On the graph of large distances

P.Erdös
Mathematical Research Institute,
Hungarian Academy of Sciences,
Budapest, H-1053 Hungary
L.Lovász
Department of Computer Science,
Eötvös Loránd University,
Budapest, H-1088 Hungary
K.Vesztergombi
Department of Mathematics,
Faculty of Electrical Engineering,
Budapest University of Technology,
Budapest, H-111 Hungary

Introduction

Let S be the set of n points in the plane. Let us denote by $d_1 > d_2 > \dots$ the different distances determined by these points, and by n_1 the number of distances equal to d_1, by n_2, the number of distances equal to d_2 etc. Denote by G(S,k) the graph on vertex set S obtained by joining x to y if their distance is at least d_k. We prove that if $n > n_o(k)$ then the chromatic number $\chi(G(S,k))$ is at most 7, and give a construction for which the equality holds for arbitrarily large n. Obviously without the assumption $n > n_o(k)$ the theorem is not true, since if we take the vertices of the regular (2k+1)-gon as our set of points then $\chi(G(S,k)) = 2k+1$.

If we assume that S is the vertex set of a convex polygon then we prove that for $n > n_1(k)$ the chromatic number $\chi(G(S,k))$ is at most 3. The problem of determining the largest possible value of the chromatic number of G(S,k) for given k (both in the convex and non-convex case) turns out quite different and we have only a partial answer. We conjecture that for fixed k the chromatic number of G(S,k) is at most 2k+1, which is the best if it is true as shown by the regular (2k+1)-gon. If it is true, this generalizes a theorem of Altman. Erdős conjectured and Altman (1963, 1972) proved that the number of distances determined by the vertices of a convex n-gon is at least $\lfloor n/2 \rfloor$. This in particular implies that in the "convex" case G(S,k) can not contain a complete subgraph of 2k+2 vertices. Perhaps in the convex case there always exists an x_i such that the degree of x_i is at most 2k. We prove that for the vertex set S of a convex polygon there exists an x_i such that the degree of x_i is at most 3k-1. From this it follows that the number of edges in G(S,k) is at most 3kn, and that its chromatic number is at most 3k.

Erdős and L. Moser conjectured that in a convex n-gon every distance can occur at most cn times. There is a construction in which the same distance occurs 5n/3 times. Hopf and Pannwitz (1934) and Sutherland (1935) proved that the maximum distance among n points occurs at most n times. Vesztergombi (1985) noticed that the kth largest distance occurs at most kn times, and in a sense described the distribution of the number of occurences of the two largest distances. In particular it follows that the number of edges in G(n,2) is at most 2n. One may conjecture that the number of edges in G(n,k) is at most kn. The result above verifies this conjecture up to a constant) and shows that the conjecture of Erdős and Moser is valid in the average for the "large" distances. Let us mention the related conjecture of Erdős that in a convex n-gon there is always a vertex x_i such that the number of distinct distances from x_i is at least n/2.

If we do not restrict ourselves to the largest k distances, we can ask the following generalization of the Erdős-Moser conjecture: what is the maximum number of times the k "favorite" distances can occur? Maybe for $k \geq 2$ the answer will be kn.

It would be nice if in the non-convex case the maximum of the chromatic number of G(S,k) for fixed k would be also equal to the largest complete graph which can be contained in some G(S,k). A 40 year old conjecture of Erdős (worth $500) implies that the number of distinct distances determined by n points is at least $cn/(\log n)^{1/2}$ (if true, this is best possible apart from the value of c). If this is true then the largest complete graph contained in G(S,k) is at most $ck(\log k)^{1/2}$. We can prove that the chromatic number is at most ck^2. $k^{1+\varepsilon}$ will not come out easily since we can not even prove that G(S,k) does not contain a complete graph on $k^{1+\varepsilon}$ vertices.

In the 1-dimensional case these problems are trivial. For large n, G(S,k) is bipartite and the chromatic number of G(S,k) can be at most k+1 which can be of course achieved.

The following problem might be of interest. Let $x_1,\ldots,x_n$ be n points in the plane and $l_1,\ldots,l_k$ are k arbitrary distances. Two points are joined by an edge if their distance is one of the l_i's. Denote by f(k) the maximum possible chromatic number of this graph. It would be nice if this would be again the largest complete graph contained in our graph.

1. The "non-convex" case

We start with a simple lemma.

1.1. Lemma. Let C be a circle with center c and radius r, and T, a set of points on the circle such that c is in the convex hull of T. Then for each point $p \neq c$ of the plane , there is a point $t \in T$ with $d(p,t) > r$.

Proof: Let ℓ be the line through c perpendicular to the line cp. Then clearly T contains a point t in the halfplane bounded by ℓ not containing p. Then the angle pct is at least 90° and hence $d(t,p) > d(c,t) = r$.

∎

Now we are able to prove the main theorem of this section.

1.2 Theorem. If $n \geq n_2(k) = 18k^2$ then $\chi(G(S,k)) \leq 7$.

Proof: Let $q \in S$ be the point of G(S,k) with largest degree. Consider the circle C with smallest radius r containing $S' = S - \{q\}$. If $r < d_k$ then we can cut the disc bounded by C into 6 pieces with diameter less than d_k, and this yields a 6-coloration of $G(S,k) - q$, and using a 7th color for q we are done.

So suppose that $r \geq d_k$. Obviously, the convex hull of $C \cap S'$ contains the center c of C. So we can choose a subset T of $C \cap S'$ with $|T| \leq 3$ such that the convex hull of T contains c. Hence by Lemma 1.1, every point in S is connected to some point in T. So T contains a point of degree more than $6k^2$, and hence by its choice, q has degree greater than $6k^2$. Now among the neighbours of q, there are more than $2k^2$ which are connected to the same point $t \in T$.

But note that these points must lie on k concentrical circles about q as well as on k concentrical circles about t. These two families of circles have at most $2k^2$ intersection points, a contradiction.

∎

Now we give a construction which shows that this upper bound for the chromatic number is sharp.

Let us take a regular 11-gon with vertices t_i, on a circle of radius 1 with center O. We take a point p for which $d(O,p) = 5$ holds (see Fig.1). We draw an arc around p with radius 5 going through O. Then on that little arc we can place the remaining points of S. Let us consider in this setting the 16 largest distances. If p is in general position then all the $d(p,t_i)$ distances are different, and another one is $d(p,O) = 5$, and also the other points on the little arc have the same difference from p, and the 4 largest chords in the regular 11-gon are the 16 largest distances . All the other distances are smaller, for arbitrarily many points. One can easily check that the t_i's need 5 color and p needs the 7^{th} color, and the remaining points are connected only to p, so one can finish by 7 color.

The threshold $n_2(k)$ in the theorem is sharp as far as the order of magnitude goes. In fact, let as modify the previous construction as follows. We construct the 11-gon an the point p as before, but now we also add a further point p' obtained by rotating p about 0 by 90°. Let us draw k-23 concentrical circles about p as well as about p' with radii very close to 5, and let as add the $(k-23)^2$ intersection points of these circles inside the 11-gon. This way we get a set S with $\approx k^2$ points such that the chromatic number of G(S,k) is 8.

It would be interesting to determine the threshold for |S| (as a function of k) where the chromatic number of G(S,k) becomes bounded. This could be settled on the basis of the previous arguments if we could answer the following question: given $t \geq 3$, what is the largest s such that G(S,k) can contain a complete bipartite graph $K_{t,s}$. In particular, can it contain a $K_{3,s}$ with $s = ck^2$? Maybe the fact that G(S,k) consists of the largest k distances has nothing to do with this question. So we obtain the following problem which is quite interesting on its own right:

1.3 Problem. Given $t \geq 2$ points $q_1, \ldots, q_t$ in the plane and k numbers $r_1, \ldots, r_k$, how many points p of the plane can exist such that each distance $d(p,q_i)$ is one of the numbers r_i?

For t=2 the answer to this question is trivially $2k^2$, but already for t=3 we do not know if the answer is $o(k^2)$.

We remark without proof that the chromatic number of G(S,k) is $O(k^2)$ for every set S in the plane. This is quite a weak bound in view of the remarks in the introduction, but we could not prove $o(k^2)$.

2. The "convex" case

In this paragraph we deal with the case when S is a set of vertices of a convex n-gon P (briefly, the "convex" case). The convexity of S gives a natural ordering of the points so throughout the proofs we refer to that ordering. Before stating the main results of this paragraph we make some simple observations.

2.1 Lemma. Suppose that $x_1, x_2, x_3, x_4 \in S$ (in this counterclockwise order) and

$$d(x_1,x_2) \geq d_k,\ d(x_2,x_3) \geq d_k,\ d(x_3,x_4) \geq d_k.$$

Then for each $y \in S$ between x_1 and x_4, at least one of the distances $d(x_i,y)$ is greater than d_k.

Proof: Since the angle x_1yx_4 is less than 180° (because S is a convex set), at least one of the angles x_iyx_{i+1} (for $i=1,2,3$) is less than 60°. Hence (x_i,x_{i+1}) cannot be the largest side of the triangle x_iyx_{i+1}, from which the lemma follows.

∎

2.2 Lemma. Suppose that x_1, x_2, x_3, y_1, y_2 are five vertices of S in this counterclockwise order, and assume that $d(x_1,x_2) \geq d_k$, $d(x_2,x_3) \geq d_k$ and $d(x_1,y_1) = d(x_1,y_2)$. Then $d(y_2,x_2) \geq d_k$.

Proof: If the semiline x_2x_3 does not intersect the semiline

y_1y_2 then the assertion is obvious. So suppose that these semilines intersect in a point z as in Figure 2. Now the angle $x_1y_1x_2$ is less than the angle $y_1x_1x_2$ because the lengths of the opposite sides of the triangle $y_1x_2x_3$ are in this order. Similarly in the triangle y_1x_3z, the angle x_2y_1z is less than the angle y_1zx_2. On the other hand, since the angle x_2x_3z is less than 180^o, the sum of the other angles in the convex quadrangle $y_1zx_3x_4$ must be more than 180^o, which means that the sum of the angles $x_2y_1x_3$ and x_3y_1z is less than 90^o, but this contradicts the fact that the angle $x_2y_1y_2$, which is the sum of the angles $x_2y_1x_3$ and x_3y_1z, is acute.

∎

2.3. Lemma. Suppose that $x_1, x_2, x_3, x_4 \in S$ (in this counterclockwise order) and

$$d(x_1,x_2) \geq d_k,\ d(x_2,x_3) \geq d_k,\ d(x_3,x_4) \geq d_k.$$

Then the number of vertices of S between x_1 and x_4 is at most $12k^2 + 4k$.

Proof: By Lemma 2.1, each vertex between x_1 and x_4 is connected in G(S,k) to at least one of the x_i's. By Lemma 2.2, there are at most k vertices between x_1 and x_4 which are connected in G(S,k) to a given x_i but no other x_j. On the other hand, all points which are connected to both x_i and x_j $(1 \leq i < j \leq 4)$ lie on k circles about x_i as well as on k circles about x_j, so their number is at most $2k^2$. This gives the bound in the Lemma.

∎

2.4. Corollary. If $n > 12k^2 + 4k$ then $G(S,k)$ contains no convex quadrilateral.

∎

2.5. Theorem. If k is fixed and $n > n_1(k) = 25000k^2$ then $\chi(G(S,k)) \leq 3$.

Proof: Let $p = \lfloor n/720 \rfloor$. Then $p > 24k^2+8k+2$ (except in the trivial case when $k=1$). We can choose $2p + 1$ consecutives vertices $a_o, \ldots, a_{2p}$ such that the angle between the vectors $a_o a_1$ and $a_{2p-1} a_{2p}$ is less than 1^o. Now we do the coloring the greedy way. We start at the point $t_1 = a_p$. We give the color 1 to the points in S going counterclockwise as long as possible, i.e. until we encounter a vertex t_2 which is connected in $G(S,k)$ to a vertex t_1' already colored with color 1. Now starting at t_2 go on using color 2, until it is possible, i.e. until we encounter a vertex t_3 connected to a vertex t_2' already colored with color 2. Going on with color 3, we either complete a 3-coloring of G, or else we find, similarly as before, vertices t_4 and t_3' connected in $G(S,k)$. Now we show that we can choose $x_1 = t_1'$, $x_2 \in \{t_2, t_2'\}$, $x_3 \in \{t_3, t_3'\}$ and $x_4 = t_4$ so that $d(x_1,x_2) \geq d_k$, $d(x_2,x_3) \geq d_k$, $d(x_3,x_4) \geq d_k$. If $t_2 = t_2'$ and $t_3 = t_3'$ then this is obvious.

Assume that $t_2 = t_2'$. Now in the convex quadrangle $t_1' t_2' t_2 t_3$ the sum of the lengths of the opposite edges (t_1', t_2') and (t_2, t_3), are of length at least $2d_k$, so at least one diagonal must be of length at least d_k. We choose x_2 accordingly, and similarly we choose x_3.

So we have the same kind of configuration as in Lemma 2.3. Thus by Lemma 2.3 there are at most $12k^2 + 4k$ vertices between x_1 and x_4. This in particular implies that $x_1 = a_i$ and $x_4 = a_j$ where

$$p - 12k^2 - 4k \leq i \leq p < j \leq p + 12k^2 + 4k + 1.$$

One of the pairs (x_1,x_3) and (x_2,x_4), say the former, is also connected in $G(S,k)$.

Now the angle $x_2x_1a_{i+1}$ cannot be larger than 91^o, or else the segments $x_2a_{i+1}, x_2a_{i+2}, \ldots, x_2a_{i+k}$ were monotone increasing and all greater than d_k, which is impossible. Similarly, the angle $a_{i-1}x_1x_3$ is less than 91^o and hence the angle $x_2x_1x_3$ is less than 2^o. Let e.g. $d(x_1,x_2) < d(x_1,x_3)$. Hence it is easy to deduce using the cosine theorem that $d(x_1,x_3) \geq 1.9d_k$. Hence

$$d(a_{2p},x_3) \geq \sin (x_3x_1a_{2p})\, d(x_1,x_3) \geq \sin 88^o\ \ 1.9\, d_k\ .$$
$$\geq 1.8\, d_k$$

But then relabelling a_{2p} by x_4, we get a contradiction at Lemma 2.3.

∎

Again, one can ask if the threshold $\text{const}\cdot k^2$ is best possible. The source of this value is Lemma 2.3, where we use (essentially) the case t=2 of Problem 1.3. It would seem that the additional information that the points considered are the vertices of a convex polygon would exclude most of the intersection points of the two families of concentric circles. But this is not the case; we can construct a set S, consisting of the vertices of a convex polygon, such that $|S| > \text{const}\cdot k^2$ and $G(S,k)$ contains a K_4 (and hence its chromatic number is larger than 3).

Let us sketch this construction. Let $a = (0,0)$, $= (1,0)$, $c = (3,0)$ and $d = (-1,0)$. Let C_0 be the circle with radius 2 about b, and let p_0 be a point on C_0 very close to c. Then the angle dp_0c is 90^o, hence the angle ap_0c is cute. Hence we can choose an interior point p_1 on the arc of C_0 between p_0 and c such that the angle ap_0p_1 is acute. We define the points $p_2, \dots p_{k-1}$ on the circle C_0 similarly so that all the angles ap_ip_{i+1} are acute. Let D_i be the circle with center a through p_i. It follows from the construction that the circel D_i contains p_{i+1} in its interior but the line tangent to D_i at p_i does not separate p_{i+1} from a.

Let ε be a very small positive number and let C_i $(i=0,\dots,k-1)$ be the circle about b with radius $2-i\varepsilon$. Let p_{ij} be the intersection point of C_i and D_j in the upper halfplane. Then the points p_{ij}, a and b form the vertices of a convex polygon and a, b, $p_{0,0}$ and $p_{k-1,k-1}$ form a complete quadrilateral in G(S, 2k+2).

Next we derive a bound on the chromatic number of G(S,k) without the hypothesis that |S| is large. First, let us define the following. Let xy be an edge of G(S,k). Let x_1 be the clockwise neighbor of x and y_1, the counterclockwise neighbor of y. If $d(x_1,y) > d(x,y)$, we say that the edge x_1y *covers* the edge xy. Similarly if $d(x,y_1) > d(x,y)$, we say that the edge xy_1 *covers* the edge xy. Starting from any edge xy, let us select an edge x'y' covering it, then an edge x"y" covering x'y' etc. In at most k-1 steps we must get stuck (by the definition of G(S,k)). Let x_0y_0 be the edge for which we could not find any edge covering it. We call x_0y_0 a *majorant* of xy. Note that in this case the angles formed by x_0y_0 and the two edges of the polygon entering x_0 and y_0 from the side opposite to xy must be acute. It is also clear that the arcs x_0x and yy_0 contain at most k-1 sides of P together.

The following proposition will not be used directly, but it seems worth formulating.

2.6 Proposition. Let (x_1,x_2) and (x_3,x_4) be two avoiding edges of G(S,k). Then either between x_2 and x_3 or between x_4 and x_1 are not more than 2k-2 sides of P (see Figure 3).

Proof: Assume that the conclusion does not hold, and let y_1y_2 be a majorant of x_1x_2 and y_3y_4, a majorant of x_3x_4. Then these majorants are also avoiding and y_1, y_2, y_3 and y_4 are in this same cyclic order on the polygon. Moreover, from the remarks made concerning the majorants it follows that all angles of the convex quadrangle $y_1y_2y_3y_4$ are acute. This is clearly impossible.

∎

2.5 Theorem. The graph G(S,k) has a point of degree at most 3k-1.

Proof: Choose $x \in S$ and let y and z be the first vertices of S in the counterclockwise and clockwise directions, respectively, that are connected to x. Choose x so that the number of points between x and y is maximal (see Figure 4). Let sv be a majorant of zx. (It is possible that v = x or s = z). Suppose there are a points between x and v and b points between z and s, then we know that $a+b \leq k-1$ holds. Then let t be the k-th point from x in the counterclockwise direction, and let u be the first vertex in the counterclockwise direction connected to t in G(S,k). Then because of the choice of x, there are not more sides of P between t and u than between x and y. Hence there are not more sides of P between y and u than between x and t, i.e., not more than a+k.

Let v's' be a majorant of tu. Obviously, v' lies on the arc vt. Just like in the proof of Proposition 2.4, the edges sv and v's' cannot be avoiding. Hence s must be on the arc us' and so the number of sides of P on the arc us is at most k-1. Hence the number of sides of P on the arc yz is at most $(a+k)+(k-1)+b \leq 3k-2$. Hence the degree of x is at most 3k-1.

∎

2.6 Corollary. The number of edges in G(S,k) is at most (3k-1)n.

∎

Moreover, by Brooks' Theorem we obtain:

2.7 Corollary. The chromatic number of G(S,k) is at most 3k.

∎

References

E.Altman (1963), On a problem of P.Erdős, *Amer. Math. Monthly* 70, 148-157.

E.Altman (1972), Some theorems on convex polygons, *Canad. Math. Bull.* 15, 329-340.

H.Hopf and E.Pannwitz (1934), Problem 167, *Jahresber. der Deut. Math. Ver.* 43, 2. Abt., 114.

J.W.Sutherland (1935), *Jahresber. der Deut. Math. Ver.* 45, 2. Abt. 33.

K.Vesztergombi (1985): On the distribution of distances in finite sets in the plane, *Discrete Math.* 57, 129-146.