SELECTIVITY OF HYPERGRAPHS

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1. INTRODUCTION

Many properties of hypergraphs were abstracted from partitions of vertices into a bounded number of classes. Examples of such properties include the B-property and the chromatic number of hypergraphs which form a possible background for Ramsey theory.

Here we study the properties of hypergraphs which stem from partitions of the vertices into unbounded number of classes. We restrict ourselves to some questions analogons of which (dealing with the concept of chromatic number) were intensively studied. To this end we introduce the notion of selective hypergraph and selective property. These concepts provide a suitable background for canonical partitions theorems which are closely related to Ramsey type theorem (see e.g. [2], [4] and [11]).

The following definitions are central for this paper and indicate the direction of our approach. They generalize some of the notions introduced in [10]:

Definition 1.1. A hypergraph (X, E) (i.e. $E \subseteq \exp X$) is called

selective hypergraph if for every mapping $c: X \to X$ there exists an edge $e \in E$ such that the mapping $c \!\restriction_e$ is either a constant or $1-1$.

Definition 1.2. Let (X, E) be a hypergraph. A hypergraph (X', E') is said to be *selective for* (X, E) if for every coloring $c: X' \to X'$ there exists an embedding $f: (X, E) \to (X', E')$ such that $c \circ f$ is either a constant or $1-1$.

Here, a $1-1$ mapping $f: X \to X'$ is said to be an *embedding* if $\{f(x); x \in e\} \in E'$ holds iff $e \in E$.

Definition 1.3. Let $\mathscr{K}$ be a class of hypergraphs. $\mathscr{K}$ is said to have *selective property* if for every $B \in \mathscr{K}$ there exists a $C \in \mathscr{K}$ such that C is selective for B.

This paper has 5 parts. This first part is introductory. In the second part we investigate the smallest number of edges which are needed for a selective k-graph. It appears that the results are similar to that for the B-property. The methods of this part are non-constructive.

In part 3 we investigate the minimal chromatic number of a selective graph (for a given hypergraph G). It appears that this number is $(\chi(G) - 1)(|V(G)| - 1) + 1$ and a construction of hypergraphs for which this bound is achieved is given.

In part 4 we further modify the construction used in part 3 ("the partite construction") and we give a construction of selective hypergraphs without short cycles. This was proved earlier in [10] by non-constructive means. (The existence of sparse selective hypergraphs found some non-combinatorial applications, see [13].) The results of parts 3 and 4 allow us to deduce the following

Theorem 1.4. *Let* $\mathfrak{A}$ *be a finite set of 2-connected graphs. Let* Forb $(\mathfrak{A})$ *be the class of all graphs which do not contain a subgraph isomorphic to any* $A \in \mathfrak{A}$. *Then the class* Forb $(\mathfrak{A})$ *has selective property in the following strong sense: For every* $G = (V, E) \in$ Forb $(\mathfrak{A})$ *there exists a graph* $H = (V', E')$ *with the following properties:*

(i) H *is selective for* G;

(ii) $\chi(H) = (|V| - 1)(\chi(G) - 1) + 1$;

(iii) $H \in \text{Forb}(\mathfrak{A})$.

A proof of this theorem is outlined in part 5.

2. SIZE OF SELECTIVE k-GRAPHS

Let X be a set and let $c: X \to X$ be a coloring of X. Alternatively, c is defined by a partition $X = C_1 \cup C_2 \cup \ldots \cup C_p$, $p \in \{1, 2, \ldots, |X|\}$.

We say that *a set* $A \subseteq X$ *is selective* with respect to the coloring c if either $c{\restriction}_A$ is $1-1$ or $c{\restriction}_A$ is a constant. In this case we also say that c *captures the set* A.

Clearly a hypergraph is selective if every coloring c captures an edge of the hypergraph.

Let $s(k)$ denote the minimal number of edges of a selective k-graph. We prove the following

Theorem 2.1. $\displaystyle \lim_{k \to \infty} \frac{s(k)^{\frac{1}{k}}}{k} = 1$.

Proof. It follows from the results proved in [5] (see also [3]), that the number of edges of a k-chromatic k-graph is at least $(1 + o(1))^k k^k$. As every selective k-graph is also at least k-chromatic we thus get that $s(k) \geqslant \geqslant (1 + o(1))^k k^k$. Hence, it suffices to prove that there exists a selective k-graph with $(1 + o(1))^k k^k$ edges. We shall use the following

Claim. *Let* n, k *be positive integers such that* $k > 18n^2$. *Then for every coloring of the set of cardinality* nk^2 *there exist at least* $\binom{nk}{k}$ k-*tuples which are either monochromatic or* $1-1$.

Proof of claim. Suppose that the set of cardinality nk^2 is decomposed into pairwise disjoint classes. We can clearly restrict ourselves to the case that all of those classes are of cardinality less than kn. On the other hand there are less than $\frac{k}{2}$ partition classes of cardinality at least $9n^2$.

(This follows as $(9n^2)^{\frac{k}{2}} = 3^k n^k > \binom{kn}{k}$ and from the fact that every p-tuple, $p \leqslant \frac{k}{2}$, which is $1-1$ can be extended to a k-tuple which is also $1-1$.) As none of these classes has cardinality bigger than kn such classes cover at most $\frac{nk^2}{2}$ points. Thus the remaining points (the number of which is at least $\frac{nk^2}{2}$) are covered by sets of cardinality at most $9n^2$. The number of such sets is therefore bigger than $\frac{k^2}{18n}$ and hence there are

more than $\binom{\frac{k^2}{18n}}{k} \geqslant \binom{nk}{k}$ $1-1$ colored sets. This proves Claim. ∎

Now we prove the theorem: let k, n be positive integers satisfying $k > 18n^2$. Let X be a set with nk^2 elements. We shall construct a selective k-graph with the vertex set X and with at most $p = \lceil k^k e^{\frac{k}{n}} k^2 n \log k^2 n \rceil$ edges by induction.

Suppose that the edges $A_1, A_2, \ldots, A_r$, $r \leqslant p$, have been chosen. Let x_r be the number of partitions capturing none of sets A_i, $1 \leqslant i \leqslant r$. If $x_r < 1$ we are done, suppose therefore $x_r \geqslant 1$. According to Claim these partitions capture at least $x_r \binom{kn}{k}$ k-tuples (where each k-tuple is counted exactly so many times, as the number of partitions, capturing the given k-tuple.). Thus, there exists a k-tuple A_{r+1} which is counted at least

$$\frac{x_r \binom{kn}{k}}{\binom{k^2 n}{k}} > x_r \left(\frac{n-1}{n}\right)^k \frac{1}{k^k} > x_r \frac{1}{k^k e^{\frac{k}{n}}}$$

times. Hence, the number of partitions which contain none of edges $A_1, \ldots, A_{r+1}$ is $x_{r+1} < x_r - \frac{x_r}{k^k e^{\frac{k}{n}}}$. As we have clearly $x_0 < (k^2 n)^{k^2 n}$ and as

$$(k^2 n)^{k^2 n} \left(1 - \frac{1}{k^k e^{\frac{k}{n}}}\right)^p < \exp\left(k^2 n \log k^2 n - \frac{p}{k^k e^{\frac{k}{n}}}\right) \leqslant 1$$

we get that after $p' \leqslant p$ steps $x_{p'} = 0 < 1$ and thus $A_1, \ldots, A_{p'}$ is selective set system. Set now e.g. $n = \lfloor k^{\frac{1}{3}} \rfloor$, then

$$p^{\frac{1}{k}} \leqslant k \exp \frac{1}{k^{\frac{1}{3}}} k^{\frac{3}{k}} = k(1 + o(1)). \quad \blacksquare$$

A hypergraph is called *simple* if any two edges intersect in at most one point.

We refine the above theorem to the case of simple selective hypergraphs.

Let $s_1^*(k)$ and $s_2^*(k)$ denote the minimal number of vertices and edges of a simple selective k-uniform hypergraph.

Theorem 2.2. $\lim\limits_{k \to \infty} \dfrac{(s_1^*(k))^{\frac{1}{k}}}{k} = 1, \quad \lim\limits_{k \to \infty} \dfrac{(s_2^*(k))^{\frac{1}{k}}}{k^2} = 1.$

Proof. Denote by $n_r^*(k)$ and $m_r^*(k)$ the minimal number of points and edges of a k-uniform $(r + 1)$-chromatic hypergraph. It was shown in [1] that

$$\lim\limits_{k \to \infty} (n_r^*(k))^{\frac{1}{k}} = r, \qquad \lim\limits_{k \to \infty} (m_r^*(k))^{\frac{1}{k}} = r^2.$$

As every selective k-graph has chromatic number $\geqslant k$ we have

$$s_1^*(k) \geqslant n_{k-1}^*(k), \qquad s_2^*(k) \geqslant m_{k-1}^*(k).$$

Thus it suffices to show that

(1) $s_1^*(k) \leqslant (1 + o(1))^k k^k,$

(2) $s_2^*(k) \leqslant (1 + o(1))^k k^{2k}.$

We get (1) and (2) as an easy combination of the following auxiliar results. The first is an immediate consequence of Theorem 1, in [1].

Lemma 2.3. *Let* $n = 80\, k^{k+5}$, $m = 1600\, k^{2k+6}$. *Then there exists a simple k-graph H on kn points with at most m edges such that each of n points contains an edge.*

Lemma 2.4. *Let* $n = 80\, k^{k+5}$ *and let* $X_1, \ldots, X_k$ *be pairwise disjoint sets of cardinality kn. Then there exists a k-graph* $(X, \mathcal{M})$, $|\mathcal{M}| \leqslant p = [4e^k k^2 n \log k^2 n]$ *such that*

(i) $X = X_1 \cup \ldots \cup X_k$,

(ii) $|E \cap X_i| \leqslant 1$ *for every* $E \in \mathcal{M}$ *and* i, $1 \leqslant i \leqslant k$,

(iii) $|E \cap E'| \leqslant 1$ *for every* $E, E' \in \mathcal{M}$, $E \neq E'$,

(iv) *for every partition* $Y_1 \cup Y_2 \cup \ldots \cup Y_t = X$ *satisfying*

(3) $|Y_j \cap X_i| \leqslant n$ *for every* i *and* j, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant t$

there exists an $E \in \mathcal{M}$ *such that* $|E \cap Y_j| \leqslant 1$ *for every* j, $1 \leqslant j \leqslant t$.

Using 2.3 and 2.4 it is easy to prove (1) and (2): Let $X_1, \ldots, X_k$ be disjoint sets each of cardinality nk. The hypergraph with vertex set $X = X_1 \cup \ldots \cup X_k$ and edge set $E(H_1) \cup \ldots \cup E(H_k) \cup \mathcal{M}$ (H_i is a copy of H with vertex set X_i, $1 \leqslant i \leqslant k$) has $80\, k^{k+7}$ vertices and at most $1600\, k^{2k+7} + kp$ edges and is selective. Thus

$$s_1^*(k) \leqslant 80\, k^{k+7} \leqslant (1 + o(1))^k k^k$$

and

$$s_2^*(k) = 1600\, k^{2k+7} + kp \leqslant (1 + o(1))^k k^{2k}. \quad \blacksquare$$

Proof of Lemma 2.4. Similarly as in the proof of Theorem 2.1 we shall proceed by induction. Suppose that the edges $E_1, \ldots, E_r$ have been constructed in such a way that (ii) and (iii) hold for every $E, E' \in \{E_1, \ldots, E_r\}$, $E \neq E'$. Let y_r be the number of partitions $P_1, P_2, \ldots$ $\ldots, P_{y_r}$, $P_i = (Y_1^i, \ldots, Y_{t_i}^i)$, satisfying (3) and capturing no one of the edges $E_1, \ldots, E_r$. If $y_r = 0 < 1$ we are done. Suppose therefore $y_r \geqslant 1$. Among all k-element sets satisfying (ii) we shall choose that one — E_{r+1} — which has at most one element intersection with all $E_1, \ldots, E_r$ and, moreover, which is contained in as many of P_i, $1 \leqslant i \leqslant y_r$ as possible.

Our aim is to show that $y_{r'} < 1$ for some $r' \leq p = \lceil 4e^k k^2 n \log k^2 n \rceil$. For every i, $1 \leq i \leq y_r$ and j_1, j_2, $1 \leq j_1 < j_2 \leq k$, denote by $\lambda(i, j_1, j_2)$ the number of $(k-2)$-tuples E, $|E \cap X_j| = 1$ for $j \in \{1, 2, \ldots, k\} - \{j_1, j_2\}$ such that $|E \cap Y_j^i| \leq 1$ for every j, $1 \leq j \leq t_i$.

Using (3) and the fact that $|X_j| = kn$ for every j, $1 \leq j \leq k$ one can derive that $\lambda(i, j_1, j_2) \geq \dfrac{k!}{2} n^{k-2}$ holds for every choice of i, j_1 and j_2.

Set

$$\lambda(i) = \max \{\lambda(i, j_1, j_2); \ 1 \leq j_1 < j_2 \leq k\}.$$

Then the number of $1-1$ k-tuples satisfying (ii) (good k-tuples) is at least $\lambda(i) n^2$.

On the other hand as every good set has 2-element intersection with at most $\binom{k}{2} \lambda(i)$ other good k-tuples which are $1-1$, the number of candidates (i.e. good k-tuples E satisfying $|E \cap E_j| \leq 1$ for every j, $1 \leq j \leq r$ and selective with respect to P_i) for choosing E_{r+1} is at least

$$\lambda(i) \left[n^2 - \binom{k}{2} r \right] > \frac{k!}{2} n^{k-2} [n^2 - k^2 p] > \frac{k!}{4} n^k.$$

Here we used that $\lambda(i) > \dfrac{k!}{4} n^{k-2}$ holds. As the total amount of good k-sets equals to $(kn)^k$ there exists one which is contained in at least

$$y_r \frac{\dfrac{k!}{4} n^k}{(kn)^k} > \frac{y_r}{4} \frac{1}{e^k} \quad \text{of } P_1, \ldots, P_{y_r}.$$

Thus $y_{r+1} \leq y_r \left(1 - \dfrac{1}{4} \dfrac{1}{e^k} \right)$ and as obviously $y_0 \leq (k^2 n)^{k^2 n}$ and $(k^2 n)^{k^2 n} \left(1 - \dfrac{1}{4} \dfrac{1}{e^k} \right)^p < e^{\left(k^2 n \log k^2 n - \frac{p}{4e^k} \right)} \leq 1$ we obtain after p steps a k-graph with the desired properties. ∎

3. EXACT BOUND FOR THE CHROMATIC NUMBER OF A SELECTIVE HYPERGRAPH

A k-graph G is a pair (X, E) where $E \subseteq [X]^k = \{Y \subseteq X; \ |Y| = k\}$.

We always assume $k \geq 2$. A *proper coloring* of G is a mapping $c: X \to \{1, \ldots, r\}$ which is non-constant on every edge $e \in E$. Minimal r for which there exists a proper r-coloring is denoted by $\chi(G)$ and called the *chromatic number* of G.

A graph (V', E') is an *induced subgraph* of (V, E) if $V' \subseteq V$ and $e \in E'$ iff $e \subseteq V'$ and $e \in E$.

Let $G = (V, E)$ be a k-graph. Recall that a k-graph $H = (W, F)$ is *selective for* G if for every coloring $c: W \to W$ there exists an induced subgraph $G' = (V', E')$ of H, G' isomorphic to G such that $c \upharpoonright_{V'}$ is either a constant or $1-1$.

The purpose of this part is to prove

Theorem 3.1. *Let* $G = (V, E)$ *be a* k-*graph. Then the following holds:*

1. $\chi(H) \geq (\chi(G) - 1)(|V| - 1) + 1$ *for every* k-*graph* H *which is selective for* G;

2. *For every* k-*graph* G *there exists a* k-*graph* H *which is selective for* G *and which satisfies* $\chi(H) = (\chi(G) - 1)(|V(G)| - 1) + 1$.

Proof. First, we prove 1. which is simpler. Let H be a selective k-graph for G and assume $\chi(H) \leq (\chi(G) - 1)(|V(G)| - 1)$. Put $\chi(G) = r$ and $|V(G)| = n$. Consequently, there exists a proper coloring $V(H) = V_1 \cup \ldots \cup V_{n-1}$ such that the k-graph H_i induced by H on the set V_i satisfies $\chi(H_i) \leq \chi(G) - 1$ for every $i = 1, \ldots, n - 1$. Consequently H_i does not contain any induced subgraph isomorphic to G. On the other side for every subgraph G' of H with at least n vertices there exists an i such that $|V_i \cap V(G')| \geq 2$. Consequently H fails to be selective for G, a contradiction.

The proof of 2. is more involved and uses a modification of the

partite construction (see [9], [12], [11]).

First, we introduce some necessary notions and terminology:

An *a-partite* *k-graph* $G = (V, E)$ is a *k*-graph together with a fixed proper coloring by means of *a*-colours. Explicitely, an *a*-partite *k*-graph is a pair $((V_i)_1^a, E)$ where

(i) $V_1, \ldots, V_a$ are mutually disjoint subsets of the set $V = \bigcup_{i=1}^{a} V_i$;

(ii) (V, e) is a *k*-graph;

(iii) $e \not\subseteq V_i$ for every edge $e \in E$, $i = 1, \ldots, a$.

Let us remark that some of the sets V_i may be empty.

Two *a*-partite graphs $((V_i)_1^a, E)$ and $((V_i')_1^b, E')$ are said to be *isomorphic* if there exists a bijection $\varphi: \bigcup_{i=1}^{a} V_i \to \bigcup_{i=1}^{b} V_i'$ such that

0. $a = b$,

1. $\varphi(V_i) = V_i'$ for $i = 1, \ldots, a$,

2. $\{\varphi(v); v \in e\} \in E'$ iff $e \in E$.

An *a*-partite *k*-graph $G = ((V_i)_1^a, E)$ is said to be a *subgraph* of $G' = ((V_i')_1^b, E')$ iff

1. there exists a monotone injection $\iota: \{1, \ldots, a\} \to \{1, \ldots, b\}$ such that $V_i \subseteq V_{\iota(i)}$, $i = 1, \ldots, a$;

2. $e \in E'$, $e \subseteq \bigcup V_i$ iff $e \in E$.

I.e. G is an induced subgraph of G' which preserves the partition into color classes (see Fig. 1).

Let G be a *k*-graph, $\chi(G) = r$, $V(G) = \{v_1, \ldots, v_n\}$. Put $a = (n - 1)(r - 1) + 1$. The existence of a *k*-graph H which is selective for G and which has chromatic number $= a$ will be proved by a chain of amalgamations of a suitable family of *a*-partite *k*-graphs $P_0, P_1, \ldots, P_a = H$.
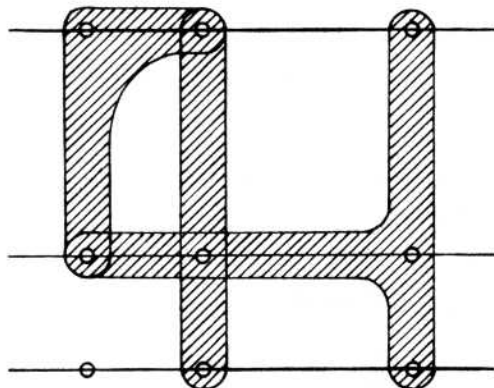
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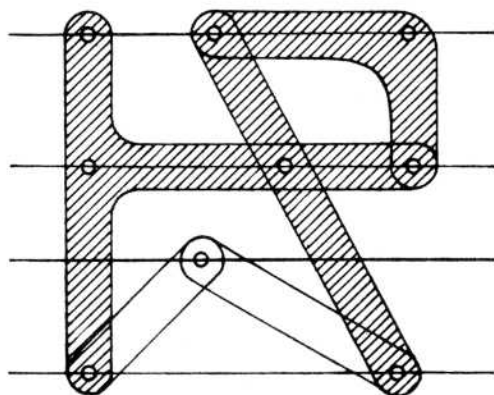
G'

Figure 1

First, let us construct the graph P_0:

Let $G_1, \ldots, G_R$, $G_i = ((V_{ij})_{j=1}^a, E_i)$ be all a-partite k-graphs arising as proper colorings of G by means of a colors (some colors need not be used).

For each $i = 1, \ldots, R$ let $G_i' = ((V_{ij}')_{j=1}^a, E_i')$ be an a-partite k-graph with the following property:

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for every set $\omega \subseteq \{1, \ldots, a\}$ and for every coloring c of the set $\bigcup_{j=1}^{a} V'_{ij}$ which is 1–1 on every set V'_{ij}, $j \in \omega$, there exists

(*) a subgraph G' of G'_i, G' isomorphic to G_i, such that $c(v) \neq \neq c(v')$ whenever at least one of the vertices v, v' belongs to the set $V(G') \cap \bigcup_{j \in \omega} V'_{ij}$ and the other one belongs to $V(G')$.

The construction of graphs G'_i is simple. Any graph of form

$$((V_{ij} \times \{1, \ldots, N\})^a_{j=1}, \{\{(v_s, j_s); v_s \in e\}; e \in E, 1 \leqslant j_s \leqslant N\})$$

will due for N sufficiently large.

Let $P_0 = ((V^0_i)^a_1, E^0)$ be the disjoint union of graphs G'_i, $i = 1, \ldots, R$. The a-partite k-graphs $P_1, \ldots, P_a$ will be defined by induction as follows:

Let $P_s = ((V^s_i)^a_1, E^s)$, $0 < s < a$ be given. Put $|V^s_{s+1}| = K$ and let $(V^{s+1}_{s+1}, \mathcal{M})$ be a selective K-graph (we may put $|V^{s+1}_{s+1}| = (K-1)^2 + 1$ and for $\mathcal{M}$ we may choose the set of all K-element subsets of V^{s+1}_{s+1}). In this situation let $P_{s+1} = ((V^{s+1}_i)^a_1, E^{s+1})$ be an a-partite k-graph with the following property: for every $M \in \mathcal{M}$ there exists a subgraph $P' = ((V'_i)^a_1, E')$ of P_{s+1} which is isomorphic to P_s and which satisfies $V'_{s+1} = M$.

P_{s+1} may be constrcucted as an amalgamation of $|\mathcal{M}|$ copies of the graph P_s with respect to the hypergraph $(V^{s+1}_{s+1}, \mathcal{M})$; in fact we may assume that $V^{s+1}_j = V^s_j \times \mathcal{M}$ for all $j \neq s + 1$. Put

$$H = \left(\bigcup_{i=1}^{a} V^a_i, E^a \right) = (W, F).$$

Obviously $\chi(H) \leqslant a = (r - 1)(n - 1) + 1$.

Claim. H is selective for G.

Proof of Claim. Let $c: W \to W$ be a coloring. By downward induction on $s = a, a - 1, \ldots, 0$ we find an a-partite subgraph $P' = ((V'_i)^a_1, E')$ of P_a with the following properties:

1. P' is isomorphic to P_0;

2. for every $i = 1, \ldots, a$ $c \restriction V_i'$ is either a constant or $1-1$.

We consider two cases: First, let there exist numbers $i_1, \ldots, i_r$, $i_1 < \ldots < i_r$ such that $c \restriction V_{i_i}$ is $1-1$, $i = 1, \ldots, r$. Consider a proper coloring of G which uses the colors $i_1, \ldots, i_r$ only and let G_{i_0} be the corresponding a-partite graph. As P' contains a subgraph isomorphic to G_{i_0}' and as G_{i_0}' possesses the above property $(*)$ there exists a subgraph G' of P', G' isomorphic to G such that c restricted to the set $V(G')$ is $1-1$. Thus assume that $c \restriction V_i'$ is $1-1$ for all $i \in \omega$ where ω is a set with at most $r-1$ elements. Put explicitely $c \restriction V_i' \equiv c_i$. As $(n-1)(r-1) + 1 - |\omega| \geqslant (n-2)(r-1) + 1$ either there exists κ, $|\kappa| = r$ such that $c_i = c_0$ for all $i \in \kappa$ or there exists κ, $|\kappa| = n-1$, $\kappa \cap \omega = \phi$ such that $c_i \neq c_j$ for all $i \neq j \in \kappa$. In the former case we get a subgraph G' of P' isomorphic to G such that $c \restriction V(G') \equiv c_0$. In the later case (again using the property $(*)$) we get a subgraph G' of P' isomorphic to G such that $c \restriction V(G')$ is $1-1$ mapping. $\blacksquare$

4. A CONSTRUCTIVE PROOF OF THE EXISTENCE OF SPARSE SELECTIVE HYPERGRAPHS

We prove here:

Theorem 4.1. *For every $2 \leqslant k, p$ there exists a k-graph $G_{k,p}$ with the following properties;*

1. $G_{k,p}$ *is selective;*

2. $\chi(G_{k,p}) = k$;

3. $G_{k,p}$ *does not contain cycles of length* $\leqslant p$.

This theorem was proved in [10] by probabilistic means. Here we give a constructive proof of this result. This solves a problem stated in [7].

The proof of Theorem 4.1 is similar to that one given above in part 3: the desired k-graph will be constructed by a chain of graphs $P_0, \ldots, P_a$

where $a = (k-1)^2 + 1$. The main difference is that the construction of the graph P_0 is more difficult.

We take time out for a lemma:

Lemma 4.2. *For every* k, p, $k \geqslant 2$ *there exists a* k-*partite* k-*graph* $G_{k,p} = ((V_i)_1^k, E)$ *without cycles of length* $\leqslant p$ *with the following properties*

(1) $|e \cap V_i| = 1$ *for every* $e \in E$, $1 \leqslant i \leqslant k$;

(2) *for every coloring* $c: \bigcup_{i=1}^{k} V_i \to \bigcup_{i=1}^{k} V_i$ *which is* $1-1$ *on each of the sets* V_i, $i \in \omega$ *for an* $\omega \subset \{1, 2, \ldots, k\}$ *there exists an edge* $e \in E$, $e = \{v_1, v_2, \ldots, v_k\}$, $v_i \in V_i$, *such that* $c(v_i) \neq c(v_j)$ *whenever one of indices belongs to* ω.

Proof. Let $G' = (V', E')$ be a k-graph without cycles $\leqslant p$ and with chromatic number bigger than 2^{k^k} (for the construction of such k-graphs see [6], [9]. Put $V' = \{v_1, v_2, \ldots, v_n\}$. Define a k-graph $G = (V, E)$ as follows: $V = V' \times \{1, 2, \ldots, k\}$; $V_i = V' \times \{i\}$; $E = \{\{(v_{i_1}, 1), \ldots, (v_{i_k}, k)\}; \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}_< \in E\}$. (If $e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ is an edge of G' and $i_1 < i_2 < \ldots < i_k$ then we write $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}_<$.) Symbolically we put $E = E' \times \{1, 2, \ldots, k\}$; see Fig. 2.

We prove that $G = (V, E)$ has the desired properties. (1) is obvious. Consider (2): choose $\omega \subset \{1, 2, \ldots, k\}$ and consider a coloring such that $c \restriction_{V_i}$ is $1-1$ for all $i = 1, 2, \ldots, k$. This coloring induces for every edge e a partition ρ_e of the set $\{1, 2, \ldots, k\}$ defined by

$$j \, \rho_e \, j' \quad \text{iff} \quad c((v_{i_j}, j)) = c((v_{i_{j'}}, j')).$$

As there are $\leqslant k^k$ possibilities for ρ_e there exists a set $E^\nu \subseteq E$ such that

(1) $\chi(V', E^\nu) > k$;

(2) the partitions ρ_e coincide for all $e \in E^\nu \times \{1, 2, \ldots, k\}$. Put $\rho_e = \rho$ for every $e \in E^\nu \times \{1, 2, \ldots, k\}$. If $\{i\}$ is the equivalence class

of the equivalence ρ for every $i \in \omega$ then there is nothing to prove (equivalently this means that $c(v_{i_j}, j) = c(v_{i_{j'}}, j')$ whenever $\{j, j'\} \cap \omega \neq$ $\neq 0$). Thus assume that there exists $j \in \omega$ such that $j \rho j'$ for a $j' \in$ $\in \{1, 2, \ldots, k\}$, $j \neq j'$. Consider the graph $G^v = (V', F^v)$ where $F^v = \{\{v_{i_j}, v_{i_{j'}}\}; \ \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}_< \in E^v\}$. As $\chi(V', E^v) > k$ it follows that $\chi(G^v) > 2$ and consequently G^v contains a cycle. Then it is easy to see that there exists edges

$$e_1 = \{v_1^1, \ldots, v_k^1\}_< \in E^v \quad \text{and} \quad e_2 = \{v_1^2, \ldots, v_k^2\}_< \in E^v$$

such that $v_{j'}^1 = v_{j'}^2$ and $v_j^1 = v_j^2$. But then $c(v_j^1, j) = c(v_{j'}^1, j') =$ $= c(v_{j'}^2, j') = c(v_j^2, j)$, a contradiction to $c \restriction V_j$ being $1-1$.
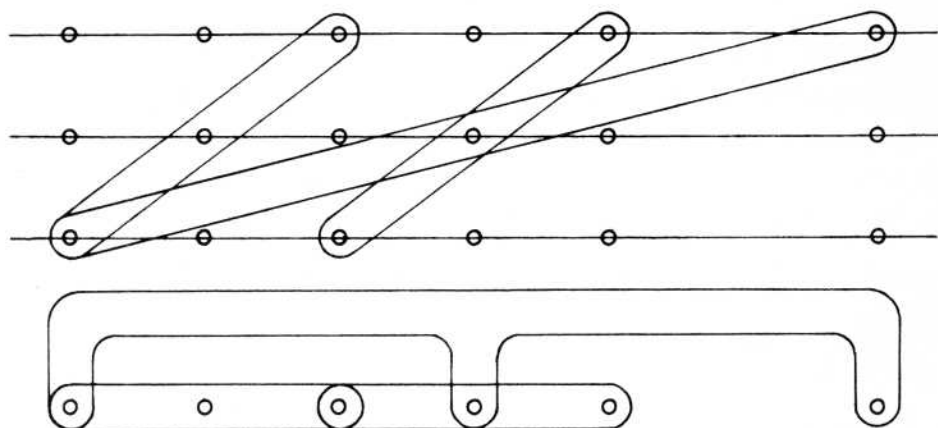

Figure 2

Proof of Theorem 4.1. We proceed by induction on p (k arbitrary ≥ 2). For $p = 1$ it suffices to take any selective k-graph. Let $k, p > 1$ be fixed. The desired k-graph will be constructed by a chain $P_0, \ldots, P_a$ of a-partite k-graphs where $a = (k - 1)^2 + 1$.

Construction of P_0

Let $P_0 = ((V_i^0)_1^a, E^0)$ be an a-partite k-graph which satisfies the following conditions:

1. For every $\omega \subseteq \{1, 2, \ldots, a\}$, $\omega = \{i_1, \ldots, i_k\}$, the subgraph of P_0 induced on the set $\bigcup_{i \in \omega} V_i^0$ contains a k-partite subgraph $G_{k,p}$ constructed in Lemma 4.2;

2. P_0 does not contain cycles of length $\leqslant p$;

3. $|e \cap V_i^0| \leqslant 1$ for every $e \in E^0$ $(1 \leqslant i \leqslant a)$.

The existence of P_0 follows immediately from Lemma 4.2 as we may take the disjoint union of all a-partite graphs arising from graphs $G_{k,p}$.

Construction of P_{s+1}

Let $P_s = ((V_i^s)_1^a, E^s)$ $(0 < s < a$ be given. Put $|V_{s+1}^s| = K$ and let $(V_{s+1}^{s+1}, \mathcal{M})$ be a selective K-graph without cycles of length $\leqslant p - 1$. It is here where the induction hypothesis on p is used.

Construct P_{s+1} as an amalgamation of $|\mathcal{M}|$ copies of the graph P_s with respect to K-graph $(V_{s+1}^{s+1}, \mathcal{M})$. Explicitely, this can be done as follows: Put $V_i^{s+1} = V_i^s \times \mathcal{M}$ for every $i \neq s + 1$ and for every $M \in \mathcal{M}$ choose a bijection $\varphi_M : V_{s+1}^s \to M$. Define E^{s+1} as the set of all sets of form

$$\{(x_i, M);\ x_i \in e - V_{s+1}^s\} \cup \{\varphi_M(e \cap V_{s+1}^s)\}$$

for an $M \in \mathcal{M}$ and $e \in E^s$. An edge of this form will be denoted as (e, M); thus, symbolically, $E^{s+1} = E^s \times \mathcal{M}$.

Put $H = \left(\bigcup_{i=1}^a V_i^a, E^a \right) = (W, F)$. Obviously $\chi(H) \leqslant k$.

Claim 1. *H is selective.*

Claim 2. *H does not contain cycles of length $\leqslant p$.*

Proof of Claim 1. Let $c : W \to W$ be a coloring. As above in Part 3, by the downward induction on $s = a, a - 1, \ldots, 0$ we find an a-partite

subgraph $P' = ((V_i')_1^a, E')$ of P_a with the following properties:

(i) P' is isomorphic to P_0 (as a-partite k-graphs);

(ii) for every i $c\restriction_{V_i'}$ is either a constant or $1-1$. Let ω be the set of all i for which $c\restriction_{V_i'}$ is $1-1$. If $|\omega| \geq k$ then the subgraph of P' induced on the set $\bigcup_{i \in \omega} V_i'$ contains a graph $G_{k,p}$ constructed in Lemma 4.2 and consequently there exists an edge $e \in E'$ such that $c\restriction_e$ is $1-1$.

If $\omega = \phi$ then clearly there exists $\kappa \subseteq \{1, \ldots, a\}$, $|\kappa| = k$, such that all constants $c\restriction_{V_i'}$, $i \in \kappa$, either coincide or are distinct. Using the construction of P_0 there exists an edge $e \in E'$, $e \subseteq \bigcup_{i \in \kappa} V_i'$. If $0 < |\omega| < k$ then $a - |\omega| \geq (k-2)(k-1) + 1$ and consequently either there exists a set $\kappa \subseteq \{1, \ldots, a\} - \omega$, $|\kappa| = k$, such that all constants $c\restriction_{V_i'}$, $i \in \kappa$, coincide or there exists a set $\kappa \subseteq \{1, \ldots, a\} - \omega$, $|\kappa| = k - 1$, such that all constants $c\restriction_{V_i'}$, $i \in \kappa$, are distinct. In the first case there exists a monochromatic edge $e \subseteq \bigcup_{i \in \kappa} V_i'$ in the second case there exists an $1-1$ edge $e \subseteq \bigcup_{i \in \kappa} V_i \cup V_{i_0}$ for every i (see the statement of Lemma 4.2).

Thus H is selective. ∎

Proof of Claim 2. We proceed by induction on s. By the construction (see Lemma 4.2) P_0 does not contain cycles of length $\leq p$. Let $s > 0$ be fixed and assume that the edges $e_1, \ldots, e_p$ form a cycle in P_s. Recall that P_s was constructed as an amalgamation of copies of P_{s-1} and that for every edge $e \in E^s$ there exists exactly one $M \in \mathcal{M}$ such that $e \subseteq M \cup \bigcup_{i \neq s} V_i^{s-1} \times \{M\}$. Denote this edge M by $\bar{e}$. It is clear that either $\bar{e}_1 = \ldots = \bar{e}_p$ or the edges $e_1, \ldots, e_p$ form a cycle in $(V_s^s, \mathcal{M})$. In the second case we have $|\{\bar{e}_1, \ldots, \bar{e}_p\}| > p - 1$ by the induction assumption (used for the K-graph $(V_s^s, \mathcal{M})$) and as there are necessarily $i \neq j$ such that $\bar{e}_i = \bar{e}_j$ we have a contradiction. If $\bar{e}_1 = \ldots = \bar{e}_p = M$ then $e_i \subseteq M \cup \bigcup_{i \neq s} V_i^{s-1} \times \{M\}$ for $i = 1, \ldots, p$ and as a consequence

the edges $e_1, \ldots, e_p$ form a cycle in a subgraph of P_s which is isomorphic to P_{s-1}. But this is a contradiction to the induction assumption. ∎

This completes the proof of Theorem 4.1. ∎

5. CONCLUDING REMARKS

I. Let us outline the proof of the theorem stated in the introduction. The theorem follows from the following slightly stronger result.

Theorem 5.1. *Let G be a k-graph and $p \geqslant 2$ an integer. Then there exists a k-graph H with the following properties:*

(1) *H is selective for G;*

(2) $\chi(H) = (\chi(G) - 1)(|V(G)| - 1) + 1$;

(3) *If the edges $e_1, e_2, \ldots, e_q$ form a cycle of length at most p in H then there exists a subgraph G' of H, G' isomorphic to G such that G' contains all edges $e_1, e_2, \ldots, e_q$.*

In order to get Theorem 1.4 let p be the maximal size of a graph belonging to $\mathfrak{A}$.

To prove Theorem 5.1 observe that because of condition (2) we cannot use standard amalgamation technique (see e.g. [11]). However we can modify the above proofs of Theorems 3.1 and 4.1. We stress the main differences only.

Sketch of the proof of 5.1. Let $G = (V, E)$ be a k-graph with n vertices; put $\chi(G) = k$. Let $V = V_1 \cup V_2 \cup \ldots \cup V_k$ be a fixed coloring of G. Assume without loss of generality that $|V_1| = |V_2| = \ldots = |V_k| = l$ (we can add some singletons if necessary) for some l. Put $a = (k - 1)(n - 1) + 1$. The resulting k-graph H will be built successively by a-partite k-graphs $P_0, P_1, \ldots, P_a = H$ using sparse selective hypergraphs (without cycles of length $\leqslant p$). This step is similar as in proofs of Theorems 3.1 and 4.1. The construction of a-partite k-graph P_0 is based on the following lemma (analogous to Lemma 4.2).

Lemma 5.2. *For every* p, r, l, $r \geqslant 2$ *there exists an* *r-partite* *rl-graph* $G_{r,l,p} = ((V_i)_1^r, E)$ *without cycles of length* $\leqslant p$ *with the following properties:*

(1) $|e \cap V_i| = l$ *for every* $e \in E$ $(1 \leqslant i \leqslant r)$;

(2) *for every coloring*

$$c: \bigcup_{i=1}^r V_i \to \bigcup_{i=1}^r V_i$$

which is $1-1$ *on each of the sets* V_i, $i \in \omega$ *for an* $\omega \subset \{1, 2, \ldots, k\}$ *there exists an edge* $e \in E$, $e = \{v_1, v_2, \ldots, v_{rl}\}$; $v_1, v_2, \ldots, v_l \in V_1$, $v_{l+1}, \ldots, v_{2l} \in V_2, \ldots, v_{(r-1)l+1}, \ldots, v_{rl} \in V_r$ *such that* $c(v_i) \neq c(v_j)$ *whenever one of the vertices belongs to* $\bigcup_{i \in \omega} V_i$.

Construction of rl-graph $G_{r,l,p}$ can be derived (similarly as in the proof of Lemma 4.2) from an rl-graph G' without cycles of length $\leqslant p$ and with chromatic number bigger than $2^{rl^{rl}}$.

II. Given a graph G denote by $\Delta(G)$ the minimal number of vertices of a graph H selective for G. Set

$$s(n) = \max \Delta(G)$$

where the maximum is taken over set of all graphs with n vertices. We can show

$$s(n) \leqslant (2 + \epsilon)^n$$

for any $\epsilon > 0$ and $n \geqslant n_0(\epsilon)$.

However we do not know any nontrivial lower bounds.

III. We have some further results concerning this topic. These concern critical selective k-graphs defined similarly as chromatic critical k-graphs. Particularly we can show that do decide whether a given 3-graph is selective is NP-complete. Surprisingly presently we do not see a simple proof of this.

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