Congruent subsets of infinite sets of natural numbers

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1. Introduction

Let $\mathbb{N}$ be the set of natural numbers (>0). If A is an infinite subset of $\mathbb{N}$ (or a strictly increasing sequence of natural numbers) and $x \in \mathbb{N}$, then we denote by $A(x)$ the number of elements of A which are $\leqq x$. Two subsets B, C of A are called *congruent*, if there exists a translation of $\mathbb{N}$ which maps B onto C. We shall prove:

If k and n are given natural numbers >1 and if $A(x) \geqq \varepsilon \cdot x^{1-\frac{1}{n}}$ for some positive ε and all x of a final segment of $\mathbb{N}$, then there exist k n-element subsets of A which are pairwise congruent.

This improves an earlier result of the first author for $k=n=2$, namely Theorem II below, which dealt with B_2-sequences. This notion arose from a paper of Sidon [6]. A strictly increasing sequence A of natural numbers $a_1, a_2, \ldots$ is called a *B_2-sequence*, if for any two pairs $(i, j) \neq (k, m)$ of natural numbers $i<j$, $k<m$ there holds $a_j - a_i \neq a_k - a_m$, in short, if A has no double-differences. (Of course it would be the same condition to presuppose that A has no double-sums: For any two pairs $(i, j) \neq (k, m)$ with $i<j$, $k<m$ there holds $a_i + a_j \neq a_k + a_m$.)

In this connection Erdös and Turán [2] introduced a function Φ as follows: For $x \in \mathbb{N}$ let $\Phi(x)$ be the maximal number m, such that there exists a B_2-sequence with m elements $\leqq x$. It satisfies:

I. $\lim\limits_{n \to \infty} \dfrac{\Phi(n)}{\sqrt{n}} = 1.$ (See [2], [3], [1].)

Further the following two theorems hold:

II. *For every B_2-sequence A one has* $\underline{\lim}_{n \to \infty} \dfrac{A(n)}{\sqrt{n}} = 0.$ (See [7], p. 133.)

III. *There exist B_2-sequences A satisfying* $\overline{\lim_{n\to\infty}} \frac{A(n)}{\sqrt{n}} \geqq \frac{1}{\sqrt{2}}$.

(See [5] and [4], p. 91)

More information on B_2-sequences can be found in the book "Sequences" of Halberstam/Roth [4].

2. Main part

At first we investigate a generalization of the function Φ:

Definition. Let k and n be natural numbers $\geqq 2$. A strictly increasing sequence A of natural numbers shall be called a B_{kn}-*sequence*, if the following holds: There are no k different n-element subsets of A, which are pairwise congruent.

Then a B_{22}-sequence is the same as a B_2-sequence in the previous sense.

Further we define a function $\Phi_{kn}\colon \mathbb{N} \to \mathbb{N}$ as follows: For $x \in \mathbb{N}$ we put $\Phi_{kn}(x) = m$, where m is the maximal number such that there exists a B_{kn}-sequence with m elements $\leqq x$.

We derive an elementary inequality concerning Φ_{kn}:

Theorem 1. *For each pair of natural numbers k, n which are $\geqq 2$, there exists a constant c_{kn}, such that for all $x \in \mathbb{N}$ the following holds:*

$$\Phi_{kn}(x) \leqq c_{kn}\, x^{1-\frac{1}{n}}. \tag{1}$$

Proof. Let x be a natural number $> n$. We subdivide the set of n-element subsets of $\{1, 2, \ldots, x\}$ into classes of pairwise congruent subsets. The number of these classes is calculated as follows: Each class contains exactly one n-element subset of $\{1, \ldots, x\}$ which contains 1. Hence the number of classes is $\binom{x-1}{n-1}$.

Now let $a_1, a_2, \ldots$ be a B_{kn}-sequence and $a_1, \ldots, a_t$ all of its elements which are $\leqq x$. Then we have:

$$\binom{t}{n} \leqq \binom{x-1}{n-1} \cdot (k-1). \tag{2}$$

This is a consequence of the fact that in each of the classes there are at most $k-1$ congruent n-subsets of $\{a_1, a_2, \ldots\}$, otherwise we would not have a B_{kn}-sequence.

A product of $n-1$ natural numbers which form a segment of $\mathbb{N}$, is $\leqq$ the $(n-1)^{\text{th}}$ power of their arithmetic mean. So we can derive from (2):

$$t(t-1)\cdot \cdots \cdot (t-n+1) \leqq n(k-1)\,(x-1)\,(x-2)\cdot \cdots \cdot (x-n+1) \leqq n(k-1)\left(x-\frac{n}{2}\right)^{n-1}. \tag{3}$$

Now we obtain:

$$t-n+1 \leqq \sqrt[n]{n(k-1)} \cdot \left(x-\frac{n}{2}\right)^{\frac{n-1}{n}}. \tag{4}$$

This is trivial for $t \leqq n-1$, and for $t > n-1$ it follows from

$$(t-n+1)^n \leqq t(t-1)\cdot \cdots \cdot (t-n+1)$$

and (3).

Since $\Phi_{kn}(x)$ is the upper limit of all possible values t, we can conclude from (4) that for all $x > n$ the following inequality is valid:

$$\Phi_{kn}(x) \leqq \sqrt[n]{n(k-1)} \cdot \left(x - \frac{n}{2}\right)^{\frac{n-1}{n}} + n - 1. \tag{5}$$

Here we can replace $\sqrt[n]{n}$ by $\sqrt[3]{3}$, since $\sqrt[n]{n}$ tends to 1 for $n \to \infty$ and from $n=3$ onwards monotonically decreasing. So each constant $c_{kn}^* \geqq \sqrt[3]{3} \cdot \sqrt[n]{k-1}$ fulfils (1) for all sufficiently large x. Then of course there also exists a constant c_{kn} satisfying (1) for all $x \in \mathbb{N}$.

Corollary. *If A is a strictly increasing sequence of natural numbers which satisfies $A(x) > c_{kn} \cdot x^{1-\frac{1}{n}}$ for at least one $x \in \mathbb{N}$, then there exist k different n-element subsets of $A \cap \{1, \ldots, x\}$, which are pairwise congruent.*

We now state a useful theorem on B_{kn}-sets:

Theorem 2. *Let k, n be $\geqq 2$ and $a_1, a_2, \ldots$ a B_{kn}-sequence. Then*

$$\sum_{m=1}^{\infty} \frac{1}{a_m} \leqq c_{kn}^{\frac{n}{n-1}} \cdot \zeta\left(\frac{n}{n-1}\right) < \infty. \quad (\zeta \text{ the Riemannian Zeta-function})$$

Proof. By the definition of Φ_{kn} we have $\Phi_{kn}(a_m) \geqq m$ for all $m \in \mathbb{N}$. By Theorem 1 there follows

$$m \leqq \Phi_{kn}(a_m) \leqq c_{kn} \cdot a_m^{1-\frac{1}{n}},$$

hence

$$a_m \geqq \left(\frac{m}{c_{kn}}\right)^{\frac{n}{n-1}} \quad \text{and} \quad \frac{1}{a_m} \leqq \left(\frac{c_{kn}}{m}\right)^{\frac{n}{n-1}}.$$

This leads to

$$\sum_{m=1}^{\infty} \frac{1}{a_m} \leqq c_{kn}^{\frac{n}{n-1}} \cdot \sum_{m=1}^{\infty} m^{-\frac{n}{n-1}} = c_{kn}^{\frac{n}{n-1}} \cdot \zeta\left(\frac{n}{n-1}\right) < \infty.$$

Let P be the set of all prime numbers. Then $\sum_P \frac{1}{p}$ is a divergent series, and so Theorem 2 gives the corollary, that the sequence corresponding to P is no B_{kn}-sequence for all $k \geqq 2 \leqq n$, in other words:

Remark. *For every two natural numbers k, n there exist k n-element sets of prime numbers which are pairwise congruent.*

As an easy consequence of Theorem 1 we mention:

Theorem 3. *Let k and n be natural numbers $\geqq 2$, x a natural number $> (2c_{kn})^n$. If $\{1, \ldots, x\} = A \cup B$, then A or B contains k pairwise congruent n-element subsets.*

Proof. Without loss of generality let A and B be disjoint. If the above statement were false, A and B would be initial segments of B_{kn}-sequences (when taken in their natural order). Hence $A(x)$ and $B(x)$ are both $\leqq \Phi_{kn}(x) \leqq c_{kn} \cdot x^{1-\frac{1}{n}}$. This would result in $x = A(x) + B(x) \leqq 2 \cdot c_{kn} x^{1-\frac{1}{n}}$ and $x \leqq (2c_{kn})^n$ contradicting the assumption.

Now we sharpen the statement of the corollary to Theorem 1 by replacing c_{kn} by an arbitrarily small positive number ε, yet under the additional condition that the inequality concerning $A(x)$ is required for all $x \in \mathbb{N}$. The proof applies the method which was already used in the proof of Theorem II of the introduction.

Theorem 4. *Let ε be a positive number and n a natural number $\geqq 2$. Let A be a strictly increasing sequence of natural numbers satisfying $A(x) > \varepsilon \cdot x^{1-\frac{1}{n}}$ for all $x \in \mathbb{N}$. Then for every $k \in \mathbb{N}$ there exist k n-element subsets of A, which are pairwise congruent.*

Proof. Let k be a fixed natural number. If there exists a number $x \in \mathbb{N}$ with $A(x) > c_{kn} \cdot x^{1-\frac{1}{n}}$, then according to the corollary the statement holds. Therefore we make the assumption, that for all $x \in \mathbb{N}$ and $\beta := c_{kn}$ the relation

$$A(x) \leqq \beta \cdot x^{1-\frac{1}{n}}$$

is valid. Now let N be a fixed natural number. To this we define the following N-element intervals I_i of $\mathbb{N}$: For $i \in \mathbb{N}$ we put

$$I_i := \{y \in \mathbb{N} \mid (i-1) \cdot N < y \leqq i \cdot N\}.$$

Then the number A_i of elements of A lying in I_i is

$$A_i = A(iN) - A((i-1) \cdot N).$$

If it is possible to choose N in such a way, that

$$\sum_{\nu=1}^{\infty} \binom{A_\nu}{n} > (k-1) \cdot \binom{N-1}{n-1}$$

is valid, then the statement of the theorem follows analogously to (2). Indeed such a choice of N is possible, as we shall prove indirectly:

Suppose that for every choice of N we always have

$$\sum_{\nu=1}^{\infty} \binom{A_\nu}{n} \leqq (k-1) \cdot \binom{N-1}{n-1}. \tag{6}$$

Now there exists a constant $\gamma > 0$ only depending on n, such that for every whole number a which is $\geqq n$ oder $= 0$, $\binom{a}{n} \geqq \frac{a^n}{\gamma}$ holds.

Next we obtain: There exists a constant d only depending on n and k, but not on N, so that

$$\sum_{\nu=1}^{N} A_\nu^n < d \cdot N^{n-1} \quad \text{for every} \quad N \in \mathbb{N}. \tag{7}$$

Indeed for every $N \in \mathbb{N}$ we have

$$\sum_{\nu=1}^{N} A_\nu^n = \sum_{\substack{\nu=1 \\ A_\nu=1}}^{N} A_\nu^n + \cdots + \sum_{\substack{\nu=1 \\ A_\nu=n-1}}^{N} A_\nu^n + \sum_{\substack{\nu=1 \\ A_\nu \geqq n}}^{N} A_\nu^n$$

$$\leqq N + 2^n \cdot N + \cdots + (n-1)^n \cdot N + \gamma \cdot \sum_{\nu=1}^{N} \binom{A_\nu}{n},$$

and this is, since (6) is supposed to be valid, $< d \cdot N^{n-1}$ with a constant d not depending on N.

Now we take $k_0 = 0$ and $k_1, k_2, \ldots$ as a sequence of natural numbers such that for all $i \in \mathbb{N}$ there holds

$$k_i \geqq \left(\frac{2\beta}{\varepsilon}\right)^{\frac{n}{n-1}} \cdot k_{i-1}.$$

Because of $\beta > \varepsilon$ this sequence is strictly increasing.

For every choice of N and every $i \in \mathbb{N}$ there holds

$$\text{(8)} \qquad \sum_{\nu=k_{i-1}+1}^{k_i} A_\nu \geqq \frac{\varepsilon}{2} \left((k_i - k_{i-1}) N\right)^{1-\frac{1}{n}}.$$

The left side indeed is

$$A(k_i N) - A(k_{i-1} N) > \varepsilon (k_i N)^{1-\frac{1}{n}} - \beta (k_{i-1} N)^{1-\frac{1}{n}} \geqq \frac{\varepsilon}{2} \cdot (k_i N)^{1-\frac{1}{n}}.$$

Now from (8) we obtain for every choice of N and every $i \in \mathbb{N}$:

$$\sum_{\nu=k_{i-1}+1}^{k_i} A_\nu^n \geqq \left(\frac{\varepsilon}{2} \frac{\left((k_i - k_{i-1}) N\right)^{1-\frac{1}{n}}}{k_i - k_{i-1}}\right)^n \cdot (k_i - k_{i-1}),$$

and hence

$$\text{(9)} \qquad \sum_{\nu=k_{i-1}+1}^{k_i} A_\nu^n \geqq \frac{\varepsilon^n}{2^n} \cdot N^{n-1}.$$

Finally we choose a natural number r with $r \cdot \frac{\varepsilon^n}{2^n} \geqq d$ and choose N as a number $\geqq k_r$. From (9) then we derive

$$\sum_{\nu=1}^{N} A^n \geqq \sum_{i=1}^{r} \sum_{\nu=k_{i-1}+1}^{k_i} A_\nu^n \geqq r \cdot \frac{\varepsilon^n}{2^n} \cdot N^{n-1} \geqq d N^{n-1}.$$

But this is a contradiction to (7).

Remark. It can easily be seen that the statement of Theorem 4 also holds if in addition we require that the k congruent n-element sets are pairwise disjoint.

For there exist also $(k-1)\cdot n\cdot(n-1)+1$ congruent n-element subsets of A, and each of them has a non-empty intersection with at most $n\cdot(n-1)$ of the others.

From Theorem 4 we can immediately derive a dual version:

Theorem 4′. *Under the assumptions of Theorem 4 there also holds: For every $k\in\mathbb{N}$ there exist n k-element subsets of A which are pairwise congruent and disjoint.*

Proof. There exist k congruent disjoint n-element subsets $A_1,\dots,A_k$ of A. For $v=1,\dots,n$ let B_v be the set of the v^{th} elements of the sets $A_1,\dots,A_k$. Then $B_1,\dots,B_n$ are congruent k-element subsets of A and disjoint.

An immediate consequence of Theorem 4 is the following generalization of II:

Theorem 5. *For any two natural numbers k, n which are $\geqq 2$ and for every B_{kn}-sequence A we have*

$$\varliminf_{x\to\infty}\frac{A(x)}{x^{1-\frac{1}{n}}}=0.$$

Remark. Of course Theorem 4 can be generalized a little by weakening the assumption in such a way, that $A(x)>\varepsilon\cdot x^{1-\frac{1}{n}}$ is only required for all x of a final segment of $\mathbb{N}$, say for all $x\geqq x_0$. For let ε be a positive number <1 (without loss of generality) and let A^* be the sequence containing all elements of A and all natural numbers $\leqq x_0$ following in their natural order. According to Theorem 4, A^* has $k+x_0$ congruent n-element subsets. From these at most x_0 have a non-empty intersection with $\{1,\dots,x_0\}$, for their smallest element is in $\{1,\dots,x_0\}$, and two of them having the same smallest element must be identical. Hence k of the $k+x_0$ congruent subsets must be subsets of A.

Remark. The notion of B_2-sequence is in a certain way opposite to the notion of an arithmetic progression. In a B_2-sequence all differences have to be different, but in an arithmetic progression as much differences as possible are equal. Namely the following statement (which was observed by N. Straus) is valid (and easily provable by complete induction on n):

If A is a finite sequence $a_1<a_2<\dots<a_n$ of natural numbers and D is the set of all differences a_j-a_i with $1\leqq i<j\leqq n$, then $|D|=|A|-1$ if and only if A is an arithmetic progression.

We will now examine the sharpness of Theorem 4. In the rest of the paper let m, n, k be integers $\geqq 2$, α a positive real number so that m^α is an integer and $>k\cdot n$.

First we establish the following

Lemma. *Let S be the set of all those m^α-element subsets of $\{1,\dots,m\}$ which contain k disjoint congruent n-element subsets. Then the number s of elements of S is*

$$\leqq\binom{m}{n+k-1}\cdot\binom{m-kn}{m^\alpha-kn}.$$

Proof. A system of k disjoint congruent n-element subsets of $\mathbb{N}$ is completely determined by the n elements of the last of the k subsets and the first elements of the first $k-1$ subsets. There are $\binom{m}{n+k-1}$ possibilities to choose $n+k-1$ elements from $\{1, \ldots, m\}$. Each of these possibilities occurs in at most $\binom{m-kn}{m^\alpha-kn}$ sets of S, which proves the assertion.

There are $\binom{m}{m^\alpha}$ subsets of $\{1, \ldots, m\}$ with m^α elements. So, if we have $\binom{m}{m^\alpha} > s$, there exists an m^α-element subset of $\{1, \ldots, m\}$, which has no k disjoint congruent n-element subsets. Therefore we have to investigate for which $\alpha > 0$ and $n \in \mathbb{N}$ there holds

$$\binom{m}{m^\alpha} > \binom{m}{n+k-1} \cdot \binom{m-kn}{m^\alpha-kn}. \tag{10}$$

(10) is equivalent to

$$\binom{m}{n+k-1} < \frac{\binom{m}{m^\alpha}}{\binom{m-kn}{m^\alpha-kn}} = \frac{m(m-1) \cdot \cdots \cdot (m-kn+1)}{(m^\alpha-kn+1) \cdot \cdots \cdot m^\alpha}. \tag{11}$$

The expression on the right side of (11) is $\geqq \frac{m(m-1) \cdot \cdots \cdot (m-kn+1)}{m^{\alpha kn}}$. Thus (11) would follow from

$$\binom{m}{n+k-1} < \frac{m(m-1) \cdot \cdots \cdot (m-kn+1)}{m^{\alpha kn}}, \tag{12}$$

which is equivalent to

$$\frac{m^{\alpha kn}}{(n+k-1)!} < (m-kn+1) \cdot \cdots \cdot (m-(n+k-1)). \tag{13}$$

The expression on the right side of (13) is $\geqq (m-kn+1)^{kn-n-k+1}$. So (13) would follow from

$$\frac{m^{\alpha kn}}{(n+k-1)!} < (m-kn+1)^{kn-n-k+1}. \tag{14}$$

This is equivalent to

$$\frac{m^\alpha}{(n+k-1)!^{\frac{1}{kn}}} < (m-kn+1)^{1-\frac{1}{k}-\frac{1}{n}+\frac{1}{kn}}. \tag{15}$$

We summarize our considerations. If

$$0<\alpha<1-\frac{1}{k}-\frac{1}{n}+\frac{1}{kn} \tag{16}$$

holds, then (15), and then also (10), is valid for all sufficiently great natural numbers m. Thus we have

Theorem 6. *If α satisfies* (16) *then for all sufficiently large natural numbers m there exists a subset of $\{1, \ldots, m\}$ which has at least m^α elements but no k disjoint congruent n-element subsets.*

Concerning the sharpness of Theorem 4 this yields:

Theorem 6′. *If $0<\alpha<1-\frac{1}{n}$ holds there exists a natural number k* (*such that* (16) *holds and*) *such that for every sufficiently large natural number m there exists a subset of $\{1, \ldots, m\}$ which has at least m^α elements but no k disjoint congruent n-element subsets.*

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