2', bp_2', bc, cd, ad.



(ii) By a construction due to Erdös (see, e.g., [**Ed** Thm. 6.18]), there exists a positive constant c_2 such that one can pick $n/2$ points and $n/2$ lines in the plane, with the property that each of the points lies on at least $c_2n^{1/3}$ of the lines and each of the lines contains at least $c_2n^{1/3}$ of the points. Let O be a point outside the plane supporting this construction.

To each point P of the construction we assign the unit vector pointing from O to P. To each line L of the construction we assign one of the two unit vectors perpendicular to the plane determined by O and L. This gives n vectors with the property that each of them is perpendicular to at least $c_2n^{1/3}$ others. The endpoints of these vectors lie on the unit sphere centered at O and meet the requirements of (ii).

Note: It follows by the methods used in [**EGS**] that the bound $n^{1/3}$ in Theorem 2(ii) cannot be improved.

Had Moser's conjecture (2) been true, it would have implied that any n-element point set P on the sphere S^2 determines at least const $\cdot\, n$ different distances. That is, using our notation (1),

$$g(P) \geqslant c'n \tag{4}$$

with an absolute constant c'. It is an intriguing open question to decide whether this weaker version of Moser's conjecture is true.

However, it is not hard to show that (4) cannot hold for all n-element subsets of any higher dimensional spheres.

THEOREM 3. *For every $d \geqslant 4$, there exists a constant c_d with the property that for infinitely many n one can find an n-element point set $P \subseteq S^{d-1}$ determining*

$$g(P) \leqslant \begin{cases} c_4 \dfrac{n}{\log \log n} & \text{if } d = 4 \\ c_d n^{2/(d-2)} & \text{if } d > 4 \end{cases}$$

different distances.

We close with some questions suggested by Theorem 2.

Our result for $\sqrt{2}$ is stronger than that for other distances, since $n^{1/3}$ grows faster than $\log^* n$. Is $\sqrt{2}$ really special, or is there a construction which gives a similar result for every $0 < \alpha < 2$? Lacking that, are there such constructions for other particular values of α? (Since $\sqrt{2}$ is the edge length of a regular octahedron inscribed in the unit sphere, perhaps the edge lengths of the other Platonic solids are worth investigating.)

We wish to thank Herbert Edelsbrunner for some valuable comments.

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