SOME NEW AND OLD PROBLEMS ON CHROMATIC GRAPHS

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This note should be considered as a preliminary announcement, some of the references are incomplete and might even be inaccurate. The reason is that some of the new results are not yet in their final form and since many of them are only a few weeks old I may not be able to give the exact authorship. All these possible errors will soon be corrected in later publications, but enough of these apologies and let us start the mathematics.

Let G be a graph of chromatic number n. Rodl defines : G is said to have local chromatic number γ $(r < n)$ if r is the smallest integer for which there is an integer p $(p > n)$ so that G has a good colouring by p colours (a colouring is called good if two vertices which are joined always get different colours) and the star of every vertex gets fewer than r colours i.e. the vertex together with its star gets at most r colours.

First of all it is not at all obvious that such graphs exist. But Rödl observed that an old graph of Hajnal and myself [1] has chromatic number n (for every finite and infinite n) and local chromatic number 3. By the way it is easy to see that if the local chromatic number is 2 then the graph must be bipartite.

Let the vertices of G be the ordered triples (i, j, k), $1 \leqslant i < j < k \leqslant N$, (i, j, k) is joined to (u, v, w) if and only if $u = j$, $v = k$. Hajnal and I proved in our old paper that the chromatic number of this graph is $\log\log_2 N - c$, where c is constant $(0 < c < 1)$. Now color the vertex (i, j, k) by the color j. Then clearly the star of (i, j, k) only gets the two colors i and k, which completes Rödl's proof. The same proof was discovered independently by P. Komjath.

Several problems can be posed. First of all let $f(n)$ be the smallest integer p for which there is a graph of chromatic number n which has a coloring by $f(n) = p$ colors which gives a local coloring number $< n$, and more generally $f(n, r)$ is the smallest integer for which there is a coloring by $f(n, r)$ colors which gives a local chromatic number $\leqslant r$. Trivially $f(n) > n$

and the biggest gap in our knowledge is that we have not yet proved $f(n) > n+1$. It is very probable that $f(n)$ increases very much faster, perhaps (though I doubt if) $f(n) < n+c$ and perhaps $f(n)$ increases exponentially or even faster. A method of Hajnal probably will give

$$f(n) < 2^{2\varepsilon n}$$

for every $\epsilon > 0$ if $n > n_0$, but at the moment $f(n) < \exp(\exp(n^{\frac{1}{2}}))$ seems beyond our reach.

Hajnal-Kamjath and somewhat later Nesetril and Rödl proved (for every $\epsilon > 0$ if $n > n_0(\epsilon)$)

$$f(n, 3) > 2^{2^{n(1-\varepsilon)}}.$$

Thus the value of $f(n, 3)$ is known with satisfactory accuracy. Hajnal further proved for $r > 3$

$$2^{2^{n/2r}} < f(n, r) < 2^{2^{c_r n}}$$

where $c_r < 1$ and the proof probably will give that $c_r \to 0$ as $n \to \infty$, but it does not seem that his proof can give more than $f(n) < \exp\exp\frac{n}{\log n}$.

So far all the constructions of graphs whose local chromatic number is less than its chromatic number are based essentially on our old graph with Hajnal, at the moment we do not know if there is some deeper reason for this.

We do not have any good criteria which would tell us when a graph G has the property that its local chromatic number is less than its chromatic number. It is of course not clear if such criteria exist.

In general : let there be given integers m, n, p, r, when is there a graph of chromatic number n local chromatic number r, the number of vertices of G is $\leqslant m$ and $\leqslant p$ colors are needed to establish that the local chromatic number is r ?

Observe that we must have $n \geqslant 4$, $1 \leqslant r < n$. It would be of some interest to find the smallest m for which there is a graph of m vertices which exhibits this phenomenon.

V. T. Sos and I proved that if G is coloured by p colors so that the star of every vertex is colored by at most $r-1$ colors then the graph has a good coloring using at most $\exp(\log p \log r)^{\frac{1}{2}}$ colors i.e. we have

$$n > \exp(\log p \log r)^{\frac{1}{2}}.$$

The proof uses simple properties of projective planes. Clearly there must be many more relations between m, n, p and r but so far we have not been successful in finding them.

Nesetril and Rödl and independently Hajnal and I proved that for every l and n there is a graph of chromatic number n local chromatic number r which has girth l.

Hajnal and Komjath found for every k and n an n-chromatic graph which has a good coloring so that for every vertex x the set of vertices y which can be reached from x by a path of length less than or equal to k has at most $2k+1$ colors, further they proved that $2k+1$ is best possible.

Hajnal and Komjath and also independently Nesetril and Rödl investigated these problems for infinite cardinal numbers too and in fact they obtained much more complete and final answers than in the finite case, but we do not discuss their results here.

I hope several papers will appear on these problems in the near future.

Now I discuss a few other problems on chromatic graphs. A well known theorem of mine [2] states that for every k and l there is a graph of chromatic number k and girth l (i.e. the smallest circuit has length l). Hajnal and I conjectured long ago that there is an $f(k, l)$ so that if G has chromatic number $\geqslant f(k, l)$ then G has a subgraph G_1 of girth $> l$ and chromatic number $\geqslant k$. Rodl [3] proved his conjecture for $l = 3$ and every k but the case $l \geqslant 4$ is still open. Probably the estimation of Rödl for $f(k, 3)$ is very far from being best possible.

Hajnal and I conjectured many years ago that if G has infinite chromatic number then

$$\sum_i \frac{1}{n_i} = \infty \qquad \dots \quad (1)$$

where $n_1 < n_2 < \dots$ are the length of the circuits occurring in G. In fact we conjectured that $\sum_i \frac{1}{n_i} = \infty$ already holds if the edge density of G is infinite. More precisely if $G(n, kn)$ is a graph of n vertices and kn edge then we conjectured that

$$\sum_i \frac{1}{n} > c \log k, \qquad \dots \quad (2)$$

(2) has recently been proved by Szemeredi *et al.*

As far as we know the following much deeper question is still open: Let G have infinite chromatic number. Is it then true that

$$\sum_i \frac{1}{n_i} = \infty. \qquad n_i \equiv 1 \pmod 2 \qquad \dots \quad (3)$$

Perhaps if G has chromatic number k then

$$\sum_i \frac{1}{n_i} > c \log k \quad n_i \equiv 1 \pmod 2) \qquad \dots \quad (4)$$

Perhaps (4) is too optimistic but perhaps we can be even more optimistic and ask if there is a $c(a, b)$ so that

$$\sum_c \frac{1}{n_i} > c(a, b) \log k \quad n_i \equiv a \pmod b. \qquad \dots \quad (5)$$

holds.

Burr and I conjectured and Bollobas [4] proved that there is a $c_1(a, d)$ so that if $G(n)$ is a graph of n vertices which has no circuit of length l with $l \equiv a \pmod d$ then the number of edges of $G(n)$ is less than $c_1(a, d)n$.

Very recently X and I asked the following question : Let $n_1 < n_2 < \dots$ be an infinite sequence of integers and assume that G has no circuit of length n_i, $i = 1, 2, \dots$. What properties of the sequence $\{n_i\}$ will insure that the chromatic number or edge density of G is finite ? The answer will not be the same for the chromatic number and for the edge density since if $n_i = 2i+1$ the chromatic number becomes 2 and the edge density can be infinite. To avoid this trivial case we can either assume that all the n_i are odd or if we can assume $n_{2i+1} = n_{2i}+1$. It is almost certain that there still will be sequences which bound the chromatic number but not the edge density.

It is known that if $\{n_i\}$ increases sufficiently fast then there is a graph G of infinite chromatic number which has no circuit of length n_i, $i = 1, 2, \dots$. But what happens if $n_i = 2^i$? Is it true that every graph G of infinite chromatic number (or of infinite edge density) contains a C_2i for infinitely many i ? We did not yet have the time to decide; these questions are really good and fruitful.

Now I give a short discussion of problems and results on chromatic number of infinite graphs.

First of all Hajnal, Rado and I proved using a modification of a construction of Specker that for every k and every infinite cardinal m there is an m-chromatic graph of power m the shortest odd circuit of which has size $2k+1$.

An old result of Tutte Ungar and Zykov states that there is a graph of chromatic number $\varkappa_0$ which contains no triangle and I proved that there is a graph of chromatic number $\varkappa_0$ which contains no circuit of size $\leqslant n$. Hajnal and I noticed that every graph of chromatic number $\varkappa_1$ contains a C_4 and in fact it contain a $K(n, \varkappa_1)$ for every finite n i.e. it contains a complete bipartite graph of n white and $\varkappa_1$ black vertices. This result which was quite unexpected at that time called attention to a profound difference between the structure of graphs of chromatic number $\aleph_0$ and greater than $\aleph_0$. Hajnal and I further proved that there is a graph of chromatic number $\aleph_1$ which does not contain a $K(\aleph_0, \aleph_0)$ and using $2^{\aleph_0} = \aleph_1$. Hajnal constructed a graph of power $\aleph_1$ chromatic number $\aleph_1$ which contains no triangle. We could not decide whether there is such a graph which also does not contain a C_5 (a circuit of 5 edges). I do not think that such a graph exists even if we do not insist that the graph should have power $\aleph_1$.

Hajnal and I also proved that every graph of chromatic number $\aleph_1$ contains a countable tree each vertex of which has degree $\aleph_0$.

Hajnal Shelah and I further proved that if G has chromatic number $\aleph_1$ then there is an n so that every graph of chromatic number $\aleph_1$ contains a C_m for every $m > n$ and Hajnal Shelah and I further conjectured in our paper that we can assume that all these C_m's have an edge in common. Subsequently this conjecture was proved by Hajnal and independently by Thomasen. Recently Hajnal and Komjath proved the following theorem of astonishing accuracy. We call a bipartite graph H a half graph if the white vertices are $X_1, X_2, \ldots,$ and the black vertices are $y_1, y_2, \ldots$; X_i is joined to y_j if $j \geqslant i$. The white vertices have infinite degree and the black vertices have degree $< \aleph_0$. Now Komjath and Hajnal proved that every graph G of chromatic number $\aleph_1$, contains a half graph and another vertex Z where Z is joined to all the X_i i.e. to all the vertices of infinite degree. On the other hand using $2^{\aleph_0} = \aleph_1$ they constructed a graph of chromatic number $\aleph_1$ what contains no half graph H and two vertices Z_1 and Z_2 where both Z_1 and Z_2 are joined to all the X_i.

Taylor asked the following fundamental problem; let G be any graph of chromatic number $\aleph_1$. Is it true that for every cardinal number m there is a graph G_m of chromatic number m all whose finite subgraphs are contained in G.

This fundamental problem is still unsolved. Hajnal Shelah and I studied this problem and we stated several related rather technical conjectures. We are not at all convinced that our conjectures stated there are correct, in fact Komjath disproved them but they might lead to interesting now insights.

In a very recent paper Hajnal, Szemeredi and I proved certain further properties of graphs of chromatic number $\geqslant \aleph_1$ and we raised many challenging problems many of which are still unsolved. We introduce some notations :

Let G be a graph $f^{(1)}(G(n))$ is the largest integer so that every subgraph of G of n vertices contain an independent set of $f^{(1)}(G(n))$ vertices. $f^{(2)}(G(n))$

is the largest integer so that every subgraph of n vertices contains a bipartite subgraph of $f^{(2)}G(n)$ vertices.

We prove that for every infinite cardinal m there is a G of chromatic number $\geqslant m$ for which

$$f^{(2)}(G(n)) > (1-\epsilon)n.$$

i.e. in some sense the graph is almost bipartite since every subgraph of m vertices can be made bipartite after the omission of $< \epsilon n$ vertices. One of the fundamental problems which we can not solve is whether the cardinal number of the vertices of G can be m. This problem is unsolved even for $m = \aleph_1$. We do not even known if $f^{(2)}(G(n)) > cn$ is possible here for some $c > 0$.

There is again a subtle difference between graphs of chromatic number $\aleph_0$ and $> \aleph_0$. We show that there is a graph of chromatic number $\aleph_0$ for which $f^{(2)}(G(n)) = n+O(n)$ but if G has chromatic number $> \aleph_0$ then there is an $\epsilon > 0$ so that

$$f^{(1)}(G(n)) < \left(\frac{1}{2}-\epsilon\right) n.$$

Very interesting problems arise if we omit edges instead of vertices. Denote by $f_e^{(k)}(G(n))$ the smallest integer so that every subgraph of n vertices can be made k chromatic by omitting at most $f_e^{(k)}G(n)$ edges. We prove that for every infinite cardinal number m there is a G of chromatic number $\geqslant m$ for which

$$f_e^{(2)}(G(n)) < 2n^{3/2}.$$

We do not know whether $2n^{3/2}$ can in fact be replaced by cn. Rodl has certain significant results which are not yet published [Rodl's paper appeared in *Combinatorica*] if we assume that G has finite chromatc number.

To finish this paper I discuss some of the problems I stated in my lecture held here two years ago if some progress has been made I will refer to my paper as I, [5].

Denote by $r(n, n)$ the smallest integer t for which if one colors the edges of a complete graph $K(t)$, $t = r(n, n)$ by two colors, there always is a monochromatic $K(n)$. The best non-probabilistic lower bound for $r(n, n)$ is due to P. Frankl who proved

$$r(n, n) > \exp(c(\log n)^2).$$

As I often stated a constructive proof of $r(n, n) > (1+c)^n$ would be very interesting and I offer 1000 rupees (or 100 dollars, whichever is worth more) for it.

Let $G(n)$ be a graph of n vertices and edge density $< c$ (i.e. for every $m \leqslant n$ every subgraph of m vertices has fewer than m edges. Burr and I conjectured 1(16) of I) that for such a $G(n)$

$$r(G(n), G(n)) < f(c)n. \qquad \ldots \quad (5)$$

Chvatal, Szemeredi and Trotter proved (5) if we assume the stronger assumption that every vertex of $G(n)$ has valency $< C$.

Paudree, Rousseau, Schelp and I define the size Ramsey number of $G(n)$, $\hat{r}(G(n), G(n))$, as follows. It is the smallest integer for which there is a G_1 of $\hat{r}(G(n), G(n))$ edges so that if we color the edges of G_1 by two colors then there always is a monocrhromatic $G(n)$. We conjectured (P_n is a path of length n, (17) of I) $\hat{r}(P_n, P_n)_{/n} \to \infty$. To our great surprise J. Beck proved (C_n is a circuit of n edges)

$$\hat{r}(C_n, C_n) < c_1 n. \qquad \ldots \quad (6)$$

This paper just appeared in *Combinatorica*.

It has not yet been decided if (5) holds for the size Ramsey number too i.e. is it true that

$$\hat{r}(G(n), G(n)) < f(c)n \qquad \ldots \quad (7)$$

holds if each vertex of $G(n)$ has valency $\leqslant c$ (or if $G(n)$ has bounded edge density). My guess would be that the answer is no.

Silverman and I asked (p. 18 of I) : Let $1 \leqslant u_1 < \ldots < u_t \leqslant X$ be a sequence of integers for which $u_i + u_j$ is never a square. Put max $t = h(X)$. Is it true that

$$\max h(x) = \frac{x}{3} + O(1) \ ? \qquad \ldots \quad (8)$$

Does (8) hold if the u's are residue classes mod d ? This conjecture, and therefore (8), was disproved by Massias. He found 11 residue classes mod 32 no two of which add up to a quadratic residue mod d.

Lagarias, Odlyzko and Shearer [6] proved that (8) holds if the u's are residue classes mod d with the only exception $d = 32$. Their proof of this result is quite difficult.

They also proved that

$$h(X) < 0{\cdot}475X. \qquad \ldots \quad (9)$$

The proof (9) is difficult. Perhaps in fact

$$h(x) = \frac{11}{32}x + O(1) \qquad \dots \quad (10)$$

if true will probably be difficult to prove.

More generally let $N = \{n_1 < n_2 < \dots\}$ be an infinite sequence of integers. Let $1 \leqslant u_1 < \dots < u_l \leqslant X$ be such that $u_i + u_j$ is never in N. Put

$$\max l = h(X, N).$$

Trivially $h(X, N) \leqslant \frac{X-N}{2}$. Equality is possible if all the n_i are odd and the u's are the even numbers. Probably if the n_i are even numbers which do not tend to infinity too fast then

$$\lim h(X, N)/X < \frac{1}{2}. \qquad \dots \quad (11)$$

The proof of Lagarias and Odlyzko will probably give that (11) holds if N is the set of k-th powers. Perhaps (11) holds if N consists of even numbers satisfying $n_{i+1/n_i} \rightarrow 1$.

P. Frankl and R. M. Wilson proved that the chromatic number of n-dimensional space is greater than $(1+c)^n$ (see p. 16, equation (21) of I). The chromatic number of E_n is the chromatic number of the graph whose vertices are the points of E_n, two points of which are joined if their distance is 1.

Finally on p. 17 of [5] I conjecture that for $n \geqslant 6$ there do not exist n points in the plane no three on a line no four on a circle which determines exactly $n-1$ distinct distances the i's of which occurs exactly i times (the distances are not ordered by size). Two Hungarian students (independently) found such sets of 6 points. Perhaps in fact for large n such a set determines at least n distinct distances. Püredi and I in fact considered the following problem : Let $f(n)$ be the smallest integer for which there is a set of n points in the plane no three on a line and no four on a circle which determine $f(n)$ distinct integers. We except that

$$f(n)_{/n} \rightarrow \infty, f(n)_{/n^2} \rightarrow 0 \qquad \dots \quad (12)$$

but could not prove anything, perhaps we overlook a simple argument.

To complete this chapter one more problem : Let G be a graph of infinite chromatic number; $h_G(n)$ is the smallest integer for which G has a subgraph of $h_G(n)$ vertices and chromatic number n. I proved that if G has chromatic number $\aleph_0$ then $h_G(n)$ can grow arbitrarily fast. Hajnal and I proved that if G has chromatic number $> \varkappa_0$ then $h_G(n)$ can increase faster than a k-fold iterated exponential for every k. We have not been able to show that whether a faster growth is possible but on the other hand we have not been able to limit the possible growth of $h_G(n)$. This problem seems fundamental to me and would deserve more attention than it has received so far.

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