On the Equality of the Grundy and Ochromatic Numbers of a Graph

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ABSTRACT

It is proved in this note that the Grundy number, $\Gamma(G)$, and the ochromatic number, $\chi^0(G)$, are the same for any graph G.

An *n-coloring* of a graph $G = (V, E)$ is a function f from V onto $N = \{1, 2, \ldots, n\}$ such that, whenever vertices u and v are adjacent, then $f(u) \neq f(v)$. An n-coloring is *complete* if for every pair i, j of integers, $1 \leq i \leq j \leq n$, there exist a pair u, v of adjacent vertices such that $f(u) = i$ and $f(v) = j$. The *chromatic number*, $\chi(G)$, and the *achromatic number*, $\psi(G)$, are the smallest and largest values n, respectively, for which G has a complete n-coloring. A complete n-coloring $g: V \to N$ is a *Grundy n-coloring* if, for every vertex $v \in V$, $g(v)$ is the smallest integer that is not assigned to any vertex adjacent to v. The *Grundy number*, $\Gamma(G)$, is the largest n for which G has a Grundy n-coloring.

Finally we define a *parsimonious proper coloring* (ppc). Let ϕ: $v_1, v_2, \ldots, v_n$ be an arbitrary ordering of the vertices V of graph $G = (V, E)$. Consider coloring the vertices of G in the following manner: the vertices are colored in the given order ϕ; when a vertex v_i is to be colored, it must be assigned one of the colors that has been used to color the vertices $v_1, \ldots, v_{i-1}$ provided a valid coloring will result; only if v_i is adjacent to a vertex of every currently used color can a new color be assigned; if v_i can be assigned more than one color, one must choose a color that results in the least number of colors being used to color G. The minimum number of colors used to color G in this way, for the

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given ordering ϕ, is called the *parsimonious proper ϕ-coloring number*, and is denoted $\chi_\phi(G)$. The *ochromatic number*, $\chi^0(G)$, is the largest value of $\chi_\phi(G)$, taken over all orderings ϕ of V.

These four coloring numbers, chromatic, achromatic, Grundy, and ochromatic, are closely related, as we shall see. In particular, we shall show that for any graph G, $\Gamma(G) = \chi^0(G)$.

The chromatic number is, of course, a very well studied parameter whose history dates back to the famous four color problem and the work on this problem by Kempe in 1879 [12] and Heawood in 1890 [10].

The achromatic number was first studied as a parameter by Harary, Hedetniemi, and Prins in 1967 [7] and later was so named and studied by Harary and Hedetniemi in 1970 [8].

Grundy functions on directed graphs date back to the work of Grundy in 1939 [6] and were popularized by Berge in 1962 [2] and later in [1]. The Grundy number was first named and studied by C. C. Christen and S. M. Selkow in 1979 [4]. It was also studied as a parameter by Cockayne and Thomason in 1981 [5] and indirectly by McDiarmid in 1979 [13] and S. M. Hedetniemi, S. T. Hedetniemi, and T. Beyer in 1982 [11].

The ochromatic number is due to Simmons, who first presented it in 1982 [14] and more recently in [15] and [16]. The ochromatic number was also studied by Hare, Hedetniemi, Laskar, and Pfaff in [9].

All of these coloring parameters, $\chi(G)$, $\Gamma(G)$, $\chi^0(G)$, and $\psi(G)$, are closely related. In fact, Simmons [16] has shown the following.

Theorem 1(Simmons). For any graph G, $\chi(G) = \min \chi_\phi(G)$.

By definition $\chi^0(G) = \max \chi_\phi(G)$. Moreover since $\chi(G)$ and $\psi(G)$ are the smallest and largest complete coloring numbers, respectively, the following results hold.

Theorem 2 (Simmons). For any graph G, $\chi(G) \leq \chi^0(G) \leq \psi(G)$.

Theorem 3 (Hedetniemi, Hedetniemi, and Beyer). For any graph G

$$\chi(G) \leq \Gamma(G) \leq \psi(G).$$

In fact, we now prove the following equality.

Theorem 4. For any graph G, $\Gamma(G) = \chi^0(G)$.

Proof. We first show that $\Gamma(G) \leq \chi^0(G)$. Let $V_1, V_2, \ldots, V_{\Gamma(G)}$ be a Grundy coloring decomposition with $\Gamma(G)$ colors such that V_i is the set of vertices colored i, $1 \leq i \leq \Gamma(G)$. An ordering ϕ of the vertices of G can be induced in the following manner. Any linear order can be assigned within each V_i and, if u is in V_i and v is in V_j with $i < j$, then u precedes v. Now a parsimonious proper ϕ-coloring on V can be induced, and this will coincide with the given $\Gamma(G)$-coloring. Hence, $\Gamma(G) = \chi_\phi(G)$. But, since $\chi^0(G)$ is the largest of all such $\chi_\phi(G)$, we have $\Gamma(G) \leq \chi^0(G)$.

But also, $\chi^0(G) \leq \Gamma(G)$. To establish this inequality, let ϕ be an ordering of V for which $\chi_\phi(G) = \chi^0(G)$. A parsimonious proper ϕ-coloring is now assigned to V with the additional rule that, when a choice of color exists to color a vertex, the smallest possible color is used. Suppose R colors are used in this way; then $\chi_\phi(G) \leq R$. Moreover, the additional rule ensures that we have a Grundy R-coloring. Thus, $R \leq \Gamma(G)$, yielding $\chi^0(G) = \chi_\phi(G) \leq R \leq \Gamma(G)$, as claimed. ∎

[Since the original writing of this note, the authors have been kindly informed by G. J. Simmons that E. Brickell (private communication) has obtained the same theorem as Theorem 4.]

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