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PROBLEMS AND RESULTS ON CONSECUTIVE INTEGERS AND PRIME FACTORS OF BINOMIAL COEFFICIENTS

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To the memory of my old friend and coworker Ernst Straus

I have published many papers both alone and with coworkers during my long life on these and related questions. I give a partial list of these papers and will refer to them with roman numerals. In the last few years two important questions on consecutive integers were settled An old conjecture of Catalan stated that 8 and 9 are the only consecutive integers which are powers. Tijdeman [11] proved that there is a computable constant c so that above c there are no more consecutive powers.

Selfridge and I proved that the product of consecutive integers is never a power [I]. Probably both results can be strengthened. Selfridge and I conjectured that for $k \geqq 4$ there is a $p > k$ for which $p \parallel \prod_{i=1}^{k}(n + i)$ ($p \parallel m$ denotes $p \mid m$, $p^2 \nmid m$). Denote by $x_1 < x_2 < \ldots$ the sequence of powers. Presumably $x_{i+1} - x_i > x_i^\varepsilon$, but no proof of $x_{i+1} - x_i \to \infty$ has been published as yet.

Nevertheless very many simple unsolved problems remain. In this note I state some of my old unsolved problems which seem interesting to me and which were neglected and which do not seem to be completely hopeless. I also state some new problems and results and outlines of proofs.

Denote by $P(m)$ the largest and by $p(m)$ the least prime factor of m. Undoubtedly $P(m)$ and $P(m + 1)$ are independent, but I can not even prove that the density of integers for which $P(m) > P(m + 1)$ is 1/2. (This problem seems very difficult and may be unattackable by our present methods.) Pomerance and I proved some preliminary results [3]. It is a simple exercise that the density of integers for which $p(m) > p(m + 1)$ is 1/2.

Let $n > k$ and put for $1 \leqq i \leqq k$

$$(1) \qquad n + i = a_{n+i}(k)b_{n+i}(k),\ P(a_{n+i}(k)) \leqq k,\ p(b_{n+i}(k)) > k,$$

i.e., (1) gives the unique decomposition of $n + i$ as the product of two

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numbers such that all the prime factors of the first are $\leq k$ and of the second $> k$. Put further

$$\min_{1 \leq i \leq k} (a_{n+i}(k)) = f(n; k). \tag{2}$$

In **[III]** I conjectured that if $k \to \infty$ then

$$f(n; k)/k \to 0 \tag{3}$$

uniformly in n. I proved by a simple averaging process that $f(n; k) < Ck$ for some absolute constant C. If (1) holds it would be interesting to estimate $f(n; k)$ as accurately as possible from above and below. Ruzsa observed that a simple argument gives that for every k and infinitely many n

$$f(n; k) > \frac{ck}{\log k}. \tag{4}$$

I overlooked this and (4) disproves some old conjectures of mine **[III]**. Here is the outline of the simple proof of Ruzsa. Denote by $k/2 < q_1 < q_2 < \cdots < q_s < k$, $s > (c\, k/\log k)$ the primes in the interval $(k/2, k)$. Let $m \equiv 0 \pmod{[k/2]!}$. Then clearly for $-k/2 \leq i \leq k/2$,

$$a_{m+i}(k) \geqq |i|, \ a_m(k) > \prod_{p < \frac{k}{2}} p. \tag{5}$$

Let further m satisfy the congruences

$$m + j \equiv 0 \pmod{q_{2j}}, \ m - j \equiv 0 \pmod{q_{2j+1}}; \tag{6}$$

(5) and (6) imply (4) by a simple argument.

I also conjectured that as $k \to \infty$

$$\sum_{i=1}^{k} \frac{1}{a_{n+i}(k)} \to \infty \tag{7}$$

uniformly in n. (7) would of course imply (3). Ruzsas proof clearly shows that for every k and infinitely many n

$$\sum_{i=1}^{k} \frac{1}{a_{n+i}(k)} < c \log \log k. \tag{8}$$

Perhaps (4) and (8) give the right order of magnitude. Unless I have again overlooked an obvious argument (3) seems to me to be a nice and non trivial conjecture.

Perhaps it would be of some interest to investigate

$$f(n, k, l) = \min_{1 \leq i \leq l} a_{n+i}(k).$$

Clearly for $l \leqq \pi(k)$ $f(n; k, l)$ can not be estimated in terms of k and l alone since if $p_1, p_2, \ldots, p_{\pi(k)}$ are the primes not exceeding k, $n + i$ can be a multiple of an arbitrarily high power of p_r. On the other hand it is easy to see that

$$f(n; k, \pi(k) + 1) = \exp\left((1+o(1))\frac{k}{\log k}\right).$$

For $k < l < (2 - \varepsilon)k$ Ruzsa's proof gives

$$f(n; k, l) > C_\varepsilon \frac{k}{\log k}.$$

I have no non trivial lower bound if $l > 2k$. The determination of the smallest l for which for every n, $f(n; k, l) = 1$ is of course a classical problem. In fact it can be reformulated in the following way. Denote by $j(k)$ the largest integer so that there are $j(k)$ consecutive integers all of which have a prime factor not exceeding k. Iwaniec proved $j(k) < ck^2$, but perhaps $j(k) < k^{1+\varepsilon}$. I do not know who first formulated this conjecture, which seems to be out of reach for the moment.

Surely further non trivial upper and lower bounds can be obtained for $f(n; k, l)$ but I have not investigated this question.

Another old and related conjecture of mine states that if $\varepsilon > 0$ and $k > k_0(\varepsilon)$ then the integers $a_{n+i}(k)$, $1 \leqq i \leqq (1 + \varepsilon)k$ can not all be distinct. Here Basil Gordon and I proved this for $1 \leqq i < (2 + o(1))k$ **[IV]**. Finally I conjectured that there is an absolute constant c so that the number of distinct integers among the $a_{n+i}(k)$, $1 \leqq i \leqq k$ is always greater than ck **[III]**. It is very annoying that I got nowhere with this attractive conjecture and I only proved it with $c\,k^{1/2} \log k$ instead of ck, but perhaps I overlooked a simple argument.

Several further questions can be asked. First of all observe that "usually" $f(n; k) = 1$. More precisely the density of integers n for which $f(n; k) > 1$ tends to 0 very fast as $k \to \infty$. In other words let $1 = u_1 < u_2 < \cdots < u_{\varphi(k!)} = k! - 1$ be the integers relatively prime to $k!$. Then the number of indices i for which $u_{i+1} - u_i > k$ is very much smaller than $k!/k^t$ $k > k_0(t)$. I expect that for every fixed t, its order of magnitude is about

$$k! \cdot \exp\left(\frac{-ck}{\log k}\right)$$

I am very far from being able to prove this. Here again I perhaps overlook a simple argument.

Here I would like to call attention to a rather striking old conjecture of mine: Let $1 = u_1 < u_2 < \cdots < u_{\varphi(n)} = n - 1$ be the integers relatively prime to n. Then

(9) $$\sum_{i=1}^{\varphi(n)-1} (u_{i+1} - u_i)^2 < C\frac{n^2}{\varphi(n)}.$$

There seems to be no doubt that for every α

(10) $$\sum_{i=1}^{\varphi(n)-1} (u_{i+1} - u_i)^\alpha < C_\alpha \frac{n^\alpha}{\varphi(n)^{\alpha-1}}$$

and perhaps for every sufficiently small β

(11) $$\sum_{i=1}^{\varphi(n)-1} \exp(\beta(u_{i+1} - u_i)) < n \exp\left(C_\beta \frac{n}{\varphi(n)}\right).$$

The last inequality is perhaps a bit too optimistic. Hooley **[5]** has many interesting results about these problems. In fact he proved (10) for every $\alpha < 2$, but (9) is still open and I offer 500 dollars for a proof or disproof of (9).

Denote by $l(n; k)$ the largest integer for which all the values $a_{n+i}(k)$, $0 \leqq i \leqq l(n; k)$ are distinct. Determine or estimate as accurately as possible the mean value, variance, and the distribution of the size of $l(n; k)$. Denote by $L(n; k)$ the largest integer for which the equation $a_{n+i}(k) = 1$, $1 \leqq i \leqq L(n; k)$ has only one solution. Clearly $L(n; k) \geqq l(n; k)$, but onewould expect that "usually" $L(n; k)$ is not much larger than $l(n; k)$, but I could not obtain any significiant results on these problems.

Some of these problems may change character if, e.g., we consider the squarefree part $q(a_{n+i}(k))$ of $a_{n+i}(k)$. Perhaps very many fewer of the $q(a_{n+i}(k))$ will be distinct. I did not investigate these and the many related questions but perhaps interesting results can be obtained.

Further questions can be asked about the distribution of the $a_{n+i}(k)$. It is not hard to prove by the second moment method that if t is fixed and $k \to \infty$ then for almost all n the number of indices i for which $a_{n+i}(k) = t$ is

$$(1 + o(1)) \frac{n}{t} \prod_{p \mid k} \left(1 - \frac{1}{p}\right).$$

I have not worked out the details of how long this formula holds if t increases with k. Perhaps more interesting is the following question: For every n and k there is a smallest $t = t_0(n; k)$ which does not occur as an $a_{n+i}(k)$. One could try to determine the distribution of the value of this t. It is easy to see that if $k \to \infty$ then for almost all n the number of indices i for which $a_{n+i}(k) > k^\alpha$ is

$$(1 + o(1)) \sum_{t > k^\alpha} \frac{k}{t} \prod_{p \leq k} \left(1 - \frac{1}{p}\right).$$

On the other hand I can not determine

$$\max_{n} \sum_{a_{n+i}(k)>k^{\alpha}} 1 = f_n(k, \alpha).$$

For $\alpha > 1$, $f_n(k, \alpha) < k$ is easy to see.

It is not difficult to show that for almost all n (as $k \to \infty$)

$$\max_{1 \leq i \leq k} a_{n+i}(k) = k^{(1+o(1)) \log k/\log\log k}. \tag{11}$$

The upper bound in (11) **[III]** follows easily from the asymptotic formula of de Bruijn **[1]** for $\psi(x, y)$ (the number of integers not exceeding x all whose prime factors are $\leq y$) and the lower bound further needs the second moment method. I overlooked this in **[III]**.

Now I discuss some problems on the prime factors of binomial coefficients. A well known theorem of Sylvester and Schur asserts that $P(\binom{n}{k}) > k$ holds for all $n \geq 2k$. I proved **[VI]** in fact that

$$P\left(\binom{n}{k}\right) > \min(n - k + 1, c\, k \log k). \tag{12}$$

In (12) very likely $ck \log k$ can be replaced by $k^{1+\varepsilon}$ and perhaps even by $\exp(k^{(1/2)-\varepsilon})$, but this, if true, is centainly out of reach at present. Put

$$\binom{n}{k} = u_k(n)v_k(n) \text{ where } P(u_k(n)) \leq k,\ p(v_k(n)) > k.$$

A well known theorem of Mahler implies that, for $k > k_0(\varepsilon)$, $u_k(n) = n^{1+\varepsilon}$. Unfortunately Mahler's theorem is not effective. I conjecture with some trepidation that in fact

$$u_k(n) < n(\log n)^{ck} \cdot e^{ck} \tag{13}$$

where c_k must tend to infinity together with k. (13) is hopelessly out of reach. $c_k \to \infty$ is also far from being known. Stewart **[10]** recently proved that for every r if $k > k_0(r)$ then for infinitely many n

$$u_k(n) > n \log n \log\log n \cdots \log_r(n) \tag{14}$$

where $\log_r(n)$ is the r-times iterated logarithm.

Stormer and Polya proved that $P(n(n + 1)) \to \infty$ and Chowla proved that $P(n(n + 1)) > c \log\log n$. Perhaps $P(n(n + 1)) > (\log n)^{2-\varepsilon}$ but for infinitely many n, $P(n(n + 1)) < (\log n)^{2+\varepsilon}$. These conjectures which are certainly vnot ery well motivated were conjectured in **[II]** and **[IV]**. Schinzel proved that for infinitely many n

$$P(n(n + 1)) < \exp(\log n/\log\log\log n).$$

Selfridge and I conjectured that if $n > k^2$, $n \neq 62$ then

$$p\left(\binom{n}{k}\right) \leq \frac{n}{k}. \tag{15}$$

We only proved (15) with k replaced by $k^\alpha (\alpha < 1)$. We easily proved that (15) holds for fixed k if $n > n_0(k)$. To see this just observe that $n - i \equiv 0 \pmod k$ for some $0 \leqq i < k$. If $n - i = ktp$ for some $p > k$ then $p \mid \binom{n}{k}$ and $p < n/k$. If on the other hand $P(n-i) \leqq k$ then, for sufficiently large n, $\binom{n}{k}$ clearly has a prime factor $p \leqq k \leqq n/k$. It would be easy to determine the largest $n = n_0(k)$ for which $p(\binom{n}{k}) > k$ and, for some $0 \leqq i < k$, $P(n-i) \leqq k$, but we leave this for the interested reader. Also it is easy to see that the density of the integers n for which

$$p\left(\binom{n}{k}\right) > k \text{ is } \exp\left((-c + o(1))\frac{k}{\log k}\right).$$

Our conjecture $p(\binom{n}{k}) \leqq n/k$ for $n > k^2$ is probably best possible. Schinzel conjectured that for every k and infinitely many n

$$n - i = (k - i)p_{i+1},\ i = 0, 1, \ldots, k-1. \tag{16}$$

(16) follows from hypothesis H of Schinzel (see [9]) and shows that (15) is the best possible. (16) is of course completely out of reach of the methods at our disposal. A slightly weaker form of Schinzel's conjecture states that for every k there are infinitely many n for which

$$\binom{n}{k} = p_{i_1} p_{i_2} \cdots p_{i_k},\ k < p_{i_1} < \cdots < p_{i_k}.$$

Selfridge and I defined the deficiency of $\binom{n}{k}$ as r if $\binom{n}{k}$ is the product of $k - r$ distinct primes greater than k. It is easy to see that for every k there are only a finite number of integers n for which the deficiency of $\binom{n}{k}$ is positive, but perhaps the following very much stronger result holds. The number of pairs n and k for which the deficiency of $\binom{n}{k}$ is positive is finite. $\binom{47}{11}$ has deficiency 4, it is the product of 7 distinct primes > 11. This is the largest deficiency we have found. On the other hand we could not exclude the existence of a $c > 0$ so that for infinitely many pairs n and k $\binom{n}{k}$ has deficiency $> ck$. It is extremely unlikely that this is possible and I outline the proof that for sufficiently small $\varepsilon > 0$ the deficiency must be less than $(1-\varepsilon)k$. First of all observe that we can assume $n > k^{1+\varepsilon}$ since it is easy to see that otherwise $p(\binom{n}{k}) < k$ **[IV]**. If $n > k^{1+\varepsilon}$ and the deficiency is $> (1-\varepsilon)k$ then $\binom{n}{k}$ would be the product of $< \varepsilon k$ primes $> k$ or $(1-\varepsilon)k$ of the integers $n, n-1, \ldots, n-k+1$ would entirely be composed of primes $\leqq k$. Thus by a simple computation their product would be greater than $k!$ and thus again $p(\binom{n}{k}) < k$ an evident contradiction. In **[IV]** I conjectured that if $p\binom{n}{k} > k$ then $n > k^c$ for every c if $k > k_0(c)$ and this, if true, would imply by the above argument that the deficiency must be $o(k)$.

Let n be such that $p(\binom{n}{k}) > k$. I wanted an n such that $p(\binom{n}{k}) > k$ and for which, for all $0 \leqq i < k$, $p(n-i) > 1$. Lacampagne and Selfridge found many such n in fact they probably can find such n for every $k \neq p^\alpha$. It seems much more difficult to prove that for every $t > 0$ there is an n and a k for which $p(\binom{n}{k}) > k$ and $p(n-i) > t$ for every $0 \leq i < k$. In fact this question is still open.

Denote by $\omega(n)$ the number of distinct prime factors of n. Perhaps if $k > k_0$ and $\omega(\binom{n}{k}) = k$ then $\binom{n}{k}$ is squarefree. This is certainly false for $k = 2$ and $k = 3$ since $\binom{10}{2} = 5.3^2$ and $\binom{50}{3} = 2^4 5^2 7^2$. Perhaps for sufficiently large k $\binom{n}{k}$ always has a prime factor $k < p < n/k$ if $n > f(k)$.

We [V] conjectured that for $n > 4$, $\binom{2n}{n}$ is never squarefree. This was proved by Sarközy for $n > n_0$ [7]. Probably for $n > n_0(k)$, $\binom{2n}{n}$ is always divisible by the k-th power of a prime.

In [V] we proved that for any two primes p and q there are infinitely many integers n for which $(p\,q, \binom{2n}{n}) = 1$. We could not extend this to every set of three primes and in fact we could not decide whether

$$\sum_{\substack{p<n \\ p \mid \binom{2n}{n}}} \frac{1}{p}$$

is bounded. Denote by $q(n)$ the number of integers $1 < k < n$ for which $\binom{n}{k}$ is squarefree. $q(n) = 0$ for infinitely many n and $q(n) = o(n)$ is easy; probably much better upper bounds for $q(n)$ can be obtained. Is it true that $q(n) = o(n^\varepsilon)$ for every $\varepsilon > 0$?

Denote by n_k the smallest integer $\geq 2k$ for which $p(\binom{n}{k}) \geq k$. n_k surely tends to infinity with k very quickly, no doubt nearly exponentially, but I could not even prove that $\log n_k / \log k \to \infty$, i.e., $n_k > k^c$ for every c if $k > k_0(c)$. Denote by k_n the largest integer $k_n < n/2$ for which $p(\binom{n}{k_n}) > k_n$. $k_n = 2$ is of course possible, e.g., for $n = 2^t$, but, for almost all n, $k_n \to \infty$. It is easy to see that $k_n = o(n)$. No doubt very much better upper estimates are possible. Perhaps k_n can tend to infinity logarithmically but not faster.

About 25 years ago I conjectured that for every k and n such that $1 \leq k \leq n/2$, $(n-i) \mid \binom{n}{k}$ for some $0 \leq i < k$. Schinzel found a counterexample and then Schinzel and I [8] disproved it for infinitely many k and Schinzel conjectured that it fails for every $k > 33$, $k \neq p^\alpha$. Probably the smallest n_k for which the conjecture fails tends to infinity nearly exponentially in k. After the failure of my conjecture perhaps the following question could be considered: Denote by $d(n;k)$ the largest divisor not exceeding n of $\binom{n}{k}$. Is it true that there is an absolute constant c for which $d(n;k) > c\,n$ and if not how small can $d(n;k)$ be?

Now I state a few problems and results on the least common multiples of consecutive integers. I conjectured some time ago that for every $k > 1$, $l > 1$, $m > n + k$

(17) $$[n+1, \ldots, n+k] \neq [m+1, \ldots, m+l].$$

I am sure that (17) holds with possibly very few exceptions. A stronger conjecture of mine stated that the two sides of (17) can not have the same prime factors if $k > 2$, $l > 2$. Here there are certainly many exceptions but perhaps for sufficiently large k and l there are none. Probably for sufficiently large k and l if $m \geqq n + k$

$$\prod_{i=1}^{k} (n + i) \neq \prod_{j=1}^{l} (m + j), \tag{18}$$

but (18) seems hopelessly out of reach.

Denote for every n and k

$$A(n; k) = \max_{m \leq n} [m, m - 1, \ldots, m - k + 1]. \tag{19}$$

$k_0(n)$ is the smallest k for which $A(n; k)$ equals the least common multiple of the integers $\leqq n$. It is easy to see that for $n > n_0$ the integers $A(n; k)$, $1 \leqq k \leqq k_0(n)$, can not all be different and it would be easy to determine the largest n_0 for which these numbers are all distinct, e.g., for $n = 10$, $A(n; 1) = 10$, $A(n; 2) = 90$, $A(n; 3) = 360$, $A(10; 4) = 2520$. It would perhaps be of some interest to determine or estimate the least $k = k_1(n)$ for which $A(n; k) = A(n; k + 1)$. At present I do not even have a good estimate for the order of magnitude of $k_1(n)$. $k_0(n) = (1 + o(1))(n/2)$ is easy to see, $k_1(n) < cn/\log n$ is also easy but probably $k_1(n)$ is much smaller.

I expect that for $n > n_0$ the m defining $A(n; k)$ can not always be n. In other words if $n > n_0$ then for some k there is a $t < n$ for which

$$[n, n - 1, \ldots, n - k + 1] < [t, t - 1, \ldots, t - k + 1]. \tag{20}$$

I expect (20) to be easy but I probably overlooked a simple argument and did not find a proof.

In a paper with Eggleton and Selfridge [4] we define

$$L(n; k) \equiv \max_{a \leq n} [a_1, a_2, \ldots, a_k] \tag{21}$$

and study various properties, of $L(n; k)$. Clearly $L(n; k) \geqq A(n; k)$ if $k \geqq 4$ and $n > n_0(k)$. Several interesting problems remain unsolved in our paper and we hope to return to them at another occasion.

I would like to call attention to a recent interesting paper of Pleasants [6] on $\omega(\binom{n}{k})$($\omega(m)$ is the number of distinct prime factors of m). He proves among others that if $1 < k \leqq n/2$ then $\omega(\binom{n}{k}) \geqq \omega(n)$. Many other results are proved in this paper and interesting problems are stated. Perhaps one could try to determine all integers s for which if $s < k \leqq n/2$ then $\omega(\binom{n}{k}) \geqq \omega(\binom{n}{s})$.

To end this paper I state a few older problems which have never been published. I hope the reader will forgive me if some of them are simpler than I expected. Selfridge define $h(n) = \min(a_k - a_1)$ where $h > 1$, $a_1 < \cdots < a_k$ and $a_1 a_2 \cdots a_k/n!$ is an integer all of whose prime factors are $\leqq n$. It is surprisingly difficult to estimate $h(n)$. We could not even prove that $h(n) > 2$, for $n > n_0$. Probably $h(n) \to \infty$ as $n \to \infty$. The difficulties are caused by the very large values of a_1. It is a well known unsolved problem that $n! = x(x+1)$ has no solutions for $n > 3$, and that $n! = (x-1)(x+1)$ has no solutions for $n > 7$. C. Spiro pointed it out to me that for infinitely many n $n! \neq x(x+1)$, but as far as I know it is not yet known that this is true for all n if we neglect a sequence of density 0. $h(n) \leqq n - 2$ is trivial and $h(n) \leqq n - 3$ certainly holds and perhaps for infinitely many n there is equality here. For almost all n, $h(n) < n - c \log n$; this is not hard, but we are not sure to what extent it can be improved.

Are there infinitely many x for which $x(x+1) = \prod_i p_i^{\alpha_i}$ where all the α_i's are distinct? In fact are there infinitely many x for which there is a k for which all exponents in the representation of $\prod_{i=0}^{k}(x+i)$ are distinct? I expect that certainly for $k > 1$ the number of solutions is finite, for $k = 1$ the number of solutions is probably infinite, e.g., I expect that there are infinitely many primes p_1 for which $p_1 + 1 = 8q^2$. There does not seem to be much hope of proving this. Put

$$\prod_{i=1}^{k}(x+i) = u_k(x)v_k(x),\ (u_k(x), v_k(x)) = 1 \tag{21}$$

where $v_k(x)$ is squarefree and all prime factors of $u_k(x)$ occur with an exponent greater than 1. The representation in (21) is clearly unique. Clearly for $k > k_0(x)$, $u_k(x) > v_k(x)$. Perhaps one can estimate the smallest $k_0(x)$ so that $u_k(x) > v_k(x)$ for all $k > k_0(x)$ quite well. I have not done this. For small values of k usually $v_k(x) > u_k(x)$. I thought that for every x there is a k for which $v_k(x) > u_k(x)$. $x = 7$ seemed a likely counterexample but if $k = 7$, $u_7(7) = 2^7 3^3 < v_7(7) = 5 \cdot 7 \cdot 11 \cdot 13$. On the other hand a simple computation shows that $n = 23$ is a counterexample, i.e., for every k, $v_k(23) < u_k(23)$. The reason for this is the existence of 24, 25, 27 and 32. I would not be surprised if 23 is the only counterexample. Perhaps in fact there is a k_0 so that for every $k > k_0$ and all $n > n_0(k)$

$$v_k(n) > u_k(n). \tag{22}$$

(22) is perhaps too optimistic. My reason for the conjecture is that, by a well known theorem of Mahler, the contribution of the primes $p \leqq k$ to $\prod_{i=1}^{k}(n+i)$ is for $n > n_0(k, \varepsilon)$ less than $n^{1+\varepsilon}$. Put

$$\binom{x+k}{k} = U_k(x)V_k(x),\ (V_k(x), U_k(x)) = 1$$

where $V_k(x)$ is squarefree and all prime factors of $U_k(x)$ occur with an exponent > 1. Here I am sure that for almost all x and all k

$$V_k(x) > U_k(x). \tag{23}$$

I do not think that the proof of (23) will be difficult but I may be wrong in this since I have not in fact proved (22). Now this paper, like every good and bad thing (except Mathematics itself) must end and I leave it to the (I hope) merciful judgment of the (I hope) non empty set of readers to judge in which class this paper belongs.

ADDENDUM: Montgomery and Vaughan have proved conjecture (9), and, in fact, (10) for all $\alpha \geqq 2$.

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