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THE ASYMPTOTIC BEHAVIOR OF A FAMILY OF SEQUENCES

P. ERDŐS, A. HILDEBRAND, A. ODLYZKO,
P. PUDAITE AND B. REZNICK

A class of sequences defined by nonlinear recurrences involving the greatest integer function is studied, a typical member of the class being

$$a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor) \quad \text{for } n \geq 1.$$

For this sequence, it is shown that $\lim a(n)/n$ **as** $n \to \infty$ **exists and equals** $12/(\log 432)$**. More generally, for any sequence defined by**

$$a(0) = 1, \qquad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor) \quad \text{for } n \geq 1,$$

where the $r_i > 0$ **and the** m_i **are integers** ≥ 2**, the asymptotic behavior of** $a(n)$ **is determined.**

1. Introduction. Rawsthorne [**R**] recently asked whether the limit $a(n)/n$ exists for the sequence $a(n)$ defined by

$$(1.1) \quad a(0) = 1, \qquad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor), \quad n \geq 1,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. If the limit exists, Rawsthorne also asked for its value. We have answered these questions [**EHOPR**]: the limit exists and equals $12/\log 432$, where, as in the rest of the paper, log denotes the natural logarithm. Our method leads to a more general result about such recursively defined sequences.

Let $a(n)$ be the sequence defined by

$$(1.2) \qquad a(0) = 1, \qquad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor), \quad n \geq 1,$$

where $r_i > 0$ and the m_i's are integers ≥ 2. Let τ be the (unique) solution to

$$(1.3) \qquad \sum_{i=1}^{s} \frac{r_i}{m_i^{\tau}} = 1.$$

We distinguish two cases: if there is an integer d and integers u_i such that $m_i = d^{u_i}$, we are in the *lattice* case, otherwise we are in the *ordinary* case. In the ordinary case, $\lim a(n)/n^{\tau}$ exists; in the lattice case, $\lim a(n)/n^{\tau}$ does not exist, but $\lim_{k \to \infty} a(d^k)/d^{k\tau}$ exists. The limit in either case is readily computable. The proof involves transforming (1.2) into a renewal equation and using the standard limit theorems for that equation. For a precise statement of our results, see Theorem 2.14 below.

We are interested also in the rapidity of convergence. We prove that $(a(n)-a(n-1))/n^{\tau}$ is greater than $\gamma \cdot (\log n)^{-(s-1)/2}$ for some $\gamma > 0$ and infinitely many n. In Rawsthorne's original sequence (1.1), this result can be strengthened (see Theorem 3.46). For $n = 432^t$,

$$\frac{a(n)-a(n-1)}{n} \sim \left(\frac{6}{5\pi t}\right)^{1/2} \quad \text{as } t \to \infty, \tag{1.4}$$

and this is, asymptotically, an upper bound. The numbers $J(m,r) := a(2^m 3^r) - a(2^m 3^r - 1)$ satisfy the so-called "square" functional equation; we use the work of Stanton and Cowan and others to help in the asymptotic analysis.

A somewhat different functional equation was studied by Erdös **[E1]**, **[E2]**: for $2 \le a_1 \le a_2 \le \cdots$ a sequence of integers, let

$$F(0) = 0,\ F(1) = 1,$$

$$F(n) = \sum_{k=1}^{\infty} F(\lfloor n/a_k \rfloor) + 1 \quad \text{for } n > 1.$$

Both the methods and results are different from ours.

2. An application of renewal theory. We fix the following notation. Let integers m_i, $2 \le m_i \le M$, and positive real numbers r_i be given. Define the sequence $a(n)$ recursively by

$$a(0) = 1, \qquad a(n) = \sum_{i=1}^{s} r_i a(\lfloor n/m_i \rfloor), \quad n \ge 1. \tag{2.1}$$

For $x \ge 0$ define

$$A(x) = a(\lfloor x \rfloor). \tag{2.2}$$

Since $[x/mn] = [[x/m]/n]$ for positive integers m and n, we may define $A(x)$ directly and, in effect, extend the sequence to a function on the positive reals:

$$A(x) = 1 \quad \text{for } 0 \le x < 1, \qquad A(x) = \sum_{i=1}^{s} r_i A(x/m_i) \quad \text{for } x \ge 1. \tag{2.3}$$

Note that the function $\phi(u) = \Sigma r_i/m_i^u$ decreases strictly on the real line from ∞ to 0 so there exists a unique $\tau > 0$ satisfying

$$\phi(\tau) = \sum_{i=1}^{s} \frac{r_i}{m_i^\tau} := \sum_{i=1}^{s} p_i = 1. \tag{2.4}$$

Now let

$$f(x) = A(x)/x^\tau \tag{2.5}$$

so that we may rewrite (2.3) as

$$f(x) = x^{-\tau}, \quad 0 < x < 1; \qquad f(x) = \sum_{i=1}^{s} \frac{r_i}{m_i^\tau} f\left(\frac{x}{m_i}\right) \quad \text{for } x \geq 1. \tag{2.6}$$

Since $p_i > 0$ and $\Sigma p_i = 1$, $f(x)$ is a convex combination of previous values of f for $x \geq 1$. It is thus unsurprising that f tends to a limit.

It is now appropriate to review some well-known (to probabilists) results about the renewal equation. We paraphrase Feller [**F**, v. 2, pp. 358–362]. Suppose h is a Riemann integrable function with compact support and $F\{dy\}$ is a probability measure with finite expectation and suppose g satisfies the renewal equation

$$g(u) = h(u) + \int_0^u g(u-\nu) F\{d\nu\}, \qquad u \geq 0. \tag{2.7}$$

If the mass of $F\{d\nu\}$ is concentrated on a set of the form $\{0, \lambda, 2\lambda, \dots\}$, we are in the *lattice* case; otherwise we are in the *ordinary* case. The following limit theorem for g is due to Erdös, Feller, and Pollard in the lattice case and Blackwell in the ordinary case.

Renewal Limit Theorem (*see* [**F**, v. 2, p. 362]).

(i) *In the ordinary case.*

$$\lim_{u\to\infty} g(u) = \frac{\int_0^\infty h(u)\,du}{\int_0^\infty yF\{dy\}}. \tag{2.8}$$

(ii) *In the lattice case, let λ be chosen to be maximal; then g does not converge, but for any fixed $x \in [0, \lambda)$,*

$$\lim_{n\to\infty} g(x+n\lambda) = \frac{\lambda\Sigma_{k=0}^\infty h(x+k\lambda)}{\int_0^\infty yF\{dy\}}, \tag{2.9}$$

where the limit in (2.9) *is taken over integral n.*

We now return to our problem. Let

$$g(u) = f(e^u). \tag{2.10}$$

Then (2.6) can be rewritten as

$$g(u) = \begin{cases} e^{-\tau u}, & u \le 0; \\ \sum_{i=1}^{s} p_i g(u - \log m_i), & u \ge 0. \end{cases} \tag{2.11}$$

Let $F\{d\nu\}$ be the probability measure with mass p_i at $\log m_i$. Then g satisfies an equation of the form (2.7), where h measures the discrepancy between the full recurrence of (2.11) and that portion provided by the convolution in (2.7). This discrepancy arises from a negative argument of g. Hence,

$$h(u) = \sum_{u < \log m_i} p_i e^{-\tau(u - \log m_i)} = \sum_{u < \log m_i} p_i m_i^{\tau} e^{-\tau u}, \tag{2.12}$$

and so

$$h(u) = \sum_{i=1}^{s} p_i m_i^{\tau} e^{-\tau u} \chi_{[0, \log m_i)}(u). \tag{2.13}$$

Having now transformed (1.2) into a renewal equation we must decide which case we are in. The mass of F is concentrated at $\{\log m_i\}$, which is a subset of $\{0, \lambda, 2\lambda, \dots\}$ for some λ (the lattice case) if and only if $m_i = d^{u_i}$ for some integers d and u_i. Alternatively, we are in the ordinary case if and only if $(\log m_i)/(\log m_j)$ is irrational for some (m_i, m_j).

We now combine these discussions into a theorem.

THEOREM 2.14. *Let $a(n)$ be defined by* (2.1) *and let τ be defined as above.*

(i) *If $\tau = 0$ then $a(n) \equiv 1$.*

(ii) *If $\tau \ne 0$ and $(\log m_i)/(\log m_j)$ is irrational for some (m_i, m_j) (the ordinary case), then*

$$\lim_{n \to \infty} \frac{a(n)}{n^{\tau}} = \frac{\sum_{i=1}^{s} p_i (m_i^{\tau} - 1)/\tau}{\sum_{i=1}^{s} p_i \log m_i}. \tag{2.15}$$

(iii) *If $\tau \ne 0$ and $m_i = d^{u_i}$, where d and the u_i's are integers and d is maximal (the lattice case), then*

$$\lim_{k \to \infty} \frac{a(d^k)}{d^{k\tau}} = \frac{\sum_{i=1}^{s} p_i (m_i^{\tau} - 1)}{\sum_{i=1}^{s} p_i \log m_i} \cdot \frac{d^{\tau} \log d}{d^{\tau} - 1}. \tag{2.16}$$

Proof. (i) If $\tau = 0$, then $\sum r_i = 1$ and it is easy to see from (2.1) that $a(n) \equiv 1$ by induction. As $u^{\tau} - 1 \approx \tau \log u$ for τ near 0, this result is consistent with the limiting behavior in (2.15) and (2.16).

(ii) From our definitions,

$$\frac{a(n)}{n^\tau} = f(n) = g(\log n), \tag{2.17}$$

so that information about the limiting behavior of $g(u)$ from the Renewal Limit Theorem can be translated into information about $a(n)/n^\tau$. In either the ordinary or lattice case,

$$\int_0^\infty yF\{dy\} = \sum_{i=1}^{s} p_i \log m_i. \tag{2.18}$$

In the ordinary case, as $\tau \neq 0$, we have by (2.13),

$$\begin{aligned}\int_0^\infty h(u)\,du &= \sum_{i=1}^{s} p_i m_i^\tau \int_0^{\log m_i} e^{-\tau u}\,du \\ &= \sum_{i=1}^{s} p_i m_i^\tau (1 - m_i^{-\tau})/\tau.\end{aligned} \tag{2.19}$$

Equation (2.15) follows from the foregoing discussion, (2.8), (2.18), and (2.19).

(iii) The period of the lattice is $\lambda = \log d$, and taking $x = 0$ in (2.9),

$$\begin{aligned}\sum_{k=0}^{\infty} h(k\lambda) &= \sum_{k=0}^{\infty} h(k \log d) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=1}^{s} p_i m_i^\tau e^{-\tau k \log d} \chi_{[0, \log m_i)}(k \log d) \right).\end{aligned} \tag{2.20}$$

Since $m_i = d^{u_i}$,

$$\begin{aligned}&\sum_{k=0}^{\infty} e^{-\tau k \log d} \chi_{[0, \log m_i)}(k \log d) \\ &\quad = \sum_{k=0}^{u_i - 1} e^{-\tau k \log d} = (1 - d^{-u_i \tau})/(1 - d^{-\tau}) \\ &\quad = (1 - m_i^{-\tau}) d^\tau/(d^\tau - 1).\end{aligned} \tag{2.21}$$

We now exchange the order of summation in (2.20) to obtain

$$\sum_{k=0}^{\infty} h(k\lambda) = \sum_{i=1}^{s} p_i m_i^\tau (1 - m_i^{-\tau}) d^\tau/(d^\tau - 1), \tag{2.22}$$

and (2.16) follows from (2.18), (2.22), and (2.9). □

In the lattice case, it is easy to show by induction that the sequence $a(n)$ is constant on intervals of the form $[d^k, d^{k+1} - 1]$. For any rational

$x = j/d^r$, $d^t < x < d^{t+1}$, $a(xd^k)$ is defined for $k \geq r$, and $a(xd^k) = a(d^{t-r+k})$. Using (2.16), one can compute $\lim a(d^k x)/(d^k x)^\tau$; we omit the details.

As a check of Theorem 2.14, consider the following simple lattice example with $s = 1$:

$$(2.23) \qquad a(0) = 1; \qquad a(n) = d^\alpha a(\lfloor n/d \rfloor), \quad n \geq 1.$$

It is easy to see in this case that $\tau = \alpha$ and $a(d^k) = d^{(k+1)\alpha}$, so that $a(d^k)/d^{k\tau} \equiv d^\alpha$. Substituting $p_1 = 1$, $m_1 = d$, and $\tau = \alpha$ into (2.16) returns d^α, as predicted.

It is perhaps worth mentioning that the existence of $\lim_{n\to\infty} f(n)$ can be proved without recourse to the Renewal Limit Theorem. Here is a sketch of the argument, without proofs. First, from (2.6), $\alpha \geq f(x) \geq \beta$ for $x \in [y, My]$, where $y \geq 1$ and $M \geq \max m_i$ implies that $\alpha \geq f(x) \geq \beta$ for all $x \geq y$. Thus $L = \overline{\lim} f(x)$ and $l = \underline{\lim} f(x)$ are positive and finite. Pick sequences $r_k \to \infty$ and $s_k \to \infty$ with $f(r_k) \to L$, $f(s_k) \to l$ and $r_k < s_k < Mr_k$. The next step in the argument is proved as in §3; $a(n) \neq a(n-1)$ if and only if $n = m_1^{e_1} \cdots m_s^{e_s}$ for some integers e_i and $\tau \neq 0$, $a(n) \geq a(n-1)$ if $\tau > 0$ and $a(n) \leq a(n-1)$ if $\tau < 0$. There is a dichotomy depending on which case arises. In the lattice case, $a(n)$ is constant on intervals $[d^k, d^{k+1} - 1]$ and the substitutions $m = d^{u_i}$, $b(k) = a(d^k)$ show that $\{b(k)\}$ satisfies a linear difference equation for k sufficiently large. By the standard method for solving linear difference equations (see [T, Ch. 4], for example) $a(d^k) = b(k) = c \cdot \beta^k + o(\beta^k) = cd^{k\tau} + o(d^{k\tau})$ for appropriate constants. (See the discussion following Corollary 3.12 for more details.)

In the ordinary case, suppose $(\log m_i)/(\log m_j) \notin Q$ and let $m = m_i$ and $\overline{m} = m_j$. For any $\varepsilon > 0$ there exists E so that every x in $[M^{-1}, M]$ is contained in an interval $[w, w(1+\varepsilon)]$, where $w = m^{e_1}/\overline{m}^{e_2}$ and e_1 and e_2 are positive integers $\leq E$. Let W be any finite set of integers of the form $m_1^{f_1} \cdots m_s^{f_s}$; for k sufficiently large and any $w \in W$, $f(r_k/w)$ is close to L and $f(s_k/w)$ is close to l. (This is proved by induction; basically, if a weighted average like (2.6) is close to the maximum then its components can't be too far off.) Now suppose $\tau > 0$ and $L > l$ and let $x = s_k/r_k$, where k is sufficiently large; let $w = m^{e_1}/\overline{m}^{e_2}$ be chosen so that $x \in [w, w(1+\varepsilon)]$. Then $r' = r_k/\overline{m}^{e_2}$ is a little less than $s' = s_k/m^{e_1}$, but $f(r')$ is close to L and $f(s')$ is close to l. As $\tau > 0$, $a(s') \geq a(r')$, and this gives a contradiction to $L > l$. (More precisely, ε is chosen so that $L - \varepsilon > (1+\varepsilon)^\tau(l + \varepsilon)$.) A similar contradiction can be wrought when $\rho < 0$. In either case, $L = l$ so the limit exists. This method, although self-contained, gives no hint about the actual value of the limit.

3. Rates of convergence. We retain the notation of the last section and continue to assume that $a(n)$ is defined by (2.1). Let

$$J(n) = a(n) - a(n-1) \tag{3.1}$$

denote the jump of the sequence at n. In this section we derive closed forms for $a(n)$ and $J(n)$ and use them to give an indication of the rate of convergence of f. Ideally, one would discuss the behavior of $|f(x) - \lim f(x)|$. As a step in that direction, we consider the "jumps" of f. It is clear from (2.2) and (2.5) that f is everywhere continuous from the right and f is continuous from the left except possibly at certain integers. Let

$$z(n) = f(n) - \lim_{\varepsilon \to 0^+} f(n-\varepsilon) = \frac{a(n) - a(n-1)}{n^\tau} = \frac{J(n)}{n^\tau}. \tag{3.2}$$

We shall show in this section that, in the ordinary case, $|z(n)| > c(\log n)^{-(s-1)/2}$ for some $c > 0$ and infinitely many integers n. In Rawsthorne's original problem, (1.1), the exponent of $\log n$ may be improved from -1 to $-1/2$.

In finding a closed form for $a(n)$, the following notation is useful. Let $\mathbf{i} = (i_1, \ldots, i_l)$, $l \geq 1$, be an l-tuple of integers, $1 \leq i_j \leq s$. Let $I(\mathbf{i})$ be the associated interval:

$$I(\mathbf{i}) = [m_{i} \cdots m_{i_{l-1}}, \ldots, m_{i} \cdots m_{i_{l-1}} m_{i_l}). \tag{3.3}$$

(If $l = 1$ in (3.3), take the left-hand endpoint to be 1.) As an inverse function to I, for $x \geq 1$, let

$$B(x) = \{\mathbf{i}\colon x \in I(\mathbf{i})\}. \tag{3.4}$$

THEOREM 3.5. *For* $x \geq 1$.

$$A(x) = \sum_{\mathbf{i} \in B(x)} r_{i_1} \cdots r_{i_l}. \tag{3.6}$$

Proof. Recall the basic recurrence (2.3):

$$A(x) = \sum_{i=1}^{s} r_i A(x/m_i).$$

Consider the infinite tree with root "x" and valence s so that each node "y" on the kth level is connected to the nodes "y/m_i", $1 \leq i \leq s$ on the $(k+1)$st level. We use this tree to iterate the recurrence (2.3) until the argument of A goes below 1 for the first time. In this way, the path from x to $x/m_{i_1}, \ldots,$ to $x/(m_{i_1} \cdots m_{i_l})$ acquires the coefficient $r_{i_1} \cdots r_{i_l}$. Since $\mathbf{i} = (i_1, \ldots, i_l)$ is in $B(x)$ by construction and $A(x/\prod m_j) = 1$, (3.6) is established. □

We now derive a recurrence for $J(n) = a(n) - a(n-1)$ and find a closed form for $J(n)$.

THEOREM 3.7.
(i)

$$J(n) = \sum_{m_i|n} r_i J(n/m_i).$$

(ii)

$$J(n) = \left(\sum_{i=1}^{s} r_i - 1\right) \sum_{m_1^{e_1} \cdots m_s^{e_s} = n} \frac{(e_1 + \cdots + e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s}.$$

Proof. (i) We have from (2.3)

$$J(n) = \sum_{i=1}^{s} r_i \left(A\left(\frac{n}{m_i}\right) - A\left(\frac{n-1}{m_i}\right)\right). \tag{3.8}$$

If $m_i \nmid n$ then $\lfloor n/m_i \rfloor = \lfloor (n-1)/m_i \rfloor$ so the ith term is zero; if $m_i | n$, then by definition, the ith term is $r_i J(n/m_i)$.

(ii) Observe that $J(1) = a(1) - a(0) = \sum_{i=1}^{s} r_i - 1$. Then, consider each representation of n as a product $m_1^{e_1} \cdots m_s^{e_s}$. The formula (ii) follows by induction from (i) and the well-known multinomial recurrence:

$$\frac{(e_1 + \cdots + e_s)!}{e_1! \cdots e_s!} = \sum_{e_i \geq 1} \frac{(e_1 + \cdots + e_s - 1)!}{e_1! \cdots (e_i - 1)! \cdots e_s!}. \qquad \square$$

We note that (ii) can also be derived from Theorem 3.5 and a consideration of $B(n) - B(n-1)$ and $B(n-1) - B(n)$. If we consider the representations $n = m_1^{e_1} \cdots m_s^{e_s}$ as formally distinct, we may let $j(f_1, \ldots, f_s)$ denote that portion of the jump $J(n)$ contributed by the representation $n = m_1^{f_1} \cdots m_s^{f_s}$. In view of Theorem 3.7 we have the following recurrences and generating function.

$$\begin{cases} j(e_1, \ldots, e_s) = \sum_{e_i \geq 1} r_i j(e_1, \ldots, e_i - 1, \ldots, e_s), \\ j(0, \ldots, 0) = \sum_{i=1}^{s} r_i - 1. \end{cases} \tag{3.9}$$

$$j(e_1, \ldots, e_s) = \left(\sum_{i=1}^{s} r_i - 1\right) \frac{(e_1 + \cdots + e_s)!}{e_1! \cdots e_s!} r_1^{e_1} \cdots r_s^{e_s}. \tag{3.10}$$

$$\begin{aligned} \mathscr{J}(z_1, \ldots, z_s) &= \sum j(e_1, \ldots, e_s) z_1^{e_1} \cdots z_s^{e_s} \\ &= \left(\sum_{i=1}^{s} r_i - 1\right)\left(1 - \sum_{i=1}^{s} r_i z_i\right)^{-1}. \end{aligned} \tag{3.11}$$

COROLLARY 3.12. *If* $\tau > 0$ *then* $J(n) > 0$ *at all* n *of the form* $m_1^{e_1} \cdots m_s^{e_s}$; *if* $\tau < 0$ *then* $J(n) < 0$ *at all such* n.

Proof. From Theorem 3.7 (ii), the sign of $J(n)$ is the sign of $\sum_{i=1}^{s} r_i - 1$, which equals $\phi(0) - \phi(\tau)$ in the notation of (2.4). Since ϕ is strictly decreasing, the conclusions follow immediately. □

We now turn our attention to the size of $z(n)$. It is convenient to dispose of the lattice case. As $f(d^k)$ converges to a limit l and $A(x)$ is constant on $[d^{k-1}, d^k)$, $z(d^k) \sim l(1 - d^{-\tau})$. It is more interesting to look at $f(d^{k+1}) - f(d^k)$. Let $m_i = d^{u_i}$ and let $u = \max u_i$. Then from (2.6),

$$f(d^k) = \sum_{i=1}^{s} p_i f(d^{k-u_i}). \tag{3.13}$$

Let $\psi(t) = t^u - \sum p_i t^{u-u_i}$ be the characteristic equation of the linear recurrence satisfied by $f(d^k)$. Clearly $\psi(1) = 0$ and, as $\lim|f(d^k)| < \infty$, it follows that the other roots of ψ have moduli less than one. Hence there exists a polynomial q of degree at most $s - 1$ and λ, $0 \le \lambda < 1$ so that

$$\left|f(d^k) - 1\right| \le q(k)\lambda^k + o(\lambda^k). \tag{3.14}$$

It follows that $|f(d^{k+1}) - f(d^k)| \le ck^r\lambda^k$ for sufficiently large k, $r \le s - 1$ and some $c > 0$.

Henceforth we assume the ordinary case and $\tau \neq 0$. We first need two approximation lemmas. The first follows directly from the Stirling approximation $\Gamma(w + 1) \sim w^w e^{-w}\sqrt{2\pi w}$ and we omit the proof. The second allows us to adjust from real numbers to integers in our asymptotic analysis.

LEMMA 3.15. *Fix $\alpha_i > 0$, $\sum_{i=1}^{s}\alpha_i = 1$ and define*

$$F(x_1, \ldots, x_s) = \frac{\Gamma(\sum_{i=1}^{s} x_i + 1)}{\prod_{i=1}^{s}\Gamma(x_i + 1)}. \tag{3.16}$$

Then, as $q \to \infty$

$$F(\alpha_1 q, \ldots, \alpha_s q) \sim (2\pi q)^{-(s-1)/2} \prod_{i=1}^{s} \alpha_i^{-(\alpha_i q + 1/2)}. \tag{3.17}$$

LEMMA 3.18. *Fix $\alpha_i > 0$, $\sum_{i=1}^{s}\alpha_i = 1$ and define*

$$\phi(q; t_1, \ldots, t_s) = \frac{F(\alpha_1 q + t_1, \ldots, \alpha_s q + t_s)}{F(\alpha_1 q, \ldots, \alpha_s q)}. \tag{3.19}$$

Then there exists $c > 0$ so that for all sufficiently large q and all choices of t_i with $|t_i| < 1$ and $\sum_{i=1}^{s} t_i = 0$,

$$c^{-1} < \phi(q; t_1, \ldots, t_s) < c. \tag{3.20}$$

Proof. From (3.16) we have

$$(3.21)\quad \phi(q;t_1,\dots,t_s) = \frac{\Gamma(q+1)}{\prod_{i=1}^s \Gamma(\alpha_i q + t_i + 1)} \Big/ \frac{\Gamma(q+1)}{\prod_{i=1}^s \Gamma(\alpha_i q + 1)} = \prod_{i=1}^s \frac{\Gamma(\alpha_i q + 1)}{\Gamma(\alpha_i q + t_i + 1)}.$$

Let

$$(3.22)\qquad H(\alpha,q,t) = \log\Gamma(\alpha q + t + 1) - \log\Gamma(\alpha q + 1).$$

As $\log\Gamma$ is convex, for $|t| < 1$, $t \neq 0$ we have

$$(3.23)\qquad \log\Gamma(\alpha q + 2) - \log\Gamma(\alpha q + 1) \geq \frac{H(\alpha,q,t)}{t} \geq \log\Gamma(\alpha q + 1) - \log\Gamma(\alpha q),$$

hence $H(\alpha,q,t) = t\log(\alpha q + p)$, $0 \leq p \leq 1$, and

$$(3.24)\quad -\log\phi(q;t_1,\dots,t_s) = \sum_{i=1}^s H(\alpha_i,q,t_i) = \sum_{i=1}^s t_i \log(\alpha_i q + p_i) = \sum_{i=1}^s t_i\log\alpha_i + \sum_{i=1}^s t_i \log q + \sum_{i=1}^s t_i \log\left(1 + \frac{p_i}{\alpha_i q}\right).$$

Since $\sum t_i = 0$, $|t_i| < 1$ and $|p_i| < 1$,

$$(3.25)\qquad |\log\phi(q;t_1,\dots,t_s)| \leq \sum_{i=1}^s t_i\log\alpha_i + \sum_{i=1}^s \frac{1}{\alpha_i q},$$

from which (3.20) follows. □

THEOREM 3.26. *In the ordinary case with $\tau \neq 0$ there exists $\gamma > 0$ so that*

$$(3.27)\qquad |z(n)| > \gamma\cdot(\log n)^{-(s-1)/2}$$

for infinitely many n.

Proof. The main idea is to let $n = (m_1^{p_1}\cdots m_s^{p_s})^q$ for large q. Large in this case means that (3.17) is a good approximation. Since $p_i q$ is not an integer in general, we need the approximation of Lemma 3.18.

To be specific, choose q large and choose integers e_i, $\sum_{i=1}^s e_i = q$ such that $|e_i - p_i q| < 1$ for all i. By Theorem 3.7 (ii), we may ignore other representations of n and

$$(3.28)\qquad |J(n)| \geq \alpha\cdot\frac{(e_1 + \cdots e_s)!}{e_1!\cdots e_s!} r_1^{e_1}\cdots r_s^{e_s},$$

where $\alpha = |\sum_{i=1}^{s} r_i - 1| = |\phi(\tau) - \phi(0)| > 0$. We now replace e_i by $p_i q$ in (3.28): $\prod r_i^{e_i}$ changes by a bounded factor, and by Lemma 3.18 we have

$$(3.29)\qquad |J(n)| \geq \beta \cdot \frac{\Gamma(q+1)}{\prod_{i=1}^{s}(p_i q + 1)} \cdot \prod_{i=1}^{s} r_i^{p_i q},$$

where β has absorbed all other constants. Finally, by Lemma 3.16,

$$(3.30)\qquad \begin{aligned} |J(n)| &\geq \beta \cdot (2\pi q)^{-(s-1)/2} \prod_{i=1}^{s} p_i^{-(p_i q + 1/2)} r_i^{p_i q}(1-\varepsilon) \\ &= \gamma q^{-(s-1)/2} \prod_{i=1}^{s} (r_i/p_i)^{p_i q} \\ &= \gamma q^{-(s-1)/2} \prod_{i=1}^{s} m_i^{\tau p_i q} = \gamma q^{-(s-1)/2} n^{\tau}. \end{aligned}$$

Since $z(n) = J(n)/n^{\tau}$ and $q = (\log n)/(\sum p_i \log m_i)$, (3.27) follows. □

It is possible to sharpen the constant slightly by noting that for any $\varepsilon > 0$ there are infinitely many q such that $p_i q$ is within ε of an integer for all i. (Standard pigeonhole principle argument.) If s were to equal 1 then (3.27) would violate the convergence of f, except that $s = 1$ is always a lattice case.

We conclude this paper by returning to Rawsthorne's original problem:

$$a(0) = 1, \quad a(n) = a(\lfloor n/2 \rfloor) + a(\lfloor n/3 \rfloor) + a(\lfloor n/6 \rfloor).$$

By Theorem 3.7 we know that $a(n)$ jumps only at numbers of the form $2^{e_1} 3^{e_2} 6^{e_3}$; that is, products of 2 and 3. Let

$$(3.31)\qquad J(m, r) =: J(2^m 3^r).$$

Then $m = e_1 + e_3$, $r = e_2 + e_3$, and by both parts of Theorem 3.7,

$$(3.32)\quad J(m, r) = 2\sum_i \frac{(m+r-i)!}{(m-i)!(r-i)!i!} = 2\sum_i \binom{m+r-i}{m-i}\binom{r}{i},$$

$$(3.33)\quad \begin{cases} J(m,r) = J(m, r-1) + J(m-1, r) + J(m-1, r-1), \\ \hfill m, r \geq 1, \\ J(0,0) = J(m, 0) = J(0, r) = 2. \end{cases}$$

Unsurprisingly, such a simply defined recurrence has a large literature; (3.33) is called the "square" functional equation and arises as a natural generalization of Pascal's triangle. (Actually, $\frac{1}{2}J$ is the standard form.) The first problem on the 19th Putnam Competition to show that

$S(n) = \frac{1}{2}\sum_i J(i, n-i)$ satisfies the recurrence $S(n+2) = 2S(n+1) + S(n)$ [**GGK**, p. 53]). This recurrence then arose in Golomb's study of sphere packing in the Lee metric [**Go**]. Stanton and Cowan [**SC**] considered (3.32) in its own right and were the first to prove Lemma 3.34 below. A. K. Gupta [**Gu1**] [**Gu2**] gave different proofs and generalized these numbers further, as did Carlitz [**Ca**] and Alladi and Hoggatt [**AH**]. The function $\frac{1}{2}J$ has a natural interpretation as the number of ways to go from $(0,0)$ to (m,r) with steps of size $(1,0)$, $(0,1)$ or $(1,1)$; see Fray and Roselle [**FR**] or Handa and Mohanty [**HM**]. Greene and Knuth [**GK**; pp. 111–113] discuss the asymptotics of $J(m,m)$.

Our analysis of $z(2^m 3^r)$ relies crucially on the following combinatorial lemma.

LEMMA 3.34.

$$J(m,r) = 2\sum_i \binom{m}{i}\binom{r}{i}2^i = 2\sum_i \binom{m+r-i}{m-i}\binom{r}{i}. \tag{3.35}$$

Proof. Consider the coefficient of x^m in

$$2(1+x)^m(1+2x)^r = 2\sum_i \binom{r}{i}x^i(1+x)^{m+r-i}. \qquad \square$$

Stanton and Cowan originally proved this lemma by a sequence of standard combinatorial substitutions. Gupta used a number of methods, including the following hypergeometric representation [**Gu**, Lemma 4]:

$$\tfrac{1}{2}J(m,r) = {}_2F_1(-m,-r;1,2). \tag{3.6}$$

Lemma 3.34 leads to a natural probabilistic interpretation of $J(m,r)$. Let

$$\alpha(m,i) = \frac{\binom{m}{i}}{2^m}, \quad \beta(r,i) = \frac{\binom{r}{i}2^i}{3^r}. \tag{3.37}$$

These denote the probabilities of i successes in m and r Bernoulli trials with $p = \frac{1}{2}$ and $\frac{2}{3}$ respectively. As

$$z(2^m 3^r) = \frac{J(m,r)}{2^m 3^r} = 2\sum_i \alpha(m,i)\beta(r,i), \tag{3.38}$$

one expects $z(2^m 3^r)$ to be largest when the probability distributions peak simultaneously; that is, when $m/2 \approx 2r/3$, cf. Proposition 3.41. As a preliminary bound, note that $\alpha(m,i) \le \alpha(m,m/2)$ and $\beta(r,i) \le \beta(r,2r/3)$, replacing factorials by Γ as necessary. By Lemma 3.15,

$\alpha(m,i) \le \gamma_0 m^{-1/2}$ and $\beta(r,i) \le \gamma_1 r^{-1/2}$ for appropriate $\gamma_i > 0$. Hence

$$z(2^m 3^r) \le \min\left(\gamma_0 m^{-1/2}, \gamma_1 r^{-1/2}\right). \tag{3.39}$$

Since $\log(2^m 3^r) = m\log 2 + r\log 3$, (3.39) implies that $z(n) \le \gamma(\log n)^{-1/2}$ for some $\gamma > 0$ and all n.

Consider now the normal approximation to the binomial distribution, see e.g. [**F**, v. 1, p. 170]. For fixed k,

$$\begin{cases} \alpha\left(m, \dfrac{m}{2} + k\dfrac{\sqrt{m}}{2}\right) \sim \dfrac{2}{\sqrt{m}}\dfrac{1}{\sqrt{2\pi}}e^{-k^2/2}, \\ \beta\left(r, \dfrac{2r}{3} + k\dfrac{\sqrt{2r}}{3}\right) \sim \dfrac{3}{\sqrt{2r}}\dfrac{1}{\sqrt{2\pi}}e^{-k^2/2}. \end{cases} \tag{3.40}$$

Let $\Delta(m,r) = |m/2 - 2r/3|$. We now show that if $\Delta(m,r)$ is comparable to $\sqrt{m}$ and $\sqrt{r}$, then $z(2^m 3^r)$ is quite a bit smaller than $\gamma(\log n)^{-1/2}$.

PROPOSITION 3.41. *Fix $k, \varepsilon > 0$ and suppose*

$$\Delta(m,r) = \left|\frac{m}{2} - \frac{2r}{3}\right| > k\left(\frac{\sqrt{m}}{2} + \frac{\sqrt{2r}}{3}\right). \tag{3.42}$$

Then for sufficiently large m and r,

$$z(2^m 3^r) \le 2(1+\varepsilon)\left(\frac{2}{\sqrt{m}} + \frac{3}{\sqrt{2r}}\right)\frac{1}{\sqrt{2\pi}}e^{-k^2/2}. \tag{3.43}$$

Proof. If (3.42) holds, then for each i at least one of the inequalities

$$\left|\frac{m}{2} - i\right| > \frac{k\sqrt{m}}{2} \quad \text{or} \quad \left|\frac{2r}{3} - i\right| > \frac{k\sqrt{2r}}{3} \tag{3.44}$$

is valid. Suppose that m and r are large enough that the approximation in (3.40) becomes an inequality after multiplication by $1+\varepsilon$. Let

$$I = \left\{i : \left|\frac{m}{2} - i\right| > \frac{k\sqrt{m}}{2}\right\};$$

then

$$\begin{aligned} z(2^m 3^r) &\le 2(1+\varepsilon)\left(\sum_{i\in I}\alpha(m,i)\right)\frac{2}{\sqrt{m}}\frac{1}{\sqrt{2\pi}}e^{-k^2/2} \\ &\quad + 2(1+\varepsilon)\left(\sum_{i\notin I}\beta(r,i)\right)\frac{3}{\sqrt{2r}}\frac{1}{\sqrt{2\pi}}e^{-k^2/2} \\ &\le 2(1+\varepsilon)\left(\frac{2}{\sqrt{m}} + \frac{3}{\sqrt{2r}}\right)\frac{1}{\sqrt{2\pi}}e^{-k^2/2}. \end{aligned} \tag{3.45}$$

□

We remark that, if $r \sim \alpha m$, where $\alpha \neq 3/4$, then this proposition implies that

$$z(n) = z(2^m 3^r) \leq h_1(\alpha)(\log n)^{-1/2} n^{-h_2(\alpha)},$$

where $h_1(\alpha)$ and $h_2(\alpha)$ are complicated positive algebraic functions of α. We spare the reader the gory details. The asymptotic behavior of

$$\sum \binom{c}{i}\binom{\alpha c}{i} x^i$$

has been studied by Laquer [**La**]; more precise information than Proposition 3.41 can be found there, as can our final estimate, whose proof we sketch.

THEOREM 3.46. *For* $n = 432^t = 2^{4t}3^{3t}$,

$$z(n) \sim \left(\frac{6}{5\pi t}\right)^{1/2} = \left(\frac{6 \log 432}{5\pi \log n}\right)^{1/2}. \tag{3.47}$$

Proof. After a reindexing, (3.38) becomes

$$z(n) = \sum_i \alpha(4t, 2t+i)\beta(3t, 2t+i). \tag{3.48}$$

By Feller [v. 1, p. 170], the estimates (3.40) are valid for $|i| \leq t^{2/3-\varepsilon}$, so the tails can be ignored. These approximations reduce to a Riemann sum:

$$z(n) \sim \sqrt{\frac{3}{2}}\,\frac{1}{\pi}\,\frac{1}{\sqrt{t}} \sum_i e^{-5i^2/4t} \sim \sqrt{\frac{6}{5\pi t}}\,. \qquad \square \tag{3.49}$$

As a measure of the slowness of convergence of f and the accuracy of (3.47), let $n = 432^5 \simeq 1.5 \times 10^{13}$. Then $f(n-1) \simeq 1.8430$, $f(n) \simeq 2.1175$, so $z(n) \simeq .2745$, whereas $(6/(25\pi))^{1/2} \simeq .2764$.

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HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, HUNGARY

UNIVERSITY OF ILLINOIS
URBANA, IL 61801

AT & T BELL LABORATORIES
MURRAY HILL, NJ 07974

AND

UNIVERSITY OF ILLINOIS
URBANA, IL 61801