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Partitions of Bases into Disjoint Unions of Bases*

PAUL ERDÖS

Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary

AND

MELVYN B. NATHANSON

Office of the Provost and Vice President for Academic Affairs, Lehman College (CUNY), Bronx, New York 10468

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Let A be an asymptotic basis of order h in the sense of additive number theory, and let $f(n)$ denote the maximum number of pairwise disjoint representations of n in the form $n = a_{i_1} + a_{i_2} + \cdots + a_{i_h}$, where $a_{i_j} \in A$ and $a_{i_1} \leqslant a_{i_2} \leqslant \cdots \leqslant a_{i_h}$. Let $t \geqslant 2$. If $f(n) \geqslant c \log n$ for c sufficiently large, then A can be written in the form $A = A_1 \cup \cdots \cup A_t$, where $A_i \cap A_j \neq \varnothing$ for $1 \leqslant i < j \leqslant t$ and A_j is an asumptotic basis of order h for $j = 1, ..., t$. If $\lim_{n \to \infty} f(n)/\log n = \infty$, then $A = \bigcup_{j=1}^{\infty} A_j$, where $A_i \cap A_j = \varnothing$ for $i \neq j$ and each A_j is an asymptotic basis of order h. These results are obtained by means of some purely combinatorial theorems. Related open problems are also discussed. © 1988 Academic Press, Inc.

In this paper we shall prove some Ramsay-like theorems about partitions of finite subsets of a countably infinite set, and then apply these results to Waring's problem and other classical topics in additive number theory.

Here is an example of the kind of results we obtain. Let A be an infinite set, and let $h \geqslant 2$. For each $n \geqslant 1$, let $S(n)$ be a collection of $f(n)$ pairwise disjoint subsets of A of cardinality at most h. If $f(n)$ tends to infinity sufficiently fast, for example, if $f(n) \geqslant c_h \log n$ for $n \geqslant n_0$, then there is a partition of A into two disjoint sets $A = A_1 \cup A_2$ such that for every $n \geqslant n_1$ there exist sets U_1, $U_2 \in S(n)$ with the property that $U_1 \subseteq A_1$ and $U_2 \subseteq A_2$. This will be shown to have the following arithmetical consequence: Let

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$k \geqslant 2$. Then there is a partition of the set of positive kth powers $\{n^k \mid n \geqslant 1\} = A_1 \cup A_2$ such that Waring's problem holds independently for both A_1 and A_2. That is, there is a number $G = G(k, A_1, A_2)$ such that for $i = 1, 2$ every sufficiently large integer is the sum of G kth powers belonging to A_i.

The paper is divided into three parts. Part 1 contains combinatorial results. Part 2 applies these results to number theory. Part 3 discusses some related unsolved problems.

1. Combinatorial Results

Let $|U|$ denote the cardinality of the set U. Let A be a countably infinite set, and let $h \geqslant 1$ be an integer. Denote by $[A]^h$ the collection of all subsets $U \subseteq A$ with $|U| = h$ and by $[A]^{\leqslant h}$ the collection of all subsets $U \subseteq A$ with $|U| \leqslant h$.

The probability of an event E is denoted prob(E).

Theorem 1. *Let A be a countably infinite set. Let $h \geqslant 1$ and $t \geqslant 2$ be integers. For each $n \geqslant 1$, let*

$$S(n) \subseteq [A]^{\leqslant h}$$

satisfy the following conditions:

(i) *If $U, V \in S(n)$ and $U \neq V$, then $U \cap V = \varnothing$,*

(ii) *There are constants c and n_0 such that*

$$c > \frac{1}{\log(t^h/(t^h - 1))}$$

and

$$f(n) = |S(n)| \geqslant c \log n$$

for all $n \geqslant n_0$. Then there exists a partition of A into t disjoint sets $A_1, ..., A_t$ such that

(iii) $S(n) \cap [A_j]^{\leqslant h} \neq 0$ *for $j = 1, ..., t$ and all $n \geqslant n_1$.*

Proof. Let $\lambda = t^h/(t^h - 1)$. Then $\lambda > 1$. By condition (ii), there exists $\delta > 0$ such that $c \log \lambda = 1 + \delta$.

We construct a probability measure on the space of all partitions

$$A = A_1 \cup \cdots \cup A_t$$

by setting

$$\operatorname{prob}(a \in A_j) = \frac{1}{t}$$

for all $a \in A$ and $j = 1, \ldots, t$.

Let $n \geqslant n_0$. If $U \in S(n)$, then $|U| = h_1 \leqslant h$ and $\operatorname{prob}(U \subseteq A_j) = t^{-h_1}$. It follows that

$$\operatorname{prob}(U \nsubseteq A_j) = 1 - \frac{1}{t^{h_1}} \leqslant 1 - \frac{1}{t^h} = \frac{1}{\lambda}$$

and so

$$\operatorname{prob}(S(n) \cap [A_j]^{\leqslant h} = \varnothing) \leqslant \frac{1}{\lambda^{f(n)}} \leqslant \frac{1}{\lambda^{c \log n}} = \frac{1}{n^{1+\delta}}.$$

Therefore,

$$\operatorname{prob}(S(n) \cap [A_j]^{\leqslant h} = \varnothing \text{ for some } j = 1, \ldots, t) \leqslant \frac{t}{n^{1+\delta}}.$$

Since the series $\sum_{n=n_0}^{\infty} t/(n^{1+\delta})$ converges, the Borel–Cantelli lemma implies that for almost all partitions $A = A_1 \cup \cdots \cup A_t$ there exists n_1 such that condition (iii) is satisfied for $j = 1, \ldots, t$ and all $n \geqslant n_1$. This proves the theorem.

THEOREM 2. *Let A be a countably infinite set. Let $h \geqslant 1$. For each $n \geqslant 1$, let*

$$S(n) \subseteq [A]^{\leqslant h}$$

satisfy the following conditions:

(i) *If $U, V \in S(n)$ and $U \neq V$, then $U \cap V = \varnothing$.*

(ii) *Define $f(n) = |S(n)|$. Then*

$$\lim_{n \to \infty} \frac{f(n)}{\log n} = \infty.$$

Then there exists a partition $A = \bigcup_{k=1}^{\infty} A_k$ such that $A_i \cap A_j \neq 0$ for $1 \leqslant i < j < \infty$, and for each k there is an integer $n_1(k)$ such that

$$S(n) \cap [A_k]^{\leqslant h} \neq \varnothing$$

for all $n \geqslant n_1(k)$.

Proof. Define a probability measure on the space of all partitions

$$A = \bigcup_{k=1}^{\infty} A_k$$

by setting

$$\mathrm{prob}(a \in A_k) = \frac{1}{2^k}$$

for all $a \in A$. Let $E_{k,n}$ denote the set of all partitions $A = \bigcup_{j=1}^{\infty} A_j$ such that $S(n) \cap [A_k]^{\leqslant h} = \varnothing$. The probability of the event $E_{k,n}$ is

$$\mathrm{prob}(E_{k,n}) \leqslant \left(1 - \frac{1}{2^{hk}}\right)^{f(n)}.$$

Define

$$\lambda_k = 2^{hk}/(2^{hk} - 1).$$

Fix $\delta > 0$. Since $f(n)/\log n$ tends to infinity, there exist integers $n_0(k)$ such that

$$1 < n_0(1) < n_0(2) < \cdots < n_0(k) < \cdots$$

and

$$f(n) \log \lambda_k \geqslant (2 + \delta) \log n$$

for all $n \geqslant n_0(k)$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} \sum_{n=n_0(k)}^{\infty} \mathrm{prob}(E_{k,n}) &\leqslant \sum_{k=1}^{\infty} \sum_{n=n_0(k)}^{\infty} \lambda_k^{-f(n)} \\
&\leqslant \sum_{k=1}^{\infty} \sum_{n=n_0(k)}^{\infty} \lambda_k^{-(2+\delta)\log n/\log \lambda_k} \\
&= \sum_{k=1}^{\infty} \sum_{n=n_0(k)}^{\infty} \frac{1}{n^{2+\delta}} \\
&\leqslant \frac{1}{1+\delta} \sum_{k=1}^{\infty} \frac{1}{(n_0(k) - 1)^{1+\delta}} \\
&\leqslant \frac{1}{1+\delta} \sum_{k=1}^{\infty} \frac{1}{k^{1+\delta}} \\
&< \infty.
\end{aligned}$$

The Borel–Cantelli implies that for almost all partitions $A = \bigcup_{k=1}^{\infty} A_k$ there exists a sequence of integers $\{n_1(k)\}_{k=1}^{\infty}$ such that

$$S(n) \cap [A_k]^{\leqslant h} \neq \varnothing$$

for all $k \geqslant 1$ and $n \geqslant n_1(k)$. This concludes the proof.

2. Applications to Number Theory

The set A of integers is an *asymptotic basis of order* h if every sufficiently large integer can be represented as a sum of h (not necessarily distinct) elements of A. The classical theorems in additive number theory assert that special sequences of integers are asymptotic bases. Lagrange's theorem, for example, states that the squares form an asymptotic basis of order 4, and Waring's problem is the assertion that the set of positive kth powers is an asymptotic basis of some order $G(k)$. We shall prove that if A is an asymptotic basis of order h and if every large integer has sufficiently many representations as a sum of h elements of A, then the basis A can be decomposed into the union of a finite or infinite number of pairwise disjoint asymptotic bases of order h.

Let A be an asymptotic basis of order h, and let

$$n = a_{i_1} + \cdots + a_{i_h} = a_{j_1} + \cdots + a_{j_h}$$

be two representations of n as a sum of h elements of A. These representations are *disjoint* if

$$\{a_{i_1}, ..., a_{i_h}\} \cap \{a_{j_1}, ..., a_{j_h}\} = \varnothing.$$

Let $U = \{a_{i_1}, ..., a_{i_h}\}$. Since the integers a_{i_k} are not necessarily distinct, it follows that $|U| = h_1$, where $1 \leqslant h_1 \leqslant h$.

In general, let $\mathscr{F} = \{F_j\}_{j \in J}$ be a family of functions $F_j = F_j(x_1, ..., x_{h(j)})$ in $h(j) \leqslant h$ variables, and let A be a countable set in the domain of $\mathscr{F}$. Two representations

$$F_j(a_1, ..., a_{h(j)}) = F_k(a'_1, ..., a'_{h(k)})$$

are *disjoint* if $\{a_1, ..., a_{h(j)}\} \cap \{a'_1, ..., a'_{h(k)}\} = \varnothing$.

Theorem 3. *Let A be an asymptotic basis of order 2, and let $f(n)$ denote the number of representations of n in the form $n = a_{i_1} + a_{i_2}$ where $a_{i_1}, a_{i_2} \in A$ and $a_{i_1} \leqslant a_{i_2}$. Let $t \geqslant 2$. If $c > \log^{-1}(t^2/(t^2-1))$ and if $f(n) \geqslant c \log n$ for all $n \geqslant n_0$, then A can be partitioned into t pairwise disjoint sets, each of which is an asymptotic basis of order 2.*

Proof. For $n \geqslant 1$, let $S(n)$ consist of all sets $\{a_{i_1}, a_{i_2}\}$ such that $a_{i_1} + a_{i_2} = n$ and $a_{i_1}, a_{i_2} \in A$. Then $|S(n)| = f(n)$ and the sets $\{a_{i_1}, a_{i_2}\}$ are pairwise disjoint. Note that if n is even and $n/2 \in A$, then $\{n/2\} \in S(n)$. If $U \in S(n)$ and $U \neq \{n/2\}$, then $|U| = 2$. Applying Theorem 1 in the case $h = 2$, we conclude that there is a partition $A = A_1 \cup \cdots \cup A_t$ such that

$$S(n) \cap [A_j]^{\leqslant 2} \neq \varnothing$$

for $j = 1, ..., t$ and all $n \geqslant n_1$. If $U = \{a_{i_1}, a_{i_2}\} \in S(n) \cap [A_j]^{\leqslant 2}$, then $n = a_{i_1} + a_{i_2}$ and so A_j is an asymptotic basis of order 2 for all $j = 1, ..., t$. This proves the theorem.

THEOREM 4. *Let A be an asymptotic basis of order h, and let $f(n)$ denote the cardinality of a maximal set of pairwise disjoint representations of n as a sum of h elements of A. Let $t \geqslant 2$. If $f(n) \geqslant c \log n$ for some constant $c > \log^{-1}(t^h/(t^h - 1))$ and all $n \geqslant n_0$, then A can be partitioned into the disjoint union of t sets, each of which is an asymptotic basis of order h.*

Proof. This follows from Theorem 1, exactly as in the proof of Theorem 3.

THEOREM 5. *Let A be an asymptotic basis of order h, and let $f(n)$ denote the cardinality of a maximal set of pairwise disjoint representations of n as a sum of h elements of A. If $\lim_{n \to \infty} f(n)/\log n = \infty$, then A can be partitioned into a countable union of pairwise disjoint sets, each of which is also an asymptotic basis of order h.*

Proof. This follows immediately from Theorem 2.

THEOREM 6. *Let $k \geqslant 2$ and let $A = \{n^k\}_{k=1}^{\infty}$. There exists an integer $s_0(k)$ such that for all $s > s_0(k)$ there is a partition $A = \bigcup_{j=1}^{\infty} A_j$ such that each set A_j is an asymptotic basis of order s.*

Proof. Let $d_{k,s}(n)$ denote the number of representations in some maximal collection of pairwise disjoint representations of n as a sum of s k-th powers. Nathanson [5, p. 304] proved that there exists an $s_0(k)$ such that for all $s > s_0(k)$ there is a constant $c > 0$ such that $d_{k,s}(n) > cn^{1/k}$ for all $n \geqslant 1$. Thus, $d_{k,s}(n)/\log n$ tends to infinity, and so the result follows immediately from Theorem 2 with $f(n) = d_{k,s}(n)$.

Lagrange proved that every positive integer is the sum of four squares, but it is easy to see that it is not possible to partition the squares into even two disjoint sets, each of which is an asymptotic basis of order 4. Let $r_4(n)$ denote the number of solutions in integers of the equation $n = a^2 + b^2 + c^2 + d^2$. Then $r_4(n) = 8 \sum'_{m|n} m$, where the summation runs over all positive divisors of n that are not divisible by 4. In particular, if $k \geqslant 1$, then

$r_4(2^{2k+1}) = 24$ and the only solutions of $2^{2k+1} = a^2 + b^2 + c^2 + d^1$ are permutations of the representation

$$2^{2k+1} = (\pm 2^k)^2 + (\pm 2^k)^2 + 0^2 + 0^2.$$

If $S(n)$ consists of all sets $U = \{a^2, b^2, c^2, d^2\}$ such that $n = a^2 + b^2 + c^2 + d^2$, then $S(2^{2k+1})$ consists of the single set $\{0, 4^k\}$ and $f(2^{2k+1}) = |S(2^{2k+1})| = 1$ for all $k \geqslant 1$. It follows that if $\{n^2 | n \geqslant 0\} = A_1 \cup A_2$ and $A_1 \cap A_2 = \varnothing$, then not both A_1 and A_2 are asymptotic bases of order 4.

If we consider only numbers not divisible by 4, however, then it is possible to establish a positive result.

THEOREM 7. *Let $T = \{n \geqslant 0 | n \not\equiv 0 \pmod 4\}$. Then there is a partition $\{n^2 | n \geqslant 0\} = \bigcup_{j=1}^{\infty} A_j$ such that for each j there is an integer n_j such that if $n \in T$ and $n \geqslant n_j$, then n is a sum of four elements of A_j.*

Proof. For $n \in T$, let $f(n)$ denote the number of representations in some maximal collection of pairwise disjoint representations of n as a sum of four squares. Erdös and Nathanson [2] proved that for every $\varepsilon > 0$ there exists a constant $c = c(\varepsilon) > 0$ such that $f(n) > cn^{(1/2) - \varepsilon}$ for all $n \in T$. The result follows immediately from Theorem 2.

THEOREM 8. *Let $F_j = F_j(x_1, ..., x_{h(j)})$ be a function in $h(j) \leqslant h$ variables, and let $\mathscr{F} = \{F_j\}_{j \in J}$. Let A be a set of integers. Let $\mathscr{F}(A)$ be the set of all numbers of the form $F_j(a_1, ..., a_{h(j)})$, where $F_j \in \mathscr{F}$ and $a_1, ..., a_{h(j)} \in A$. Let $W = \{w_n\}_{n=1}^{\infty} \subseteq \mathscr{F}(A)$. Let $f(n)$ denote the maximum number of pairwise disjoint representation of w_n in the form $w_n = F_j(a_1, ..., a_{h(j)})$. If $f(n) \geqslant c \log n$ for some $c > \log^{-1}(t^h/(t^h - 1))$ and all $n \geqslant n_0$, then there is a disjoint partition $A = \bigcup_{j=1}^{t} A_j$ such that*

$$\{w_n\}_{n=n_1}^{\infty} \subseteq \mathscr{F}(A_j)$$

for some integer n_1 and all $j = 1, ..., t$.

If $\lim_{n \to \infty} f(n)/\log n = \infty$, then there is a disjoint partition $A = \bigcup_{j=1}^{\infty} A_j$ and integers $n_1(j)$ such that

$$\{w_n\}_{n=n_{1(j)}}^{\infty} \subseteq \mathscr{F}(A_j)$$

for all $j = 1, 2, ...$

Proof. This follows immediately from Theorems 1 and 2.

3. OPEN PROBLEMS

1. In Theorem 1 the condition that $f(n) \geqslant c \log n$ is best possible in the following sense. In the case $A = \{1, 2, 3, ...\}$ and $h = t = 2$, R. L. Graham

(personal communication) has constructed for each n a collection $S(n)$ of pairs of integers such that $f(n)=|S(n)|\geqslant c\log n$ for some $c>0$ and all $n\geqslant 1$, but there is no partition $A=A_1\cup A_2$ such that $S(n)\cap[A_i]^{\leqslant 2}\neq\varnothing$ for $i=1$ and $i=2$ and all $n\geqslant n_1$. Let $c(h,t)$ denote the infimum of all real numbers c such that the conclusion of Theorem 1 holds whenever $f(n)\geqslant c\log n$. Calculate $c(h,t)$.

2. Let A be an asymptotic basis of order 2 and let $f(n)$ denote the number of representations of n in the form $n=a_i+a_j$, where $a_i, a_j\in A$ and $a_i\leqslant a_j$. According to Theorem 3, if $f(n)\geqslant c\log n$ for some $c>\log^{-1}(4/3)$ and all $n\geqslant n_0$, then A can be partitioned into the disjoint union of two asymptotic bases of order 2. Can the condition that $f(n)\geqslant c\log n$ be weakened? In particular, if we assume only that $\lim_{n\to\infty}f(n)=\infty$, does $A=A_1\cup A_2$, where $A_1\cap A_2=\varnothing$ and A_1 and A_2 are both asymptotic bases of order 2?

3. An asymptotic basis A of order h is *minimal* if no proper subset of A is an asymptotic basis of order h. Härtter [3] and Nathanson [4] proved that there exist asymptotic bases that do not contain any minimal asymptotic bases. On the other hand, Erdös and Nathanson [1] proved that if A is an asymptotic basis of order 2 such that $f(n)\geqslant c\log n$ for some $c>\log^{-1}(4/3)$ and all $n\geqslant n_0$, then A contains a minimal asymptotic basis of order 2. The proof is similar to the proof of Theorem 1, but seems to work only in the case $h=2$. It is not known whether an asymptotic basis A of order $h>2$ for which $f(n)\geqslant c\log n$ for some constant c sufficiently large must necessarily contain a minimal asymptotic basis of order h. An old problem of Erdös and Nathanson [1] is the following: If A is an asymptotic basis of order h such that $\lim_{n\to\infty}f(n)=\infty$, then does A contain a minimal asymptotic basis of order h? This is open even in the case $h=2$.

4. It is not clear if there is a relationship between asymptotic bases that contain minimal asymptotic bases and asymptotic bases that can be decomposed into the disjoint union of two asymptotic bases. For example, let A_1 and A_2 be asymptotic bases of order 2 such that $A_1\cap A_2=\varnothing$. Let $A=A_1\cup A_2$. Does A contain a minimal asymptotic basis of order 2?

5. Under the conditions of Theorem 1, let $f(n)>c'\log n$ for some sufficiently large constant c'. Then there exists $\delta>0$ such that

$$|S(n)\cap[A_j]^{\leqslant h}|>\delta\log n$$

for $j=1,\ldots,t$ and $n\geqslant n_1$. The proof is essentially the same as the proof of Theorem 1. Estimate the size of the constant c'.

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