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Extremal Subgraphs for Two Graphs

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In this paper we study several interrelated extremal graph problems:

(i) Given integers n, e, m, what is the largest integer $f(n, e, m)$ such that every graph with n vertices and e edges must have an induced m-vertex subgraph with at least $f(n, e, m)$ edges?

(ii) Given integers n, e, e', what is the largest integer $g(n, e, e')$ such that any two n-vertex graphs G and H, with e and e' edges, respectively, must have a common subgraph with at least $g(n, e, e')$ edges?

Results obtained here can be used for solving several questions related to the following graph decomposition problem, previously studied by two of the authors and others.

(iii) Given integers n, r, what is the least integer $t = U(n, r)$ such that for any two n-vertex r-uniform hypergraphs G and H with the same number of edges the edge set $E(G)$ of G can be partitioned into $E_1,..., E_t$ and the edge set $E(H)$ of H can be partitioned into $E'_1,..., E'_t$ in such a way that for each i, the graphs formed by E_i and E'_i are isomorphic.

I. INTRODUCTION

For a graph G with vertex set $V(G)$ and edge set $E(G)$ let $f(G, m)$ denote the maximum number of edges in an induced subgraph of G on m vertices and define

$$f(n, e, m) = \min \{ f(G, m): |V(G)| = n, |E(G)| = e \}.$$

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In other words, $f(n, e, m)$ denotes the largest value k such that every graph on n vertices and e edges must contain an induced subgraph on m vertices having k edges.

Now suppose we have two graphs G and H (not necessarily having the same number of vertices or edges). Let $g(G, H)$ denote the maximum number of edges in a graph which is a subgraph of both G and H. We define

$$g(n, e, e') = \min \{ g(G, H): |V(G)| \leqslant n, |V(H)| \leqslant n, |E(G)| = e, |E(H)| = e \}.$$

Therefore any two graphs on n vertices and e, e' edges must have a common part of $g(n, e, e')$ edges.

In this paper we will determine $f(n, e, m)$ and $g(n, e, e')$ (up to within a constant factor) for various ranges of e. These values turn out to be useful in considering the following problem of graph decomposition [1, 2].

For two graphs (or r-uniform hypergraphs) G and H, let $U(G, H)$ denote the least integer t such that $E(G)$ can be partitioned into $E_1, ..., E_t$ and $E(H)$ can be partitioned into $E'_1, ..., E'_t$ in such a way that the graphs formed by E_i and E'_i are isomorphic for each i. (Note than an r-uniform hypergraph H is just a collection $E = E(H)$ of r-element subsets (called edges) of a set $V = V(H)$.) We define $U(n, r) = \max \{ U(G, H)$: G and H are r-uniform hypergraphs, $|V(G)| = |V(H)| = n$ and $|E(G)| = |E(H)| \}$.

It was proved in [1] that

$$U(n, 2) = \tfrac{2}{3}n + o(n).$$

For $r \geqslant 3$, in [3] it was shown that

$$c_1 n^{4/3} \log\log n/\log n < U(n, 3) < c_2 n^{4/3}$$

$$c_3 n^{r/2} \leqslant U(n, r) \leqslant c_4 n^{r/2}$$

for r even and

$$c_5 n^{(r-1)^2/(2r-3)} \log\log n/\log n \leqslant U(n, r) \leqslant c_4 n^{r/2}$$

for r odd.

We will prove

$$c_1 n^{4/3} \log\log n/\log n < U(n, 3) < c_2 n^{4/3} (\log\log n/\log n)^{1/6}.$$

In [2, 3] the simultaneous decomposition of more than two graphs is also investigated.

Another related problem is the determination of the largest unavoidable graphs. A graph G is called (n, e)-unavoidable if G is contained in every graph on n vertices and e edges. Exact values and sharp bounds for the largest (n, e)-unavoidable graphs for graphs and 3-uniform hypergraphs can be found in [4]. These values serve as lower bounds for $g(G, H)$. However it is not surprising that the value of $g(G, H)$ is in general much larger than the number of edges in an unavoidable graph.

II. On $f(n, e, m)$

Bounds for $f(n, e, m)$ for certain values of e and m can be found in the literature [6, 7]. The most often seen lower bound for $f(n, e, m)$ can be obtained by a standard averaging method (see [2]).

Fact 1. $f(n, e, m) \geqslant cm^2e/n^2$.

However, in certain situations $f(n, e, m)$ can be much larger than m^2e/n^2 (i.e., the ratio of $f(n, e, m)$ and m^2e/n is unbounded). For example, every graph of $n^{5/3}$ edges has an induced subgraph on $n^{1/3}$ vertices with $n^{1/3} \log n/\log \log n$ edges! For general n, e, m we have the following:

THEOREM 1. *$ckm < f(n, e, m)$ if*

$$\left(\frac{kn^2}{2em}\right)^k < \frac{2n}{m} \text{ and } f(n, e, m) < c'km \text{ if } \left(\frac{50\,kn^2}{em}\right)^{50k} > \frac{n}{m}.$$

Proof. First we derive the lower bound. Let G denote a graph on n vertices and e edges. For a vertex v and a subset $S \subseteq V(G)$ with $|S| = m' = m/2$ we define[1]

$$g(v, S) = 1 \quad \text{if } v \text{ is adjacent to } k \text{ vertices in } S,$$
$$= 0 \quad \text{otherwise.}$$

Obviously for $\deg(v_i) = d_i$ we have

$$\sum_S g(v_i, S) \geqslant \binom{d_i}{k}\binom{n-d_i-1}{m'-k}$$

and

$$\sum_{i,S} g(v_i, S) \geqslant \sum_{i=1}^{n} \binom{d_i}{k}\binom{n-d_i-1}{m'-k}.$$

Let V_1 denote all the vertices v_i with $d_i \leqslant kn/m$ and $V_2 = V - V_1$. We have $\sum_i d_i = 2e$. Now we consider the following two possibilities:

[1] We remark that although m' may not be integral such statements are always made with the implicit understanding that the graphs (and quantities) involved may have to be adjusted slightly by adding or deleting (asymptotically) trivial subgraphs (and amounts) so as to make the stated inequalities true.

Case 1. $\sum_{v_i \in V_1} d_i \geqslant e$. We note that the function $f(x) = \binom{x}{k}\binom{n-x-1}{m'-k}$ is convex if $x \leqslant kn/2m'$ since $f(x+1) + f(x-1) \geqslant 2f(x)$ if $(m^2+m)x^2 - 2kmnx + (k-1)^2 n^2 \geqslant 0$. Therefore we have

$$\sum_{i,S} g(v_i, S) \geqslant n \binom{\bar{d}}{k} \binom{n-\bar{d}-1}{m'-k},$$

where $\bar{d} = e/n$. Therefore there exists an S_0 such that

$$\sum_i g(v_i, S_0) \geqslant \frac{n\binom{\bar{d}}{k}\binom{n-\bar{d}-1}{m'-k}}{\binom{n}{m'}}$$

$$\geqslant n\left(\frac{2em}{kn^2}\right)^k \geqslant m/2.$$

Therefore G contains an induced subgraph G' on S_0 together with $m/2$ additional vertices each of which has k edges to S_0. Thus G' has $km/2$ edges.

Case 2. $\sum_{v_i \in V_2} d_i \geqslant e$. Let $d_1, \dots, d_{m'}$ be the m' largest degrees in G. If $m' < |V_2|$, then

$$\sum_{i=1}^{m'} d_i \geqslant kn/2.$$

If $m' \geqslant |V_2|$, then again we have $\sum_{i=1}^{m'} d_i \geqslant e \geqslant kn/2$. Let $\omega(v_i)$ denote the number of neighbors of v_i in $\{v_1, \dots, v_{m'}\}$. Then we have

$$\sum_{i=1}^{n} \omega(v_i) = \sum_{i=1}^{m'} d_i \geqslant kn/2.$$

Let V_3 denote the $m' v_i$'s with the largest values of $\omega(v_i)$. Then

$$\sum_{v \in V_3} \omega(v) \geqslant km'/2.$$

Therefore the induced graph G' on $\{v_1, \dots, v_{m'}\} \cup V_3$ has at least $km'/4$ edges.

To establish the upper bound we will establish the existence of a graph G_0 on n vertices and e edges with the property that every induced subgraph on m vertices has at most $100\,km$ edges. We consider the family F of all graphs on n vertices and e edges. We say a graph $G \in F$ is bad if there is an induced subgraph on m vertices having at least $100\,km$ edges. The total number of bad graphs is at most

$$\binom{n}{m}\binom{\binom{m}{2}}{100\,km}\binom{\binom{n}{2}-100\,km}{e-100\,km}$$

which is fewer than the total number of graphs in F, since $(n/m)^m (me/50\,n^2k)^{50\,km} < 1$. Therefore there is a good graph in F and

$$f(n, e, m) \leqslant 100\,km.$$

The proof of Theorem 1 is completed.

Theorem 2 follows immediately from Theorem 1.

THEOREM 2. *Suppose* $m = o(n)$

(a) *For* $m \geqslant n^2 \log n/e$, *we have*

$$c\frac{m^2e}{n^2} < f(n, e, m) < c'\frac{m^2e}{n^2}.$$

(b) *For* $n^2/e < m \leqslant n^2 \log n/e$ *we have*

$$\frac{cm \log n}{\log(n^2 \log n/me)} < f(n, e, m) < \frac{c'm \log n}{\log(n^2 \log n/me)}.$$

(c) *For* $n^2/(e \log n \log\log n) < m \leqslant n^2/e$ *we have*

$$c\frac{m \log n}{\log\log n} < f(n, e, m) < c'\frac{m \log n}{\log\log n}.$$

(d) *For* $m \leqslant n^2/(e \log n \log\log n)$ *we have*

$$\frac{cm \log n}{\log(n^2/me)} < f(n, e, m) < \frac{c'm \log n}{\log(n^2/me)},$$

where c, c' *denote appropriate constants independent of* n *and* e.

Proof. Choose an appropriate value of k in each case and apply Theorem 1. ∎

Here are a few easy observations:

Fact 2. $c(n^2 \log n/e \log\log n) < f(n, e, n^2/e) < c'(n^2 \log n/\log\log n)$.

Fact 3. For $m = cn$ we have $f(n, e, m) \geqslant c'e$.

Fact 4. For any m, we have $f(n, e, m) \geqslant \min\{m/2, e\}$.

Proof. Consider the maximum star forest G' (i.e., the vertex disjoint union of stars) on m vertices. Either G' does not contain isolated vertices (thus has at least $m/2$ edges) or all e edges are in G'.

Now we consider r-uniform hypergraphs. We can ask the question of determining the largest value of $f_r(n, e, m)$ such that every r-uniform hypergraph on n vertices and e edges must contain an induced subgraph on m vertices having $f_r(n, e, m)$ edges.

THEOREM 3. *For $e \leqslant \binom{n}{2}$ we have*

$$ckm < f_r(n, e, m) < c'km \quad \text{if} \quad \frac{1}{2}\left(\frac{kn^r}{em^{r-1}}\right)^k \leqslant \frac{n}{m} < \left(\frac{50\,kn^r}{em^{r-1}}\right)^{50k}$$

where c and c' are constants depending on r.

Proof. The method for obtaining the bounds is quite similar to that in Theorem 1 and will be omitted.

III. ON $g(n, e, e')$

Suppose G and H are two graphs on n vertices and e and e' edges. In [1] it is proved that there is a common subgraph of $ee'/\binom{n}{2}$ edges.

FACT 5 [1]. $g(n, e, e') \geqslant ee'/\binom{n}{2}$.

In this section we will prove that in some cases $g(n, e, e')$ is much larger than $ee'/\binom{n}{2}$ (by a factor of powers of $\log n$).

THEOREM 4. *For $n^{-2}ee' \log n \log\log n < (e')^{1/2}(\log n/\log\log n)^{1/2} \leqslant n^2/e \leqslant e/n$ we have*

$$c\frac{ee'}{\binom{n}{2}}\frac{\log n}{\log\log n} \leqslant g(n, e, e') \leqslant c'\frac{ee'}{\binom{n}{2}}\frac{\log n}{\log\log n}.$$

Proof. Let ω denote $(\log n/\log\log n)^{1/2}$. Let G and H denote two graphs on n vertices and e, e' edges, respectively. We consider two possibilities:

Case 1. H has at least $(e')^{1/2}\omega$ nonisolated vertices. We can then find a star forest F in H with $(e')^{1/2}\omega/2$ edges. In G there are at least e/n vertices with degree $\geqslant e/n$. Since $e/n \geqslant (e')^{1/2}\omega$, F can be embedded in G. Therefore

$$g(G, H) \geqslant (e')^{1/2}\omega \geqslant \frac{cee'}{\binom{n}{2}}\omega^2.$$

Case 2. H has at most $(e')^{1/2}\omega$ nonisolated vertices. Using Theorem 2 there is an induced subgraph G' in G on $(e')^{1/2}\omega$ vertices with $(e')^{1/2}\omega$ $(\log n/\log\log n)$ edges. By Fact 5, H and G' have a common subgraph with

$$e'(e')^{1/2}\omega\frac{\log n}{\log\log n}\Big/\binom{(e')^{1/2}\omega}{2}\geqslant(e')^{1/2}\omega$$

$$\geqslant\frac{cee'}{\binom{n}{2}}\omega^2\text{ edges.}$$

Thus we have proved that

$$g(n,e,e')\geqslant\frac{ee'}{\binom{n}{2}}\frac{\log n}{\log\log n}.$$

For the upper bound, we can choose G to be a graph with all induced subgraphs on $\sqrt{2e'}$ vertices having at most $f(n,e,\sqrt{2e'})$ edges and H to be a graph on $\sqrt{2e'}$ vertices together with $n-\sqrt{2e'}$ isolated vertices. Therefore a common subgraph of G and H can have at most $f(n,e,\sqrt{2e'})$ edges. For e' in the indicated range, we have (by Theorem 2) that

$$\begin{aligned}g(n,e,e')&\leqslant f(n,e,\sqrt{2e'})\\&\leqslant c'\sqrt{2e'}\log n/\log\left(\frac{n^2}{(e')^{1/2}e}\log n\right)\\&\leqslant c''\sqrt{e'}\frac{\log n}{\log\log n}\end{aligned}$$

Therefore Theorem 3 is proved. We have also proved the following:

Fact 6. $g(n,e,e')\leqslant f(n,e,\sqrt{2e'})$.

IV. The Common Subgraph of two 3-Uniform Hypergraphs

First we will state a few auxiliary facts.

Fact 7 [5]. Any 3-uniform hypergraph of n vertices and e triples contains a subgraph of $\sqrt{e/n}-1$ triples which form a strong Δ-system denoted by $S(\sqrt{e/n}-1)$ (i.e., there is a single vertex that is the intersection of any two of these $\sqrt{e/n}-1$ triples.)

Fact 8 [3]. Any two 3-uniform hypergraphs G and H on n vertices and e, e' triples has a common subgraph of $ee'/\binom{n}{3}$ triples.

Fact 9 [3]. A 3-uniform hypergraph with e triples either has x pairwise disjoint triples or has maximum degree y if $3xy \leqslant e$.

For certain values of e, we can get a better lower bound for the maximum number of edges in a common subgraph of two 3-uniform hypergraphs than that in Fact 8.

THEOREM 5. *For $n^{5/3}/(\log n/\log\log n)^{1/3} < e \leqslant n^{5/3}(\log n/\log\log n)^{1/6}$ any two 3-uniform hypergraphs on n vertices and e triples have a common subgraph with $c\sqrt{e/n}\,(\log n/\log\log n)^{1/4}$ triples.*

Proof. Suppose G and H are two 3-uniform hypergraphs with e edges, where e is in the indicated range. Set $t = \log n/\log\log n$. Suppose G and H do not have a common subgraph of $\sqrt{e/n}\, t^{1/4}$ triples. We may assume G does not contain $\sqrt{e/n}\, t^{1/4}$ disjoint triples. Suppose there is a vertex u in H with degree $nt^{1/2}$. By Fact 9, G has a vertex v with degree $\sqrt{ne}/t^{1/4}$. But two 2-graphs with $\sqrt{ne}/t^{1/4}$ and $nt^{1/2}$ edges must have a common subgraph of size $\sqrt{e/n}\, t^{1/4}$ (by Fact 5), this is a contradiction. Thus we may assume H has maximum degree at most $nt^{1/2}$ and contains at least $e/3nt^{1/2}$ disjoint triples. Suppose G has degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$. Let s denote the smallest integer satisfying $\sum_{i \leqslant s} d_i \geqslant e/2$. We consider the following possibilities. The first two cases are quite easy. The third case is somewhat complicated.

Case 1. $s \leqslant t^{1/4}/2$. In G there is a vertex u with degree $e/t^{1/4}$. In H there is a vertex v, with degree e/n. Now we follow the proof of Theorem 4. Let $\bar{G}$ and $\bar{H}$ denote the 2-graphs formed by triples containing u and v in G and H, respectively. If $\bar{H}$ has at least $\sqrt{e/n}\, t^{1/4}$ nonisolated vertices, then we have $g(\bar{G}, \bar{H}) \geqslant \sqrt{e/n}\, t^{1/4}$. If $\bar{H}$ has at most $\sqrt{e/n}\, t^{1/4}$ nonisolated vertices, then by using Theorem 2 $\bar{G}$ contains a subgraph on $\sqrt{e/n}\, t^{1/4}$ vertices and $\sqrt{e/n}\, t^{5/4}$ edges. Therefore by Fact 2, we have $g(\bar{G}, \bar{H}) \geqslant \sqrt{e/n}\, t^{1/4}$.

Case 2. $s > \sqrt{e/n}\, t^{1/4}$. Consider a maximum set T of x vertex-disjoint triples in G. Suppose $x \leqslant \sqrt{e/n}\, t^{1/4} < s$. The number of triples containing any vertex in T is fewer than $\sum_{i=1}^{s-1} d_i < e/2$. Thus there is a triple disjoint from T. This contradicts the maximality of T. Therefore G contains $\sqrt{e/n}\, t^{1/4}$ disjoint triples, which is again a contradiction.

Case 3. $t^{1/4}/2 < s < \sqrt{e/n}\, t^{1/4}$. If there is a vertex in G with degree at least $e/t^{1/4}$, then we can proceed as in the same way as that in Case 1. Thus we may assume that

Property 1. All vertices in G have degree at most $e/t^{1/4}$.

From Fact 7 we know that any 3-uniform graph on e edges contains a strong Δ-systems $S(\sqrt{e/n} - 1)$. Since H has maximum degree at most $nt^{1/2}$, we can prove by greedy algorithm to obtain the following:

Property 2. H contains any set of disjoint unions of $S(m_i)$ satisfying $\sum_i m_i \leqslant e/10nt^{1/2}$ and $m_i \leqslant \sqrt{e/2n} - 1$.

Now we consider the subgraph G' of G formed by $e/2$ edges each of which is incident to a vertex in $v_1, ..., v_s$, the s vertices with largest degrees. Let T denote a set of vertex disjoint $S(k_i)$, $i = 1, ..., \omega$, in G' with the property that $\sum_{i=1}^{\omega} k_i$ is maximum and $k_i \leqslant \sqrt{e/2n}$. (If there are two such sets we choose the one with larger ω.) If $\sum_i k_i \geqslant \sqrt{e/n}\, t^{1/4}/10$, G and H have a common subgraph of the desired size. We may assume $\Sigma_i k_i < \sqrt{e/n}$ $t^{1/4}/10$. Let W denote the union of all vertices in T.

Property 3. In G' there are at least $e/4$ triples each of which contains two vertices of W.

Proof. Suppose the contrary. There are $e/4$ triples in G' each of which contains exactly one vertex in W, which must be one of the $\{v_1, ..., v_s\}$. Because of the maximality of T, any v_i, $1 \leqslant i \leqslant s$, in a triple with two vertices not in W must be a center of $S(\sqrt{e/2n} - 1)$. Since $\sum k_i \leqslant \sqrt{e/n}\, t^{1/4}/10$, there are fewer than $t^{1/4}/5$ such v_i. Since the maximum degree is at most $e/t^{1/4}$, there are fewer than $e/5$ such triples, a contradiction.

Therefore there are at least $e/4$ triples in G' which form a subgraph G'' of G' with the property that any triple in G'' contains one vertex in $\{v_1, ..., v_s\}$, and one vertex in $W = \{v_1, ..., v_{s'}\}$, where $s' < \sqrt{e/n}\, t^{1/4}$.

Property 4. In H there is a subgraph H' with $\sqrt{e/n} \cdot t^{5/4}/5$ triples and a subset W' of $V(H)$, $|W'| \leqslant \sqrt{e/n}\, t^{1/4}$, such that any triple in H' has two vertices in W'.

Proof. For any set $S \subseteq V(H)$ with $|S| = \frac{1}{2}\sqrt{e/n}\, t^{1/4} = m$ we define

$$q(v, S) = 1 \quad \text{if } |\{E \in E(H): v \in E, E \cap S \neq \varnothing\}| \geqslant t' = t/5,$$
$$= 0 \quad \text{otherwise.}$$

Then for $\deg(v_i) = d_i$ we have

$$\sum_S q(v_i, S) \geqslant \binom{d_i}{t'}\binom{n - 2d_i - 1}{m - t'}$$
$$\sum_{i,S} q(v_i, S) \geqslant \sum_i \binom{d_i}{t'}\binom{n - 2d_i - 1}{m - t'}.$$

Let V_1 denote all the vertices with $d_i \leqslant tn/m$ and $V_2 = V - V_1$. We have $\sum_i d_i \geqslant 3e$. Now we consider two possibilities.

Case 1. $\Sigma_{v_i \in V_1} d_i \geqslant e$. The function $f_1(x) = \binom{x}{t}\binom{n-2x-1}{m-t}$ can easily be checked to be convex for $x \leqslant tn/(4m)$.

Therefore we have

$$\sum_{i,S} q(v_i, S) \geqslant n \binom{\bar{d}}{t'} \binom{n-2\bar{d}-1}{m-t'},$$

where $\bar{d} = e/n$.

Therefore there exists a set S_0 such that

$$\sum_i q(v_i, S_0) \geqslant \frac{n\binom{\bar{d}}{t'}\binom{n-2\bar{d}}{m-t'}}{\binom{n}{m}}$$

$$\geqslant n\left(\frac{em}{t'n^2}\right)^{t'}$$

$$\geqslant n\left(\frac{e^{3/2}}{2n^{5/2}t^{3/4}}\right)^{t'}$$

$$\geqslant n\left(\frac{1}{2t}\right)^{t/4}$$

$$\geqslant m.$$

Therefore we can choose W' to be the union of S_0 and m vertices v_i having $q(v_i, S_0) = 1$. The number of triples containing two vertices in W is at least $mt = \sqrt{e/n}\, t^{5/4}/5$.

Case 2. $\sum_{v_i \in V_2} d_i \geqslant e$. Let $d_1, \dots, d_m$ be the m largest degrees in G. If $m \leqslant |V_2|$, then $\sum_{i=1}^m d_i \geqslant tn/4$. If $m > |V_2|$, then $\sum_{i=1}^m d_i \geqslant e \geqslant tn/4$.

Let $\omega(v_i)$, $1 \leqslant i \leqslant n$, denote the number of neighbors of v_i in $\{v_1, \dots, v_m\}$. Then we have

$$\sum_{i=1}^{n} \omega(v_i) \geqslant tn/4.$$

Let V_3 denote the m v_i's with the largest values of $\omega(v_i)$. Then we have

$$\sum_{v \in V_3} \omega(v) \geqslant tm/4.$$

Therefore the number of triples containing two vertices in $\{v_1, \dots, v_m\} \cup V_3$ is at least $mt/24 = \sqrt{e/n}\, t^{5/4}/24$.

Now let H' denote the graph formed by these triples.

Property 5. G'' and H' have a common subgraph with at least $\sqrt{e/n}\, t^{3/4}/4000$ triples.

Proof. Order the vertices in G'' and H' so that $V(G'')=\{v_1,..., v_n\}$ with $W\subseteq\{v_1,..., v_\omega\}$; and $V(H')=\{v'_1,..., v'_n\}$ with $W'\subseteq\{v'_1,..., v'_\omega\}$, where $\omega=\sqrt{e/n}\,t^{1/4}$. Let p denote a permutation on $\{1,..., \omega\}$ and q denote a permutation on $\{1,..., n\}$. For $1<j\leqslant\omega<k$, we define

$$F_{p,q}(i,j,k)=1 \quad \text{if } \{v_i, v_j, v_k\}\in V(G'') \text{ and } \{v_{p(i)}, v_{p(j)}, v_{q(k)}\}\in V(H'),$$
$$=0 \quad \text{otherwise.}$$

Then for fixed (i, j, k) with $\{v_i, v_j, v_k\}\in V(G'')$,

$$\sum_{p,q} F_{p,q}(i,j,k)\geqslant e'2(\omega-2)!\,(n-1)!,$$

and

$$\sum_{\substack{(i,j,k)\\ p,q}} F_{p,q}(i,j,k)\geqslant ee'(\omega-2)!\,(n-1)!/4,$$

where $e'=|V(H')|=\sqrt{e/n}\cdot t^{5/4}/6$.

Therefore there exist p_0 and q_0 such that

$$\sum_{(i,j,k)} F_{p_0,q_0}(i,j,k)\geqslant\frac{ee'(\omega-2)!\,(n-1)!}{4\cdot\omega!\,n!}$$
$$\geqslant\frac{e\cdot\sqrt{e/n}\,t^{5/4}/5}{150(\sqrt{e/n}\,t^{1/4})^2\,n}$$
$$=\sqrt{e/n}\,t^{3/4}/750.$$

Since every triple can be counted at most 6 times, there is a common subgraph with $\sqrt{e/n}\,t^{3/4}/4000$ triples. This completes the proof of Theorem 6.

V. On $U_2(n, 3)$

In this section we will improve the upper bound for $U_2(n, 3)$, the number of subgraphs in the simultaneous decompositions of two 3-graphs on n vertices as defined in Section I.

THEOREM 6. $cn^{4/3}\log\log n/\log n\leqslant U_2(n,3)\leqslant c'n^{4/3}(\log\log n/\log n)^{1/6}$.

Proof. The lower bound is proved in [3]. We will only work on the upper bound. Now we consider two 3-uniform hypergraphs G, each with n vertices and e triples. We will successively remove isomorphic subgraphs F

from G and H, thereby decreasing the number e of triples currently remaining in each of the original graphs. The subgraph $F=F(e)$ removed will depend on the current value of e.

Again t denotes $\log n/\log\log n$. We distinguish three ranges of e:

(i) $e \geqslant n^{5/3}t^{1/6}$. In this case we repeatedly remove a common subgraph $F(e)$ having at least $e^2/\binom{n}{3}$ triples. The existence of such an $F(e)$ is guaranteed by Fact 8. Let e_i denote the number of triples remaining in each hypergraph after i such subraphs have been removed. Then we have

$$e_i+1 \leqslant e_i - e_i^2 \Big/ \binom{n}{3}.$$

Setting $\alpha_i = e_i/\binom{n}{3}$, we have

$$\alpha_{i+1} \leqslant \alpha_i - \alpha_i^2.$$

Since $\alpha_i < 1$ and $1/i - 1/i^2 < 1/(i+1)$, it follows by induction that $\alpha_i \leqslant 1/i$ for all i. Thus, after $n^{4/3}t^{-1/6}$ steps, the remaining graphs have at most $n^{5/3}t^{1/6}$ triples.

(ii) $n^{5/3}t^{1/6} > e \geqslant n^{5/3}t^{-1/3}$. For this range, we repeatedly remove a common subgraph $F(e)$ with $c_1\sqrt{e/n}\,t^{1/4}$ triples (guaranteed by Theorem 5). Let e_0 denote the number of triples in each graph at the beginning of this process. In general, if e_i denotes the number of triples remaining after removing i such subgraphs then we have

$$e_{i+1} \leqslant e_i - c_1\sqrt{\frac{e_i}{n}}\,t^{1/4}$$

Setting $\alpha_i = e_i n/(c_1^2 t^{1/2})$, we have

$$\alpha_{i+1} \leqslant \alpha_i - \sqrt{\alpha_i}$$

and $\alpha_0 \leqslant n^{8/3}t^{-1/3}/c_1^2$. Suppose

$$\alpha_i \leqslant (n^{4/3}t^{-1/6}/c_1 - i/2)^2 \qquad \text{for some } i \geqslant 0.$$

Then

$$\begin{aligned}\alpha_{i+1} &\leqslant (n^{4/3}t^{-1/6}/c_1 - i/2)^2 - n^{4/3}t^{-1/6}/c_1 + i/2\\ &\leqslant (n^{4/3}t^{-1/6}/c_1 - (i+1)/2)^2.\end{aligned}$$

Therefore, after at most $n^{4/3}t^{-1/6}/c_1$ steps, the remaining graphs can have at most $n^{5/3}t^{-1/3}$ triples.

(iii) $e < n^{5/3} t^{-1/3}$. Here we repeatedly apply Fact 7 and remove $F(e)$ with $\sqrt{e/n} - 1$ triples. Define e_0 and e_i as before. We have

$$e_{i+1} \leqslant e_i - \sqrt{\frac{e}{2n}}.$$

Again we can prove by induction that

$$e_i n \leqslant (2n^{4/3} t^{-1/6} - i/2)^2.$$

Therefore after at most $2n^{4/3} t^{-1/6}$ steps, all edges in each graph will have been removed. We have proved

$$U_2(n, 3) \leqslant c' n^{4/3} (\log \log n / \log n)^{1/6}.$$

We remark that the power $\frac{1}{6}$ of $(\log \log n / \log n)$ for the upper bound can probably be improved slightly by careful examination of more cases. However the main intent here is to show that $U_2(n, 3)$ is much smaller than $c' n^{4/3}$. We remark that the averaging argument used here does not seem to be able to bring down the upper bound to $cn^{4/3} \log \log n / \log n$. Some new idea is needed to close the gap.

References

1. F. R. K. Chung, P. Erdős, R. L. Graham, S. M. Ulam, and F. F. Yao, Minimal decompositions of two graphs into pairwise isomorphic subgraphs, *in* "Proceedings of the Tenth Southeastern Conference on Combinatorics Graph Theory and Computing," 1979, pp. 3–18.
2. F. R. K. Chung, P. Erdős, and R. L. Graham, Minimal decompositions of graphs into mutually isomorphic subgraphs, *Combinatorica* **1** (1981), 13–14.
3. F. R. K. Chung, P. Erdős, and R. L. Graham, Minimal decompostions of hypergraphs into mutually isomorphic subhypergraphs, *J. Combin. Theory Ser. A* **32** (1982), 241–251.
4. F. R. K. Chung and P. Erdős, On unavoidable graphs, *Combinatorica* **3** (1983), 167–176.
5. F. R. K. Chung, Unavoidable stars in 3-graphs, *J. Combin. Theory Ser. A* **35** (1983), 252–262.
6. P. Erdős and A. Szemerédi, On a Ramsey type problem, *Period. Math. Hungar.* **2** (1972), 295–299.
7. P. Erdős and V. T. Sós, Some remarks on Ramsey and Turán theorems, combinatorial theory and its applications, *Colloq. Math. Soc. János Bolyai* **4** (1970), 395–404.