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ON THE IRRATIONALITY OF CERTAIN SERIES: PROBLEMS AND RESULTS

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During my long life I have spent lots of time thinking about irrationality of series. The reader with a little maliciousness may say "spent and wasted" since I have never discovered any new general methods nor had any spectacular success like Apéry. Nevertheless I hope to convince the reader that not all of it was completely wasted. First of all, I state some of my previous results several of which were obtained with E. Straus. I state many unsolved problems and also prove some new theorems.

I proved more than 30 years ago [1] that for every integer $t > 1$,

$$\sum_{n=1}^{\infty} \frac{d(n)}{t^n} = \sum_{n=1}^{\infty} \frac{1}{t^n - 1}$$

($d(n)$ is the number of divisors of n) is irrational. Chowla conjectured that the same holds for every rational $t > 1$. This is almost certainly true but is unattackable by my methods. It is very annoying that I cannot prove that $\sum_{n=1}^{\infty} \frac{1}{2^n-3}$ and $\sum_{n=2}^{\infty} \frac{1}{n!-1}$ are both irrational (one of course expects that $\sum_{n=1}^{\infty} \frac{1}{2^n+t}$ and $\sum_{n=1}^{\infty} \frac{1}{n!+t}$ are irrational and in fact transcendental for every integer t.) Peter Borwein just informed me (June 1987) that he proved that $\sum \frac{1}{2^n+r}$ is irrational for every rational r. Denote by $\nu(n)$ the number of distinct prime factors of n; $\varphi(n)$ is Euler's φ function and $\delta_k(n)$ the sum of the k-th powers of divisors of n. It is very annoying that I cannot prove that $\sum_{n=1}^{\infty} \frac{\nu(n)}{2^n}$ is irrational; perhaps here I am overlooking a simple argument. $\sum_{n=1}^{\infty} \frac{\varphi(n)}{2^n}$ and $\sum_{n=1}^{\infty} \frac{\delta(n)}{2^n}$, $\delta(n) = \delta_1(n)$, are no doubt also irrational but this is probably unattackable by my methods.

Kac and I [2] proved that $\sum_{n=1}^{\infty} \frac{\delta_k(n)}{n!}$ is irrational for $k = 1$ and $k = 2$. Our proof does not seem to work for $k > 2$, but perhaps we again are overlooking a simple argument.

Straus and I [3] proved that if $1 < a_1 < a_2 < \ldots$ is a sequence of integers then

$$\sum_{n=1}^{\infty} \frac{d(n)}{a_1 a_2 \dots a_n}$$

is irrational; we conjectured that it suffices to assume that $a_n \to \infty$ but could not prove it. We also conjectured that if $a_{n+1} \geq a_n$ then

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{a_1 \dots a_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\delta(n)}{a_1 \dots a_n} \tag{1}$$

are irrational, but we could only prove (1) if we made some further assumptions on the growth properties of the a's. Observe that $a_n = \varphi(n) + 1$, $a_n = \delta(n) + 1$ shows that $a_n \to \infty$ does not suffice for the irrationality of (1). I further proved that if $p_1 < p_2 < \dots$ is the sequence of primes then $\sum_{n=1}^{\infty} \frac{p_n^k}{n!}$ is irrational for every k [4]. I could not prove that $\sum \frac{p_n^k}{2^n}$ is irrational for every k. This is probably very difficult already for $k = 1$. It seems reasonable to expect that if $g_n \geq 2$, $g_n/p_n \to 0$ then

$$\sum_{n=1}^{\infty} p_n / g_1 \cdots g_n \tag{2}$$

is irrational, but I can prove the irrationality of (2) only under much more restrictive conditions; $g_n = p_n + 1$ shows that some growth condition is needed for the irrationality of (2).

A few years ago I proved [5] that if $n_{k+1} - n_k \to \infty$ then

$$\sum_{k=1}^{\infty} \frac{n_k}{2^{n_k}} \tag{3}$$

is irrational. The proof is not entirely trivial. I think (3) remains true if we assume only that $n_k/k \to \infty$, but unless I am overlooking a simple argument my proof breaks down. In this connection there are two questions which I could not settle and which particularly annoy me. The first question states: Does there exist a sequence $n_1 < n_2 < \dots$ for which $\limsup n_{k+1} - n_k = \infty$ but $\sum_{k=1}^{\infty} \frac{n_k}{2^{n_k}}$ is rational? The answer is almost certainly affirmative. The second question states: Let $q_1 < q_2 \dots$ be the sequence of square free numbers. Then $\sum q_n/2^{q_n}$ surely must be irrational, and in fact this should hold if the q's are an arbitrary subsequence of the square free numbers.

Several related problems are stated in my book with Graham [6]. Does the equation

$$\frac{n}{2^n} = \sum_{k=1}^{t} a_k/2^{a_k}, \qquad t > 1,$$

have a solution for infinitely many n,† or perhaps for all n? Is there a rational x for which

$$x = \sum_{k=1}^{\infty} a_k/2^{a_k}$$

has two solutions?

It is a simple exercise to prove that if $n_k^{1/2^k} \to \infty$ then $\sum \frac{1}{n_k}$ is irrational, and it is easy to see that the condition $n_k^{1/2^k} \to \infty$ cannot be weakened. On the other hand I proved [7] that if

$$\limsup n_k^{1/2^k} = \infty \quad \text{and} \quad n_k > k^{1+\varepsilon} \tag{4}$$

for all k then $\sum \frac{1}{n_k}$ is irrational. My proof is not entirely trivial. It is probable that many other theorems of this kind can be proved. In (4) the condition $n_k > k^{1+\varepsilon}$ is essentially best possible.

Once I asked: Assume that $\sum \frac{1}{n_k}$ and $\sum \frac{1}{n_k - 1}$ are both rational. How fast can n_k tend to infinity? I was (and am) sure that $n_k^{1/k} \to \infty$ is possible but $n_k^{1/2^k}$ must tend to 1. Unfortunately almost nothing is known. David Cantor observed that

$$\sum_{k=3}^{\infty} \frac{1}{\binom{k}{2}} \quad \text{and} \quad \sum_{k=3}^{\infty} \frac{1}{\binom{k}{2}+1}$$

are both rational and we do not know any sequence with this property which tends to infinity faster than polynomially. Stolarsky asked the following pretty question: is it true that if $\sum \frac{1}{n_k} < \infty$ then there is always an integer t, $t \neq n_k$, for which $\sum_{k=1}^{\infty} \frac{1}{n_k - t}$ is irrational?

Straus and I proved that the set of points in the plane of the form

$$x = \sum \frac{1}{n_k}, \qquad y = \sum \frac{1}{n_k + 1}$$

† A simple proof of this statement was communicated to me by Cusick (June 1987). The question for all n remains open.

contains open sets. Probably the analogous result holds for r dimensions. We never published our proof since we did not work out the r-dimensional case.

Straus and I proved [8] that if $\limsup n_k^2/n_{k+1} \le 1$ and $N_k = [n_1, \ldots, n_k]$ and

$$\limsup \frac{N_k}{n_{k+1}} \left(\frac{n_k^2}{n_{k+1}} - 1 \right) \le 0 \tag{5}$$

then $\sum 1/n_k$ is irrational except if $n_{k+1} = n_k^2 - n_k + 1$ for all $k > k_0$. Perhaps our result remains true without the assumption (5).

We defined a sequence $n_1 < n_2 < \ldots$ to be an *irrationality sequence* if, for every sequence of integers t_k, $\sum_{k=1}^{\infty} \frac{1}{t_k n_k}$ is irrational. Observe that $n!$ is not an irrationality sequence since $\sum \frac{1}{(n+2)n!} = 1$. We conjectured that and I later proved, [7], that $n_k = 2^{2^k}$ is an irrationality sequence.

It is not clear if the irrationality sequence must increase very rapidly. I have not been able to find an irrationality sequence for which $n_k^{1/2^k} \to 1$. Graham and I observed that if n_k is an irrationality sequence then $n_k^{1/k} \to \infty$. We do not know if there is an irrationality sequence $n_1 < n_2 < \ldots$ for which $(n_i, n_j) = 1$ and $\limsup n_k^{1/2^k} < \infty$.

Graham and I modified the definition of an irrationality sequence. Let us try to call a sequence $a_1 < a_2 \ldots$ an irrationality sequence if, for every $b_n/a_n \to 1$, $\sum_{n=1}^{\infty} \frac{1}{b_n}$ is irrational. The trouble with this definition is that we do not know a simple non-trivial irrationality sequence, for example, we cannot prove that 2^{2^n} is an irrationality sequence of this kind. On the other hand it is not difficult to show that if $\liminf n_k^{\frac{1}{2^k}} > 1$ and $\lim n_k^{\frac{1}{2^k}}$ does not exist then $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is irrational and hence $\{n_k\}$ is an irrationality sequence of this kind; but perhaps it is not an irrationality sequence with our original definition.

Another possibility would be to call $\{a_n\}$ an irrationality sequence if for every $|b_n| < C$, $\sum_{n=1}^{\infty} \frac{1}{a_n + b_n}$ is irrational. In this case we proved that 2^{2^n} is an irrationality sequence but we cannot decide if 2^n or $n!$ is an irrationality sequence. Is there an irrationality sequence a_n of this type which increases exponentially? It is not hard to show that it cannot increase slower than exponentially. As stated previously, Borwein showed that 2^n is an irrationality sequence of this kind.

The following further problems stated in [6] are perhaps interesting: let $n_1 < n_2 < \ldots$. Is it then true that $\sum \frac{1}{2^{n_k} - 1}$ is irrational, or perhaps $\sum \frac{1}{2^{n_k} - t_k}$ is irrational for every $|t_k| < C$?

Let $n_k \to \infty$ rapidly; then $\sum_{k=1}^{\infty} \frac{1}{n_k n_{k+1}}$ is irrational. Probably $\liminf n_k^{1/2^k} > 1$ should suffice.

It is not hard to prove that $\sum \frac{1}{2^{n_k}}$ is transcendental if $n_k/k^\ell \to \infty$ for every ℓ. Perhaps the weaker condition $\frac{1}{k}n_k \to \infty$ suffices. On the other hand we do not know of any algebraic number for which $\limsup(n_{k+1} - n_k) = \infty$, but in fact one would expect that every algebraic number which is irrational has this property. Many of these problems seem hopeless at present, but perhaps one can prove that if $n_k > ck^2$ then $\sum_{k=1}^{\infty} \frac{1}{2^{n_k}}$ is not the root of any quadratic polynomial.

Let $p_1 < p_2 < \ldots$ be an infinite sequence of primes. It is a simple exercise to prove that if $a_1 < a_2 < \ldots$ is the sequence of integers composed of the p's then

$$\sum_{n=1}^{\infty} \frac{1}{[a_1, \ldots, a_n]}$$

is irrational, where $[a_1, \ldots, a_n]$ is the least common multiple of $a_1, \ldots, a_n$. This result probably remains true if the number of primes p_i is finite but of course greater than 1.

We are going to prove the following Theorem. Let $a_1 < a_2 < \ldots$ be an infinite sequence of integers. Assume that for every $x > x_0$ and some $\varepsilon > 0$

$$A(x) = \sum_{a_i < x} 1 > (1 - \log 2 + \varepsilon)x. \tag{6}$$

Then

$$\sum_{n=1}^{\infty} \frac{1}{c(n)}$$

is irrational, where $c(n)$ is the least common multiple of the integers $a_i < n$.

We present the proof of Halberstam who simplified and clarified my somewhat carelessly presented proof.

We need the following simple lemma.

Lemma. Denote by $P(n)$ the greatest prime factor of n; then if $\eta = \eta(\varepsilon) > 0$ is sufficiently small we have

$$t = t(x) = \sum_{\substack{a_i < x \\ P(a_i) > x^{\frac{1}{2}+\eta}}} 1 > \tfrac{1}{2}\varepsilon x.$$

The proof follows easily from

$$\sum_{x^{1/2}<p<x} \frac{1}{p} = \log 2 + o(1).$$

The details are left to the reader.

Let $a_{n_1} < \dots < a_{n_t} \le x$, $t > \frac{1}{2}\varepsilon x$, be the integers for which $P(a_{n_i}) > x^{\frac{1}{2}+\eta}$. By the Lemma these a's exist. Let now p be a prime greater than $x^{\frac{1}{2}+\eta}$. There clearly are at most $\frac{x}{p} \le x^{\frac{1}{2}-\eta}$ of the a_{n_i} which are multiples of p and since an integer not exceeding x is divisible by at most one of these primes there are at least $\frac{1}{2}\varepsilon x^{\frac{1}{2}(1+\eta)}$ distinct primes p_i for which there is an integer a_{n_i} satisfying $P(a_{n_i}) = p_i > x^{\frac{1}{2}+\eta}$ and we can assume that a_{n_i} is chosen minimally.

Denote now by I_r the interval

$$(rx^{\frac{1}{2}}, (r+1)x^{\frac{1}{2}}), \qquad 1 \le r \le x^{\frac{1}{2}}.$$

There clearly is at least one r, say r_0, for which there are at least $u > \frac{1}{2}\varepsilon x^{\frac{\eta}{2}}$ integers a_{n_i} in I_{r_0} each of which have a prime factor $p_i > x^{\frac{1}{2}+\eta}$ and for distinct i's the p_i's are distinct. Denote these a's by $r_0 x^{\frac{1}{2}} < b_1 < \dots < b_u < (r_0+1)x^{\frac{1}{2}}$.

Now we are ready to prove our Theorem. Assume that

$$\sum_{n=1}^{\infty} \frac{1}{c(n)} = \ell_1/\ell_2, \qquad (\ell_1, \ell_2) = 1. \tag{7}$$

Multiply both sides by $\ell_2 c(b_1 - 1)$. We immediately obtain from (7)

$$\ell_2 c(b_1 - 1) \sum_{n \ge b_1} \frac{1}{c(n)} \ge 1. \tag{8}$$

Now by the definition of b_1, $b_1 \equiv 0 (\operatorname{mod} p)$ for some $p > x^{\frac{1}{2}+\eta}$ and b_1 is the smallest $a \equiv 0 (\operatorname{mod} p)$. Thus

$$\frac{c(b_1 - 1)}{c(b_1)} \le \frac{1}{p} \le \frac{1}{x^{\frac{1}{2}+\eta}}. \tag{9}$$

Write now (8) in the form

$$\ell_2 c(b_1 - 1)(\Sigma_1 + \Sigma_2) \ge 1, \tag{10}$$

where we place in Σ_1 the integers n in I_{r_0}; each such n satisfies $n \geq b_1$ and there are at most $x^{\frac{1}{2}} + 1$ such n's. Thus

$$c(b_1 - 1)\Sigma_1 < \frac{x^{\frac{1}{2}} + 1}{x^{\frac{1}{2}+\eta}} < 2x^{-\eta}. \tag{11}$$

Now we have to estimate $c(b_1 - 1)\Sigma_2$. If n is in Σ_2 we can of course assume $n > (r_0 + 1)x^{1/2}$, i.e. n lies beyond I_{r_0}. But since $b_1, b_2, \ldots, b_u$ are in I_{r_0} and each is divisible by a distinct prime $> x^{\frac{1}{2}+\eta}$, we have (for large x)

$$\frac{c(b_1 - 1)}{c(n)} < \left(\frac{1}{x^{1/2}}\right)^u < \left(\frac{1}{x^{1/2}}\right)^{\frac{1}{2}\varepsilon x^{\eta/2}} < x^{-10}.$$

Thus we evidently have

$$c(b_1 - 1) \sum_{(r_0+1)x^{1/2} < n \leq x^2} \frac{1}{c(n)} < \frac{x^2}{x^{10}} = x^{-8}. \tag{12}$$

Finally, suppose $n > x^2$. Write

$$\sum_{n > x^2} \frac{1}{c(n)} = \sum_{r=1}^{\infty} \sum_{x^{2^r} < n \leq x^{2^{r+1}}} \frac{1}{c(n)}. \tag{13}$$

We obtain from our Lemma and by the argument we just used that

$$c([y^{1/2}]) \sum_{y < n \leq y^2} \frac{1}{c(n)} < y^{-8}. \tag{14}$$

Thus from (13) and (14) we obtain

$$\sum_{n \geq x^2} \frac{c(b_1 - 1)}{c(n)} < \sum_{r=1}^{\infty} (x^{2^r})^{-8} < x^{-8}. \tag{15}$$

Then (11), (12) and (15) clearly contradict (10) which completes the proof of our Theorem. It can be shown without much difficulty that the Theorem does not remain true if $A(x) < x(1 - \log 2 + \varepsilon)$.

References

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For some further results on irrationality see P. Erdős, On the irrationality of certain series, *Indagationes Math.* **19** (1957), 212–219 and *Math. Student* **36** (1968), 222–226.