Products of integers in short intervals

by

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1. Introduction. In this paper we discuss various properties distinct integers $n_1, \ldots, n_f$ taken from a short interval may have, such as

$\prod_{i=1}^{f} n_i \in N^m$ for some $m \in N$, $m \geq 2$: the product of $n_1, \ldots, n_f$ is a perfect power;

$\prod_{i \in I_1} n_i = \prod_{i \in I_2} n_i$ for distinct subsets I_1, I_2 of $\{1, \ldots, f\}$: there exist two distinct subsets of $\{n_1, \ldots, n_f\}$ that yield the same result if their elements are multiplied;

$\prod_{i \in I_1} n_i^{m_i} = \prod_{i \in I_2} n_i^{m_i}$ for distinct subsets I_1, I_2 of $\{1, \ldots, f\}$ for certain $m_i \in N$, $i \in I_1 \cup I_2$: there exist two distinct subsets of $\{n_1, \ldots, n_f\}$ that yield the same result if their elements are multiplied, when repetitions are allowed. Stated differently: $n_1, \ldots, n_f$ are multiplicatively dependent.

$\omega(\prod_{i=1}^{f} n_i) < f$: the total number of distinct prime divisors in the prime factorizations of the integers $n_1, \ldots, n_f$ is less than the number of integers.

By short intervals we mean intervals $[n, n+k(n)]$, where $k(n)$ is a 'small' function of n (such as $\sqrt{n}$, or $\log n$), for arbitrary $n \geq 1$.

Our results can be summarized as follows: the above properties never occur in 'very short' intervals, sometimes in 'short' intervals and always in 'large' intervals.

For example, distinct sets of integers from

$$[n, n+c_1(\log n)^2(\log\log n)^{-2}], \quad \text{for any } n \geq 3,$$

have distinct products, for infinitely many $n \in N$ this also holds for $[n, n+\exp(c_2(\log n \log\log n)^{1/2})]$, but for infinitely many $n \in N$ there exist two distinct sets of integers in $[n, n+\exp(c_3(\log n \log\log n)^{1/2})]$ with equal products and for all $n \in N$ the latter holds for $[n, n+c_4 n^{0.496}]$. The c_1, c_2, c_3, c_4 are absolute positive constants.

Acknowledgement. We gratefully acknowledge R. L. Graham (Bell Laboratories) and J. L. Selfridge (Mathematical Reviews) for some of the ideas in this paper.

2. Basic lemmas and notation.

Notation. For primes p and $n \in \mathbf{N}$ we define the non-negative integers $v_p(n)$ by $n = \prod_p p^{v_p(n)}$. For $n \in \mathbf{N}$ the number of distinct primes dividing n is $\omega(n)$ and the greatest prime dividing $n \geqslant 2$ is $P(n)$, while $P(1) := 1$. As usual, $\pi(x)$ is the number of primes not exceeding x, $\gcd(n_1, n_2)$ denotes the greatest common divisor of n_1 and n_2 and $\operatorname{lcm}(n_1, \ldots, n_f)$ is the least common multiple of $n_1, \ldots, n_f$. In proofs we sometimes use the familiar Landau symbols O and o, as well as $\ll$ (having the same meaning as O), for convenience. In the statements of our theorems we shall not use these symbols and we reserve the symbols $c, c_0, c_1, \ldots, k_0, k_1, \ldots, n_0, n_1, \ldots$ for certain absolute positive constants. If m divides n we write $m|n$. We denote the number of elements of a set S with $|S|$. We write $\mathbf{N}^m$ for the set $\{n^m\colon n \in \mathbf{N}\}$.

To prove our main results in Sections 3, 4, 5 and 6 we need upper and lower bounds for the number of integers in 'short' intervals which are composed of 'small' primes. The purpose of this section is to derive such bounds. To be more specific we need the following definition.

DEFINITION. For k, $n \in \mathbf{N}$ we define

$$f(n, k) = \sum_{\substack{n < v \leqslant n+k \\ P(v) \leqslant k}} 1.$$

We shall be interested in upper and lower bounds for $f(n, k)$ in terms of k, with k equal to various functions of n. Note that for $k \geqslant n$ we clearly have $f(n, k) = k - (\pi(n+k) - \pi(k))$, so

$$k - 2n/\log n \leqslant f(n, k) \leqslant k \quad \text{for} \quad k \geqslant n.$$

Our interests are in the cases where $k < n$.

LEMMA 2.1. *For $k = n^\alpha$, where $0 < \alpha < 1$,*

$$f(n, k) \leqslant \alpha k + 2k/\log k.$$

Proof. Let $\{n_1, \ldots, n_f\} = \{n < v \leqslant n+k\colon P(v) \leqslant k\}$. For every prime $p \leqslant k$ delete one integer from $n_1, \ldots, n_f$ with $v_p(-)$ maximal. The resulting product is at most

$$\prod_{p \leqslant k} p^{\sum_{j=1}^{\infty} [(k-1)/p^j]} \leqslant k! \leqslant k^k$$

and at least $n^{f - \pi(k)}$, so that $f \leqslant (k \log k)/\log n + \pi(k)$.

Note that Lemma 2.1 does not give an upper bound less than $2k/\log k$, even when k becomes very small in comparison to n. The next lemma gives a better upper bound for $f(n, k)$ for such 'small' k (i.e. $k \leqslant \exp(\varepsilon_0(\log n)^{1/2})$, where ε_0 is some positive absolute constant).

LEMMA 2.2. *For $k = \exp(\Delta^{-1}(\log n)^{1/2})$, where $\Delta \geqslant 3$,*

$$f(n, k) \leqslant \max\left\{1,\, c_0 \frac{k}{\log k} \frac{\log\log \Delta}{\log \Delta}\right\},$$

where c_0 is an absolute constant.

Proof. See [12], p. 37, 3.10.4. The proof involves a theorem on lower bounds for lineair forms in logarithms of rational numbers.

The next lemma shows that $\sum_{\substack{n < v \leqslant n+n^{\alpha} \\ P(v) \leqslant n^{\beta}}} 1 \leqslant (1-\gamma(\alpha, \beta)-\varepsilon)n^{\alpha}$ for sufficiently large n and $\beta \geqslant \alpha > 2/5$. For $\beta = \alpha$ actually Lemma 2.1 is somewhat stronger, but we shall use Lemma 2.3 only for $\beta > \alpha$.

LEMMA 2.3. *For $2/5 < \alpha \leqslant 1$ put*

$$\delta(\alpha) = \begin{cases} \frac{5}{3}\alpha - \frac{2}{3} & \text{for} \quad \frac{2}{5} < \alpha \leqslant \frac{1}{2}, \\ \alpha - \frac{1}{3} & \text{for} \quad \frac{1}{2} \leqslant \alpha \leqslant 1 \end{cases}$$

and for $\beta \geqslant \alpha$ put

$$\gamma(\alpha, \beta) = 1 - \alpha - (\beta - \alpha)(\beta + \alpha)/\delta(\alpha).$$

Then for any $\gamma < \gamma(\alpha, \beta)$ we have, with N_0 a constant depending only on α, β and γ,

$$(*) \qquad \sum_{\substack{N < n \leqslant N+N^{\alpha} \\ P(n) > N^{\beta}}} 1 \geqslant \gamma N^{\alpha} \quad \text{for} \quad N \geqslant N_0.$$

Proof. We follow the method of Ramachandra in [8]; we use the same notation as in [8].

We have

$$\begin{aligned} \sum_{\substack{x < m \leqslant x+x^{\alpha} \\ P(m) > x^{1-\beta}}} 1 &= \sum_{n \leqslant x^{\beta}} \left\{\pi\left(\frac{x+x^{\alpha}}{n}\right) - \pi\left(\frac{x}{n}\right)\right\} \\ &\geqslant \sum_{n \leqslant x^{1-\alpha}} \left\{\pi\left(\frac{x+x^{\alpha}}{n}\right) - \pi\left(\frac{x}{n}\right)\right\} \frac{\log(x/n)}{\log x} - \\ &\quad - \sum_{x^{\beta} < n \leqslant x^{1-\alpha}} \left\{\pi\left(\frac{x+x^{\alpha}}{n}\right) - \pi\left(\frac{x}{n}\right)\right\} \frac{\log(x/n)}{\log x} \\ &=: \Sigma_1 - \Sigma_2. \end{aligned}$$

By Lemma 1 in [8] we have, provided that $1/3 < \alpha \leqslant 1$,

$$\Sigma_1 = (1-\alpha)\, x^\alpha + O(x^\alpha/\log x).$$

To estimate Σ_2 we divide $[x^\beta, x^{1-\alpha}]$ into N segments $[x^{\beta_i}, x^{\beta_{i+1}}]$ where $\beta_0 = \beta$, $\beta_N = 1-\alpha$ (assuming $\beta \leqslant 1-\alpha$, otherwise $\Sigma_2 = 0$). By the method of Lemma 3 in [8] we have, for $z \geqslant 3$,

$$\sum_{x^{\beta_i} \leqslant n \leqslant x^{\beta_{i+1}}} \left\{\pi\left(\frac{x+x^\alpha}{n}\right) - \pi\left(\frac{x}{n}\right)\right\}$$
$$\leqslant \frac{2x^\alpha}{\log z} \log(x^{\beta_{i+1}-\beta_i}+2)\cdot\left(1+O\left(\frac{1}{x^{\beta_i}}+\frac{1}{\log z}\right)\right) + O(z \max_{d \leqslant z} |R_d|),$$

where the remainder terms R_d can be estimated by Lemma 2 in [8]. We obtain

$$|R_d| = O\left(x^{(1-\alpha)/2}\log x + x^{(1-\alpha)3/2}\left(\frac{x}{d}\right)^{-1/2} + \left(\frac{x}{d}\right)^{1/3}\right).$$

Choosing $z = x^\delta$ we get

$$\max_{d \leqslant z} \{z|R_d|\} = O(\max\{x^{(1-\alpha)/2+\delta}\log x,\ x^{3(1-\alpha)/2-1/2+3\delta/2},\ x^{1/3+\delta}\}).$$

This is $o(x^\alpha)$ if $\delta < \delta(\alpha)$, $2/5 < \alpha \leqslant 1$.

Since $\log(x/n) \leqslant (1-\beta_i)\log x$ for $x^{\beta_i} \leqslant n \leqslant x^{\beta_{i+1}}$ we obtain

$$\Sigma_2 \leqslant \sum_{i=0}^{N-1} \left\{(1-\beta_i)\frac{2x^\alpha}{\delta \log x}(\beta_{i+1}-\beta_i)\log x \cdot\left(1+O\left(\frac{1}{\log x}\right)\right) + o(x^\alpha)\right\}$$
$$= \bigl(1+o(1)\bigr)\frac{2}{\delta}\, x^\alpha \sum_{i=0}^{N-1} (1-\beta_i)(\beta_{i+1}-\beta_i).$$

Note that

$$\sum_{i=0}^{N-1} (1-\beta_i)(\beta_{i+1}-\beta_i) \to \tfrac{1}{2}(1-\beta-\alpha)(1-\beta+\alpha)$$

when $\max\limits_{0 \leqslant i \leqslant N-1} (\beta_{i+1}-\beta_i) \to 0$.

Combining the bounds for Σ_1 and Σ_2 we obtain that

$$\sum_{\substack{x < m \leqslant x+x^\alpha \\ P(m) > x^{1-\beta}}} 1 > \left(1-\alpha-\frac{(1-\beta-\alpha)(1-\beta+\alpha)}{\delta}-\varepsilon\right)x^\alpha$$

for any $\varepsilon > 0$ and any $0 < \delta < \delta(\alpha)$ for x sufficiently large. Changing $1-\beta$ into β and choosing $\gamma < \gamma(\alpha, \beta)$ now gives the assertion.

We use Lemma 2.3 to obtain a lower bound for $f(n, k)$ when $k = n^\alpha$, $\alpha \geqslant \alpha_0$, where α_0 is a certain constant less than $1/2$ ($\alpha_0 = 0.49509\ldots$). We use the specific dependence of $\gamma(\alpha, \beta)$ in Lemma 2.3 on α and β to obtain such a bound.

LEMMA 2.4. *For every $\alpha \geqslant \alpha_0 (= 0.49509\ldots)$ there exist a $c(\alpha) > 0$ and a $n_0(\alpha)$ such that*

$$f(n, k) > c(\alpha)k \quad \textit{for} \quad k = n^\alpha, \; n \geqslant n_0(\alpha).$$

For $\alpha > \frac{1}{2}$ this actually holds for any $c(\alpha) < 2-\alpha^{-1}$.

Proof. Let α, β, γ satisfy the conditions of Lemma 2.3, hence, the inequality $(*)$. Then, for $k = n^\alpha$,

$$\left(\frac{2en}{k}\right)^k \geqslant \frac{(n+1)\ldots(n+k)}{k!} \geqslant \prod_{\substack{p>k \\ p|(n+1)\ldots(n+k)}} p > (k^{\beta/\alpha})^{\gamma k} \prod_{i=1}^{s} p_i,$$

w'ere $k < p_1 < \ldots < p_s$ are the first s primes exceeding k and $s = \omega\big((n+ +1)\ldots(n+k)\big) - \pi(k) - \gamma k$.

It follows that

$$\omega\big((n+1)\ldots(n+k)\big) \leqslant \big(\alpha^{-1} - 1 - \gamma(\beta/\alpha - 1) + O\big((\log k)^{-1}\big)\big)k.$$

Since $f(n, k) \geqslant k - \omega\big((n+1)\ldots(n+k)\big) + \pi(k)$ we infer that

$$f(n, k) \geqslant \big(2 - \alpha^{-1} + \gamma(\beta/\alpha - 1) + o(1)\big)k.$$

Let α_0 be the constant defined by: $2/5 < \alpha_0 < 1/2$ and for $\alpha \geqslant \alpha_0$ there exists a $\beta > \alpha$ with $\gamma(\alpha, \beta) > (1-2\alpha)/(\beta-\alpha)$. (We have $\alpha_0 = 0.49509\ldots$).

Then for $\alpha \geqslant \alpha_0$ there exists a $\gamma < \gamma(\alpha, \beta)$ with $2-\alpha^{-1}+\gamma(\beta/\alpha-1) > 0$, which implies the first assertion of Lemma 2.4. The second assertion follows by taking in the above discussion the trivial values $\gamma = 0$, $\beta = \alpha$.

Remark 2.4. Plausibly, for every $\alpha > 0$ there exists a $c^*(\alpha) > 0$ such that $f(n, k) > c^*(\alpha)k$ for $k = n^\alpha$, $n \geqslant n_0(\alpha)$.

This certainly holds for infinitely many $n \in \mathbf{N}$, as can be seen as follows. We have

$$\sum_{\substack{n \leqslant x \\ P(n) < n^\alpha/2}} 1 \sim \varrho(\alpha^{-1})x \quad \text{for} \quad x \to \infty,$$

where $\varrho(\alpha^{-1}) > 0$ is the Dickman function. Let $c < \varrho(\alpha^{-1})$ and x large, then there exists an interval $[t, t+t^\alpha] \subset [1, x]$ with $t \in \mathbf{N}$ large with at least ct^α integers n with $P(n) < \frac{1}{2}n^\alpha$. As $\frac{1}{2}n^\alpha < \frac{1}{2}(t+t^\alpha)^\alpha < t^\alpha$ the assertion follows.

LEMMA 2.5. *Let $n \geqslant 3$ and $t \leqslant 0.9(\log n)/\log\log n$. Then the number $\Psi(n, n^{1/t})$ of positive integers $v \leqslant n$ with $P(v) \leqslant n^{1/t}$ equals $n/t^{t(1+o_t(1))}$.*

Proof. See [1], Corollary of Theorem 3.1.

LEMMA 2.6. *For every $c < 1/\sqrt{2}$ there exist infinitely many $n \in \mathbf{N}$ such that the interval $[n, n+k^*(n)]$, with $k^*(n) = \exp\big(c(\log n \log\log n)^{1/2}\big)$, contains only integers which are divisible by a prime $p > k^*(n)$ but not by p^2.*

Proof. The number of integers in $[1, n]$ which are divisible by a square x^2 with $x > n^{1/t}$ is at most $\sum_{x > n^{1/t}} [n/x^2] = n/(1+o(1))n^{1/t}$.

By Lemma 2.5, there exist at most $n/t^{t(1+o(1))}$ integers in $[1, n]$ which are not divisible by a prime exceeding $n^{1/t}$. Take t such that $(1+o(1))n^{1/t} = t^{t(1+o(1))}$, then

$$t = (1+o(1))(2 \log n/\log\log n)^{1/2}.$$

Call the above integers in $[1, n]$ bad. Since their number is at most $2(1+o(1))n/n^{1/t}$ there must exist at least $[\frac{1}{3}n^{1/t}]$ consecutive integers $m+1, \ldots, m+[\frac{1}{3}n^{1/t}]$ which are not bad, i.e. divisible by a prime $p > n^{1/t}$ but not by p^2. Provided that n is sufficiently large, we have $[\frac{1}{3}n^{1/t}] \geqslant k^*(m)$. In this manner we obtain infinitely many $m \in \boldsymbol{N}$ for which $[m, m+k^*(m)]$ has the desired property.

In the next lemma we use the notation $o(1)$ for several functions of n tending to zero as $n \to \infty$.

Lemma 2.7. *For every $\lambda \geqslant 1$ there exist infinitely many $n \in \boldsymbol{N}$ such that the interval $[n, n+k^*(n)]$, with*

$$k^*(n) = \exp\left\{\frac{1+\lambda}{\sqrt{2}}(1+o(1))(\log n \log\log n)^{1/2}\right\},$$

contains distinct integers $n_1, \ldots, n_f$ with

$$\omega(n_1 \cdot \ldots \cdot n_f) < f^{1/\lambda} \quad \textit{and} \quad f > k^*(n)^{\lambda(1+o(1))/(1+\lambda)}.$$

Proof. By Lemma 2.5 there exist $\Psi(m, m^{1/t}) = m/t^{(1+o(1))t}$ integers v in $[1, m]$ with $P(v) \leqslant m^{1/t}$. Suppose every interval $[\sigma k, (\sigma+1)k]$, $\sigma \in \boldsymbol{N}$, contained in $[m/2t^{(1+o(1))t}, m]$ contains at most $m^{\lambda/t}$ integers v with $P(v) \leqslant m^{1/t}$. Then

$$\Psi(m, m^{1/t}) \leqslant m/2t^{(1+o(1))t} + (m/k)m^{\lambda/t}.$$

Choosing

$$t = (1+o(1))(2 \log m/\log\log m)^{1/2}$$

and

$$k = 3m^{\lambda/t} \cdot t^{(1+o(1))t} = \exp\left\{\frac{\lambda+1}{\sqrt{2}}(1+o(1))(\log m \log\log m)^{1/2}\right\}$$

we obtain the contradiction $\Psi(m, m^{1/t}) < m/t^{(1+o(1))t}$. Hence there exists an interval $[n, n+k]$, with $n \geqslant m/2t^{(1+o(1))t}$, which contains distinct integers $n_1, \ldots, n_f$ with $P(n_i) \leqslant m^{1/t}$ and $f > m^{\lambda/t}$. We have

$$k = \exp\left\{\frac{1+\lambda}{\sqrt{2}}(1+o(1))(\log n \log\log n)^{1/2}\right\} = k^*(n)$$

and

$$\omega(n_1 \cdot \ldots \cdot n_f) \leqslant \pi(m^{1/t}) \leqslant m^{1/t} < f^{1/\lambda}$$

while

$$f > m^{\lambda/t} = k^*(n)^{\lambda(1+o(1))/(1+\lambda)}.$$

3. Integers composed of few primes.

DEFINITION 3.1. The positive integers $n_1, \ldots, n_f$ are said to be *composed of few primes* if $\omega(n_1 \cdot \ldots \cdot n_f) < f$.

DEFINITION 3.2. The positive integers $n_1, \ldots, n_f$ are said to be *composed of few integers* if there exists $p_1, \ldots, p_\omega \in \boldsymbol{N}$ with

$$n_i = \prod_{j=1}^{\omega} p_j^{v_{ij}}$$

fo certain $v_{ij} \in Z$ with $v_{ij} \geqslant 0$ $(1 \leqslant i \leqslant f,\ 1 \leqslant j \leqslant \omega)$, while $\omega < f$.

Note that the p_j in Definition 3.2 are not required to be prime, which makes the difference with Definition 3.1. We shall also consider, more generally, the properties $\omega(n_1 \cdot \ldots \cdot n_f) < F(f)$, resp. $\omega < F(f)$, where $F\colon \boldsymbol{N} \to \boldsymbol{N}$ is some given function with $F(f) \leqslant f$. This last restriction is a natural one since any f positive integers are composed of f integers, namely themselves (take $p_j = n_j$, $v_{ij} = \delta_{ij}$ in Definition 3.2). Being composed of few integers is really weaker than being composed of few primes: m^2, $m(m+1)$ and $(m+1)^2$ are composed of few integers but not of few primes (for most $m \in \boldsymbol{N}$). A still weaker property is being multiplicatively dependent (see § 6), which is equivalent to Definition 3.2 without the stipulations $v_{ij} \geqslant 0$. The property of being composed of few integers (primes) is a basic one in the context of this paper. From the existence of a set with $\omega(n_1 \cdot \ldots \cdot n_f) < F(f)$ we infer the existence of a subset with certain desired properties in several instances (5.1, 5.2, 6.1).

We also recall a relation between the property of being composed of few primes and another multiplicative property of consecutive integers (see [9]):

There exists no subset $\{n_1, \ldots, n_f\}$ of $\{n+1, n+2, \ldots, n+k\}$ with $\omega(n_1 \cdot \ldots \cdot n_f) < f \Leftrightarrow$ There exist distinct primes $p_1, \ldots, p_k$ with $p_i | n+i$ for $i = 1, \ldots, k$.

The following theorem shows that short intervals do not contain integers composed of few integers.

THEOREM 3.1. *Suppose $n_1, \ldots, n_f$ are distinct integers in $[n, n+k]$ composed of $p_1, \ldots, p_\omega \in \boldsymbol{N}$* (i.e. $n_i = \prod_{j=1}^{\omega} p_j^{v_{ij}}$ with $v_{ij} \geqslant 0$), *where $f, n, k \in \boldsymbol{N}$. Then* ($c_0, c_1, \varepsilon_0$ *are absolute positive constants*):

(1) *if $\omega < f$ then $k \geqslant n^{1/\omega} \geqslant n^{1/(f-1)}$,*

(2) *if $\omega < f - \sqrt{2f}$ then $k \geqslant n^{1/\sqrt{(2f)}}$,*

(3) *if* $\omega < \sqrt{f}$ *then* $k > c_0(\log n/\log\log n)^6$,

(4) *if* $\omega < f$ *then* $k > n^{\varepsilon_0/\sqrt{(2f)}}$,

(5) *if* $\omega < f$ *then* $k > c_1(\log n/\log\log n)^3$.

Proof. The first two results are special cases of

$$k \geqslant n^{1/\min_{1 \leqslant \lambda \leqslant f-\omega}(\omega/\lambda+(\lambda-1)/2)},$$

which follows from

$$\prod_{i=1}^{\lambda} n_i \leqslant \operatorname{lcm}(n_1, \ldots, n_\lambda) \prod_{1 \leqslant i < j \leqslant \lambda} \gcd(n_i, n_j).$$

See [12], p. 17.

The third result is elementary, too, but more involved. See [13] or [12], Theorem 2.8, p. 23. On the other hand, (4) and (5) are non-elementary (a lower bound for linear forms in logarithms of rational numbers is used). See [13] and [12], p. 35. Note that (1), (2) and (4) give a trivial conclusion if f is large in comparison to n, but that the lower bound for k in (5) is independent of f. This bound (5) was first proven in [9] in the case $\omega(n_1 \cdot \ldots \cdot n_f) < f$.

The next theorem is the main result of this section.

THEOREM 3.2. *For* $n \in \mathbf{N}$ *let* $k(n) := \min\{k \in \mathbf{N}$: $[n, n+k]$ *contains distinct integers composed of few primes*$\}$. *Let* $\varepsilon > 0$. *Then* (c_0, c_1 *are absolute positive constants*):

(1) $k(n) > c_0(\log n/\log\log n)^3$ *for all* $n \in \mathbf{N}$ *with* $n \geqslant 3$,

(2) $k(n) > \exp\left(\left(\frac{1}{\sqrt{2}}-\varepsilon\right)(\log n \log\log n)^{1/2}\right)$ *for infinitely many* $n \in \mathbf{N}$,

(3) $k(n) < \exp((\sqrt{2}+\varepsilon)(\log n \log\log n)^{1/2})$ *for infinitely many* $n \in \mathbf{N}$,

(4) $k(n) < c_1 n^{0.496}$ *for all* $n \in \mathbf{N}$.

Proof. See for (1), Theorem 3.1(5). From Lemma 2.6 we infer (2): the primes $p > k^*(n)$ must all be distinct. Lemma 2.7 immediately gives (3). From the proof of Lemma 2.4 we see that $\omega((n+1)\ldots(n+k)) < k$ if $k \geqslant n^{\alpha_0}$, $n \geqslant n_0$, which implies (4).

When the number of elements f of a set $\{n_1, \ldots, n_f\} \subset [n, n+k]$ with $\omega(n_1 \cdot \ldots \cdot n_f) < f$ is restricted, then better lower bounds for the length k of the interval than $k \gg (\log n/\log\log n)^3$ can be obtained. When f is small in comparison to the size n of the integers involved then 3.1 (1) and 3.1 (4) are superior to 3.1 (5). When $f \geqslant f_0 = 2/\varepsilon_0^2$ then 3.1 (5) is better than 3.1 (1). If $f \leqslant k^{2/3}$ then 3.1 (4) gives a better bound for k than 3.1 (5), e.g. when $f = k^\alpha$, $0 < \alpha \leqslant 2/3$, then $k \gg (\log n/\log\log n)^{2/\alpha}$. In the extreme case when $f = k+1$ (i.e. $n_1, \ldots, n_f$ are the consecutive integers n, $n+1, \ldots, n+k$) we have $k > \exp(c(\log n)^{1/2})$. Actually we have the following results about this important special case of consecutive integers.

THEOREM 3.3. *There exist absolute positive constants c_1, c_2, c_3, c_4 such that*

(1) $\omega((n+1)\dots(n+k)) < k$ *for all* $(n, k) \in \boldsymbol{N} \times \boldsymbol{N}$ *with* $k \geqslant c_1 n^{0.496}$,

(2) $\omega((n+1)\dots(n+k)) \geqslant k$ *for all* $(n, k) \in \boldsymbol{N} \times \boldsymbol{N}$ *with* $k < \exp(c_2 (\log n)^{1/2})$,

(3) $\omega((n+1)\dots(n+k)) \geqslant k$ *for infinitely many* $(n, k) \in \boldsymbol{N} \times \boldsymbol{N}$ *with* $k \geqslant c_3 n^{1/e}$,

(4) $\omega((n+1)\dots(n+k)) < k$ *for infinitely many* $(n, k) \in \boldsymbol{N} \times \boldsymbol{N}$ *with* $k < c_4 n^{1/e}$.

Proof. For (1) we refer to the proof of Theorem 3.2 (4). To prove (2), note that, since every prime exceeding k divides at most one integer in $[n, n+k]$, we have $\omega((n+1)\dots(n+k)) \geqslant k - f(n, k) + \pi(k)$. So it is sufficient to show that $f(n, k) < \pi(k)$ for $k < \exp(c_2 (\log n)^{1/2})$. This follows from Lemma 2.2 if c_2 is sufficiently small. In [3] an averaging argument is given that proves (3). Actually this argument can be used to prove both (3) and (4), as we show now. For $n, k \in \boldsymbol{N}$ with $n > k > 1$ we put $t := [n/k]$ and we denote by $\omega_k(m)$ the number of distinct primes exceeding k that divide $m \in \boldsymbol{N}$. Since every prime $> k$ divides at most one integer among k consecutive integers we have

$$(*) \qquad \sum_{i=0}^{t-1} \omega_k\Big(\prod_{j=1}^{k} (n+ik+j)\Big) = \sum_{p>k} \Big(\sum_{\substack{n<v\leqslant n+tk \\ p\mid v}} 1\Big).$$

The right side of $(*)$ equals

$$\sum_{k<p\leqslant n+tk} \frac{tk}{p} + O\Big(\sum_{k<p\leqslant n+tk} 1\Big) = tk\left(\log\left(\frac{\log n}{\log k}\right) + O\left(\frac{1}{\log k}\right)\right).$$

Put

$$\min_{0\leqslant i\leqslant t-1} \omega_k\Big(\prod_{j=1}^{k} (n+ik+j)\Big) =: m \quad \text{and} \quad \max_{0\leqslant i\leqslant t-1} \omega_k\Big(\prod_{j=1}^{k} (n+ik+j)\Big) =: M.$$

Since the left side of $(*)$ is at least mt and at most Mt it follows that

$$m \leqslant k\left(\log\left(\frac{\log n}{\log k}\right) + \frac{C_1}{\log k}\right) \quad \text{and} \quad M \geqslant k\left(\log\left(\frac{\log n}{\log k}\right) - \frac{C_2}{\log k}\right),$$

where C_1 and C_2 are certain absolute positive constants. Take $0 < c < \exp(-C_2)$. Then for all sufficiently large $n \in \boldsymbol{N}$ and $k := [cn^{1/e}]$ there exists an $0 \leqslant i \leqslant t-1$ with

$$M = \omega_k\Big(\prod_{j=1}^{k} (n+ik+j)\Big) > k.$$

This implies (3), if $c_3 < c \cdot 2^{-1/e}$. Now take $c_4 > \exp(C_1+2)$. Then for all sufficiently large $n \in \boldsymbol{N}$ and $k := [c_4 n^{1/e}]$ there exists an $0 \leqslant i \leqslant t-1$ with

$$m = \omega_k\Big(\prod_{j=1}^{k} (n+ik+j)\Big) < k - 2k/\log k.$$

Since $\pi(k) \leqslant 2k/\log k$ this implies (4).

Finally we remark that for every $k \in \boldsymbol{N}$ we have

$$\omega\big((n+1)\ldots(n+k)\big) \geqslant k+\pi(k)-1$$

for all sufficiently large n, e.g. $n \geqslant \exp\exp(Ck)$, where C is an absolute constant. See [12], p. 38. On the other hand, for every $k \in \boldsymbol{N}$ there exist, though only conjecturally for $k \geqslant 2$, infinitely many $n \in \boldsymbol{N}$ with $\omega\big((n+1)\ldots$ $\ldots(n+k)\big) = k+\pi(k)$. See [5].

4. Multiplicative dependence.

Definition 4.1. The positive integers $n_1, \ldots, n_f$ are *multiplicatively dependent* if there exist $m_1, \ldots, m_f \in Z$, not all zero, with $\prod_{i=1}^{f} n_i^{m_i} = 1$.

Equivalently, $n_1, \ldots, n_f$ are multiplicatively dependent if they can be divided into two sets having equal products, where repetitions are allowed. Also, $n_1, \ldots, n_f$ are multiplicatively dependent iff there exist $p_1, \ldots, p_\omega \in \boldsymbol{N}$ with $\omega < f$ such that

$$n_i = \prod_{j=1}^{\omega} p_j^{v_{ij}} \quad \text{with} \quad v_{ij} \in Z \quad (1 \leqslant i \leqslant f,\ 1 \leqslant j \leqslant \omega).$$

Note that being composed of few integers (Section 3) implies being multiplicatively dependent.

Lemma 4.1. *Suppose $n_1, \ldots, n_f$ are distinct ($f \geqslant 2$) integers in $[n, n+k]$ which are multiplicatively dependent. Then $k \geqslant n^{1/(f-1)}$.*

Proof. We have $\prod_{i \in I} n_i^{m_i} = \prod_{j \in J} n_j^{m_j}$ with $m_t \in \boldsymbol{N}$ for $t \in (I \cup J) \subset \{1, \ldots, f\}$. We may assume that $I \cap J = \emptyset$. Let $\max\{m_t\colon t \in I \cup J\} = m_{t_0}$. By symmetry we may assume that $t_0 \in I$. Then $n_{t_0}^{m_{t_0}}$ divides $\prod_{j \in J} n_j^{m_j}$, hence

$$n_{t_0}^{m_{t_0}} = \gcd\big(n_{t_0}^{m_{t_0}}, \prod_{j \in J} n_j^{m_{t_0}}\big) \Big| \prod_{j \in J} \gcd(n_{t_0}, n_j)^{m_{t_0}}.$$

Since $\gcd(n_{t_0}, n_j)$ divides $|n_{t_0} - n_j| \in \{1, \ldots, k\}$ we conclude that

$$n^{m_{t_0}} \leqslant k^{|J| m_{t_0}} \leqslant k^{(f-1)m_{t_0}}.$$

Theorem 4.1. *For $n \in \boldsymbol{N}$ let $k(n) := \min\{k \in \boldsymbol{N}\colon [n, n+k]$ contains distinct integers which are multiplicatively dependent$\}$. Let $\varepsilon > 0$ be arbitrary and let c_0, c_1 be certain absolute positive constants. Then*

(1) $k(n) > c_0 \log n \log\log n(\log\log\log n)^{-1}$ *for all* $n \in \boldsymbol{N}$ *with* $n \geqslant 15$.

(2) $k(n) > \exp\left(\left(\frac{1}{\sqrt{2}}-\varepsilon\right)(\log n \log\log n)^{1/2}\right)$ *for infinitely many* $n \in \boldsymbol{N}$.

(3) $k(n) < \exp((\sqrt{2}+\varepsilon)(\log n \log\log n)^{1/2})$ *for infinitely many* $n \in \boldsymbol{N}$.

(4) $k(n) < c_1 n^{0.496}$ *for all* $n \in \boldsymbol{N}$.

Proof. Suppose $[n, n+k]$ contains distinct integers $n_1, \ldots, n_f$ which are multiplicatively dependent: $\prod_{i=1}^{f} n_i^{m_i} = 1$ for certain $m_i \in Z$ with $m_i \neq 0$ (without loss of generality). Then $P(n_i) \leqslant k$ for $i = 1, \ldots, f$, hence $f \leqslant f(n, k)$. To prove that $k \gg \log n \log\log n(\log\log\log n)^{-1}$ we may assume that $k \leqslant (\log n)^2$ and then we have, by Lemma 2.2, that $f(n, k) \ll k(\log 3k)^{-2} \log\log(3k)$. Combining this with $f \log k \geqslant \log n$ (Lemma 4.1) we obtain (1).

To prove (2) we invoke Lemma 2.6: these intervals $[n, n+k^*(n)]$ do not contain integers n_i with $P(n_i) \leqslant k^*(n)$. The third result (3) follows from Lemma 2.7: $\omega(n_1 \cdot \ldots \cdot n_f) < f$ implies that $n_1, \ldots, n_f$ are multiplicatively dependent.

Similarly, (4) follows from Theorem 3.3 (1).

5. Equal products. In this section we investigate intervals which contain distinct subsets of integers S_1 and S_2 with equal products: $\prod_{s \in S_1} s = \prod_{s \in S_2} s$.

Note that this property is stronger than multiplicative dependence: the latter guarantees the existence of distinct subsets S_1 and S_2 with $\prod_{s \in S_1} s^{m(s)} = \prod_{s \in S_2} s^{m(s)}$ for certain $m(s) \in \boldsymbol{N}$, $s \in S_1 \cup S_2$. Observe that integers in $S_1 \cap S_2$ can be deleted from both S_1 and S_2 without destroying the equality of the products, so we may always assume that S_1 and S_2 are disjoint.

Lemma 5.1. *Suppose* $n_1, \ldots, n_f$ *are distinct* ($f \geqslant 2$) *positive integers with* $\omega(n_1 \cdot \ldots \cdot n_f) < f \log 2/(\log(fv))$, *where* $v = \max_{\substack{1 \leqslant i \leqslant f \\ p\,\mathrm{prime}}} \{1 + v_p(n_i)\}$. *Then there exist distinct disjoint subsets* S_1 *and* S_2 *of* $\{n_1, \ldots, n_f\}$ *with equal products.*

Proof. For every subset $S \subset \{n_1, \ldots, n_f\}$ put

$$p(S) = \prod_{s \in S} s = \prod_{p} p^{v_p(S)}.$$

Then

$$v_p(S) = \sum_{s \in S} v_p(s) \leqslant (v-1)|S| \leqslant (v-1) f,$$

so the number of distinct integers $p(S)$, $S \subset \{n_1, \ldots, n_f\}$, is at most $(1+(v-1) f)^{\omega} \leqslant (vf)^{\omega} < 2^f$. The number of distinct S equals 2^f, hence the conclusion (elements in $S_1 \cap S_2$ can be deleted from both S_1 and S_2).

COROLLARY 5.1. *In the above situation, let $f_1 \in \mathbf{N}$ be minimal with $2^{f_1} > (vf_1)^{\omega}$. Then there exist disjoint subsets T_1 and T_2 of $\{n_1, \ldots, n_f\}$ with equal products and $|T_1 \cup T_2| > f - f_1$.*

Proof. Choose any subset F_1 of $\{1, \ldots, f\}$ with $|F_1| = f_1$ (if this is impossible take $T_1 = T_2 = \emptyset$). This gives disjoint S_1 and S_2 in F_1 with $\prod_{i \in S_1} n_i = \prod_{i \in S_2} n_i$. Remove n_i, $i \in S_1 \cup S_2$, from $\{n_1, \ldots, n_f\}$ and start again. This gives sets $S_1, S_2, S_3, S_4, \ldots$, disjoint from each other, with $\prod_{i \in S_{2t-1}} n_i = \prod_{i \in S_{2t}} n_i$ $(t = 1, 2, \ldots)$. The process stops when there are less than f_1 elements left. Take $T_1 = \bigcup_{i \text{ odd}} S_i$ and $T_2 = \bigcup_{i \text{ even}} S_i$.

In the case when $\{n_1, \ldots, n_f\}$ is the set $\{n < v \leqslant n+k\colon P(v) \leqslant k\}$ we can relax the condition in Lemma 5.1 to get equal products:

LEMMA 5.2. *Let $n, k \in \mathbf{N}$ with $k \geqslant k_0$ and suppose*

$$f(n, k) > 2 \frac{k}{\log k} \log\log\log k.$$

Then there exist two disjoint subsets of $\{n+1, \ldots, n+k\}$ with equal products (and at least $f(n, k) - 2k \log\log\log k/\log k$ elements).

Proof. Let $\{n_1, \ldots, n_f\} \subset \{n < v \leqslant n+k\colon P(v) \leqslant k\}$ with

$$\text{(A1)} \qquad f \geqslant 2k \frac{\log \infty(k)}{\log k},$$

where $\infty(k)$ shall be chosen later. Delete all n_i with $P(n_i) > k/\infty(k)$. The number of deletions is at most

$$\sum_{k/\infty(k) < p \leqslant k} (1 + [k/p]) = \big(1 + o(1)\big) k \frac{\log \infty(k)}{\log k}.$$

Hence $S_0 = \{n_i\colon P(n_i) \leqslant k/\infty(k)\}$ has more than $f/3$ elements. For all $S \subset S_0$ we define

$$p(S) = \prod_{s \in S} s = \prod_{p \in P_1} p^{v_p(S)} \prod_{p \in P_2} p^{v_p(S)} =: p_1(S) \cdot p_2(S),$$

where $P_1 = \{p \leqslant k/\log k\}$ and $P_2 = \{k/\log k < p \leqslant k/\infty(k)\}$. We have

$$v_p(S) = \sum_{s \in S} v_p(s) \leqslant \max_s \{v_p(s)\} \sum_{\substack{s \in S \\ p \mid s}} 1 \leqslant (\log k)^c \sum_{\substack{s \in S \\ p \mid s}} 1,$$

since $v_p(s) \leqslant \dfrac{\log(n+k)}{\log 2}$ and $k > \exp\big((\log n)^{1/2}\big)$ (this follows from our assumption on $f(n, k)$ and Lemma 2.1).

For $p \in P_1$ the trivial bound $\sum_{\substack{s \in S \\ p|s}} 1 \leqslant k$ gives $v_p(S) \leqslant k(\log k)^{O(1)}$. For $p \in P_2$ we have $\sum_{\substack{s \in S \\ p|s}} 1 \leqslant 1 + [k/p] \leqslant 1 + \log k$, hence $v_p(S) \leqslant (\log k)^{O(1)}$.

The number of distinct integers $p(S) = p_1(S)\, p_2(S)$ is therefore at most

$$\{k(\log k)^{O(1)}\}^{|P_1|}\, \{(\log k)^{O(1)}\}^{|P_2|} = \exp\left(\frac{k}{\log k}\left(\frac{\log\log k}{\infty(k)} + O(1)\right)\right).$$

Since the number of distinct $S \subset S_0$ equals $2^{|S_0|} > 2^{f/3}$ we can infer the existence of two distinct S_1 and S_2 in S_0 with $p(S_1) = p(S_2)$ if

$$\text{(A2)} \qquad f \geqslant \frac{3}{\log 2}\, \frac{k}{\log k}\left(\frac{\log\log k}{\infty(k)} + O(1)\right).$$

Now choose $\infty(k) = 3(\log\log k)(\log\log\log k)^{-1}$, then (A1) and (A2) are satisfied if $f \geqslant 2\dfrac{k}{\log k}\log\log\log k$.

As in the proof of Corollary 5.1 it follows that there exist two disjoint subsets of $\{n < v \leqslant n+k\colon P(v) \leqslant k\}$ with equal products and at least $f(n, k) - 2\dfrac{k}{\log k}\log\log\log k$ elements.

Lemma 5.3. *Suppose $[n, n+k]$ contains f distinct integers which can be divided into two distinct sets having equal products, where $n, k, f \in \mathbf{N}$ with $n \geqslant 2$. Then*

$$\frac{2\log n}{\log k} \leqslant f \leqslant 2\frac{k\log k}{\log n}.$$

Proof. Let $\prod_{i \in I} n_i = \prod_{j \in J} n_j$, where $\{1, \ldots, f\} = I \cup J$ with I, J disjoint (without loss of generality). Then for $i \in I$, $n_i = \gcd(n_i, \prod_{j \in J} n_j)$ divides $\prod_{j \in J}\gcd(n_i, n_j)$, hence $n \leqslant k^{|J|}$. Similarly, $n \leqslant k^{|I|}$. Since one of $|I|$ or $|J|$ does not exceed $[f/2]$ we obtain the first inequality. For any set $\{n_i\}$ of integers in $[n, n+k]$ we write, for every prime p, $\max_i v_p(n_i) = v_p = v_p(n_{i(p)})$. Then we have

$$\sum_{i \neq i(p)} v_p(n_i) = \sum_{j=1}^{v_p} |\{n_i\colon i \neq i(p),\ p^j \text{ divides } n_i\}|$$
$$\leqslant \sum_{j=1}^{v_p} [k/p^j] \leqslant v_p(k!).$$

Now if $\prod_{i\in I} n_i = \prod_{j\in J} n_j$, where $I \cap J = \emptyset$, then we have, for every p with $i(p)\in I$, that p^{v_p} divides $\prod_{i\in J} p^{v_p(n_i)}$. Hence

$$n^{|I|} \leqslant \prod_{i\in I} n_i = \prod_{p} \{p^{v_p} \prod_{\substack{i\neq i(p)\\ i\in I}} p^{v_p(n_i)}\}$$

$$\leqslant \prod_{p} \{ \prod_{\substack{i\neq i(p)\\ i\in I\cup J}} p^{v_p(n_i)}\} \leqslant \prod_{p} p^{v_p(k!)} = k!.$$

Similarly $n^{|J|} \leqslant k!$ ($\leqslant k^k$). Since one of $|I|$ or $|J|$ is at least $f/2$ we obtain the second inequality.

THEOREM 5.1. *For $n\in \boldsymbol{N}$ let $k(n) := \min\{k\in \boldsymbol{N}\colon [n, n+k]$ contains two distinct subsets of integers with equal products$\}$. Then, for arbitrary $\varepsilon > 0$ and a certain absolute constant c,*

(1) $k(n) > \frac{1}{4}\left(\frac{\log n}{\log\log n}\right)^2$ *for all $n\in\boldsymbol{N}$ with $n \geqslant 4$,*

(2) $k(n) > \exp\left(\left(\frac{1}{\sqrt{2}}-\varepsilon\right)(\log n \log\log n)^{1/2}\right)$ *for infinitely many $n\in\boldsymbol{N}$,*

(3) $k(n) < \exp((\sqrt{2}+\varepsilon)(\log n \log\log n)^{1/2})$ *for infinitely many $n\in\boldsymbol{N}$,*

(4) $k(n) < cn^{0.496}$ *for all $n\in\boldsymbol{N}$.*

Proof. From Lemma 5.3 it follows that if $[n, n+k]$ has two distinct subsets of integers with equal products then $k \geqslant ((\log n)/\log k)^2$ which implies (1). Since $\prod_{i\in I} n_i = \prod_{j\in J} n_j$ with $I\cap J = \emptyset$, and all $n_t\in[n, n+k]$, implies that $P(n_t) \leqslant k$ for all t, Lemma 2.6 immediately gives (2). To prove (3), choose $1 < \lambda < 1+\varepsilon\sqrt{2}$, then, by Lemma 2.7, for all n in an infinite subset N of $\boldsymbol{N}$ there exist distinct integers $n_1, \ldots, n_f$ in $[n, n+\exp((\sqrt{2}+\varepsilon)(\log n \log\log n)^{1/2})]$ with $f > k^*(n)^{(\lambda+o(1))/(1+\lambda)}$ and $\omega(n_1\cdot\ldots\cdot n_f) < f^{1/\lambda}$. Now we can use Lemma 5.1: we have $v \leqslant (\log 2n)/\log 2+1$ hence $\omega(n_1\cdot\ldots\cdot n_f) < f^{1/\lambda} < (f\log 2)/\log(fv)$ for all $n\in\boldsymbol{N}$ with at most finitely many exceptions.

To prove (4) we use Lemma 5.2 and Lemma 2.4: if $k \geqslant n^{0.496}$ and $n \geqslant n_1$ then the assumptions of Lemma 5.2 are satisfied hence $k(n) \leqslant n^{0.496}$. To include $n < n_1$ we simply take c sufficiently small.

In view of Remark 2.4 it is plausible that $k(n) = O_\varepsilon(n^\varepsilon)$ for all $\varepsilon > 0$.

Note that the lower bound $k \gg (\log n/\log\log n)^2$ for the length of an interval $[n, n+k]$ containing f ($\geqslant 1$) distinct integers which can be divided into two disjoint sets with equal products, can be improved if the number f of integers involved differs appreciably from $k^{1/2}$ (use Lemma 5.3): e.g., if f is bounded then $k \geqslant n^{2/f}$; if $f \leqslant k^{\alpha}$, $0 < \alpha \leqslant 1/2$ then $k \gg (\log n/\log\log n)^{1/\alpha}$; if $f \geqslant \varepsilon k$, $0 < \varepsilon \leqslant 1$, then $k \geqslant n^{\varepsilon/2}$.

We also observe that for $\alpha \geqslant \alpha_0$ there exists a $c_\alpha > 0$ such that there exist equal disjoint products in $[n, n+k]$, $k = n^{\alpha}$, with at least $c_\alpha k$ terms (and this is probably true for $\alpha > 0$). This follows from Lemma 5.2 and Lemma 2.4. On the other hand, for $\alpha < 1$ there exists a $c'_\alpha < 1$ such that there do not exist equal disjoint products in $[n, n+k]$, $k = n^{\alpha}$, with $c'_\alpha k$ or more terms. This follows from Lemma 2.1 (with $c'_\alpha = \alpha + o(1)$).

6. Power products. In this section we investigate sets of distinct integers $n_1, \ldots, n_f$ with the property that there exists a non-trivial way to multiply them that yields a perfect power: $\prod_{i=1}^{f} n_i^{m_i} \in \boldsymbol{N}^m$ for certain $m, m_1, \ldots, m_f \in \boldsymbol{N}$ with $m \geqslant 2$ and $m \nmid m_i$ for $i = 1, \ldots, f$. A variant results when one does not allow for repetitions ($m_i = 1$ for $i = 1, \ldots, f$): distinct integers the product of which is a perfect power. Before turning to results on power products in short intervals we give some results related to the well known Erdös–Selfridge theorem ([4]) which states that the product of two or more consecutive positive integers is never a perfect power.

What happens if one deletes one (or more) integers from a product of consecutive integers? It is trivial to show that if one deletes one integer from a product of three consecutive positive integers then the resulting product is never a perfect square (it can be a perfect power but it can be proven that the only instance is 2.4). Deleting one out of four does not give a square either (as we hope to prove soon). However, deleting one out of nine (or ten) positive consecutive integers does produce a square sometimes: $(1\cdot)\,2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 8\cdot 9\cdot 10$ is a square. We shall prove (see Corollary 6.1) that there exists a constant $k(1)$ such that if one deletes 1 integer from a product of $k(1)$ or more consecutive positive integers then the resulting product is never a perfect power.

Another natural question is: do there exist (infinitely many) products of consecutive positive integers which are twice a perfect power? Since $x^2 - 2y^2 = 1$ has infinitely many solutions $x, y \in \boldsymbol{N}$ there exist infinitely many $n \in \boldsymbol{N}$ with $n(n+1) \in 2\boldsymbol{N}^2$. Theorem 6.1 implies that, apart from these infinitely many products $n(n+1) \in 2\boldsymbol{N}^2$, there exist at most finitely many other products $n(n+1)\ldots(n+k)$ with $n, k \in \boldsymbol{N}$ which are twice a perfect power.

THEOREM 6.1. *Let $0 \leqslant \delta < 1/2$ and $a \in \boldsymbol{N}^2$. Let $n_1, \ldots, n_f$ be two or more integers obtained by deleting at most $\delta k/\log k$ integers from k consecutive positive integers, where $k \in \boldsymbol{N}$ ($k \geqslant 2$) is arbitrary. Then*

$$\prod_{i=1}^{f} n_i^{m_i} \notin a\boldsymbol{N}^m \quad \text{for any } m, m_1, \ldots, m_f \in \boldsymbol{N}$$

with $m \geqslant 2$ *and* $\gcd(m_i, m) = 1$ *for* $i = 1, \ldots, f$, *except for at most finitely many such sets* $\{n_1, \ldots, n_f\}$. *If* $a \in \mathbf{N}$, $a \notin \mathbf{N}^2$ *then the same is true but then there are also the infinitely many exceptions* n_1, n_2 *with* $n_1 n_2 \in a\mathbf{N}^2$, $1 \leqslant |n_2 - n_1| \leqslant 2$.

Proof. Suppose the conditions of Theorem 6.1 are satisfied and, moreover, $k \geqslant \max\{2P(a), k_0(\delta)\}$, where $k_0(\delta)$ is some (large) constant depending only on δ. We shall prove that $\prod_{i=1}^{f} n_i^{m_i} \in a\mathbf{N}^m$ gives a contradiction. The cases with $k < \max\{2P(a), k_0(\delta)\}$ shall be treated at the end of the proof. Let $n_1, \ldots, n_f$ be contained in $(n, n+k]$, where $n \in \mathbf{N} \cup \{0\}$.

Suppose $k \geqslant n$. Then there exist more than $\delta k/\log k$ primes p in $((n+k)/2, n+k] \subset (n, n+k]$, hence $n_i = p$ for some i. Since $2p > n+k$ we have $p \nmid n_j$ for $j \neq i$ and since $p > k/2 \geqslant P(a)$ we obtain a contradiction from $\prod_{i=1}^{f} n_i^{m_i} \in a\mathbf{N}^m$. So $k < n$.

Suppose $n^{2/3} \leqslant k$ $(< n)$. By the well known theorem of Ingham, the number of primes p in $(n, n+k]$ is asymptotically $k/\log n$, hence exceeding $\delta k/\log k$. So $n_i = p$ for some i and since $p > n > k$ we have $p \nmid n_j$ for $j \neq i$ and we obtain a contradiction from $\prod_{i=1}^{f} n_i^{m_i} \in a\mathbf{N}^m$ as above. So $k < n^{2/3}$.

For $k_0 \leqslant k < n^{2/3}$, where k_0 is an absolute constant, the number of integers v in $(n, n+k]$ with $P(v) > k$ exceeds $\frac{1}{6}k$ $(> \delta k/\log k)$ by Lemma 2.1. Hence $P(n_i) = p > k$ $(> P(a))$ for some i. Since $p > k$ we have $p \nmid n_j$ for $j \neq i$ and we deduce from $\prod_{j=1}^{f} n_j^{m_j} \in a\mathbf{N}^m$ and $\gcd(m_i, m) = 1$ that $p^m | n_i$. This implies $(k+1)^m \leqslant p^m \leqslant n_i \leqslant n+k$, hence $k < n^{1/m}$.

Put $n_i = a_i x_i^m$, with $a_i \in \mathbf{N}$ m-free (i.e. $v_p(a_i) < m$ for all p), for $i = 1, \ldots, f$. We distinguish two cases now.

Case 1: $m \geqslant 3$. We refer to the paper of Erdös and Selfridge [4]; it is easy to see that, since $k < n^{1/m}$ and $m \geqslant 3$, all products $a_i a_j$ $(1 \leqslant i, j \leqslant f)$ are distinct. This implies ([4]) that $\sum_{a_i \leqslant x} 1 \leqslant x(\log x)^{-1}\left(1 + O\left((\log x)^{-1}\right)\right)$. Assuming without loss of generality that $a_1 < \ldots < a_f$ we infer that $a_t > t \log t + t \log\log t + O(t)$, in particular, $a_t \geqslant t \log t$ for $t \geqslant t_0$ (an absolute constant). So, for $T \geqslant 2$,

$$(*) \quad \prod_{t=1}^{T} a_t \geqslant \exp\left(\sum_{t=2}^{T} \log(t \log t) + O(1)\right) = \exp\left(T \log T + T \log\log T + O(T)\right).$$

Choose for every prime p dividing the product $a_1 \cdot \ldots \cdot a_f$ an integer $n(p) \in \{n_1, \ldots, n_f\}$ with $\max_{1 \leqslant i \leqslant f} v_p(n_i) = v_p(n(p))$.

Then

$$\prod_{\substack{i=1\\ n_i\neq n(p)\forall p}}^{f} a_i=\prod_p p^{\sum_{j\geqslant 1}|\{1\leqslant i\leqslant f:n_i\neq n(p),\, p^j \text{ divides } a_i\}|}$$

$$\leqslant \prod_p p^{\sum_{j=1}^{m-1}|\{1\leqslant i\leqslant f:n_i\neq n(p),\, p^j \text{ divides } n_i\}|}\leqslant \prod_p p^{\sum_{j=1}^{m-1}[k/p^j]}\leqslant k!.$$

Note that every prime p dividing $\prod_{i=1}^{f} a_i$ does not exceed k: if $p|a$ then $p\leqslant P(a)<k$ and if $p\nmid a$, $p|a_i$ then, since $\prod_{i=1}^{f} a_i^{m_i}\in a\mathbf{N}^m$, we have $p|a_j$ for some $j\neq i$, hence $p|\gcd(a_i, a_j)|\gcd(n_i, n_j)|\,|n_i-n_j|\in\{1, \ldots, k\}$. So there are at most $\pi(k)$ primes dividing $\prod_{i=1}^{f} a_i$.

Put $f^*=f-\pi(k)$ ($\geqslant 2$). We have

$$\prod_{t=1}^{f^*} a_t\leqslant \prod_{\substack{i=1\\ n_i\neq n(p)\forall p}}^{f} a_i\leqslant k!\leqslant k^k.$$

Combining this with $(*)$ (with $T=f^*$) gives

$$f^*\leqslant k\left(1-\frac{\log\log k}{\log k}+O(1/\log k)\right).$$

This contradicts $f\geqslant k-\delta k/\log k$, since $k\geqslant k_0(\delta)$.

Case 2: $m=2$. As we saw above, $\prod_{i=1}^{f} a_i$ divides $(\prod_{p\leqslant k} p)k!$ Hence it divides, in fact,

$$(\prod_{p\leqslant k} p)k!\prod_{p\leqslant P} p^{v_p(\prod a_i)-v_p(k!)-1}\qquad \text{for any } 2\leqslant P\leqslant k.$$

Now

$$\sum_{i=1}^{f} v_p(a_i)=\sum_{\substack{i=1\\ p|a_i}}^{f} 1=\sum_{\substack{i=1\\ v_p(n_i)\,\text{odd}}}^{f} 1\leqslant \sum_{\substack{n<v\leqslant n+k\\ v_p(v)\,\text{odd}}} 1=k/(p+1)+O\left((\log k)/\log p\right)$$

for all $p\leqslant k$.

Also,

$$v_p(k!)=k/(p-1)+O\left((\log k)/\log p\right)\qquad \text{for all } p\leqslant k.$$

Hence

$$\prod_{p \leqslant P} p^{v_p(\prod a_i) - v_p(k!) - 1} \leqslant \exp\left(-k \sum_{p \leqslant P} \frac{2 \log p}{p^2 - 1} + O(\pi(P) \log k)\right)$$
$$= \exp\left(-\sigma k + O(k/P) + O((P \log k)/\log P)\right),$$

where $\sigma = \sum_{p \text{ prime}} \frac{2 \log p}{p^2 - 1}$. Since $k! \prod_{p \leqslant k} p = \exp(k \log k + O(k/\log k))$ we conclude, choosing $P = k/\log k$, that

$$\prod_{i=1}^{f} a_i \leqslant \exp(k \log k - \sigma k + O(k/\log k)).$$

On the other hand, the a_i are square-free and (without loss of generality) $a_1 < \ldots < a_f$. Hence $a_i \geqslant di$ for any $d < \pi^2/6$ and $i \geqslant i_0(d)$, a constant depending only on d. Hence, for some constant $\varepsilon_0 > 0$,

$$\prod_{i=1}^{f} a_i \geqslant d^f f!\, \varepsilon_0 = \exp(f \log f - (1 - \log d) f + O(\log f)).$$

Combining the estimates for $\prod_{i=1}^{f} a_i$ gives

$$f \leqslant k - (\sigma - 1 + \log d) k/\log k + O(k/(\log k)^2).$$

Since $\sigma - 1 + \log(\pi^2/6) > 1/2$ we obtain a contradiction with $f \geqslant k - \delta k/\log k$, $\delta < 1/2$ and $k \geqslant k_0(\delta)$.

Now we consider, finally, the cases for which $2 \leqslant k < k_0 := \max\{2P(a), k_0(\delta)\}$. Suppose we have f distinct integers $n_1, \ldots, n_f$ in an interval $[n, n+k]$, where $n, k \in \mathbf{N}$, such that $\prod_{i=1}^{f} n_i^{m_i} \in a\mathbf{N}^m$ for certain $m, m_1, \ldots, m_f \in \mathbf{N}$ with $m \geqslant 2$ and $\gcd(m_i, m) = 1$ for $i = 1, \ldots, f$. In [14] it was proven that this implies $k > c \log\log\log(n+15)$, where $c = c(a)$ is some positive constant depending only on a, provided $f \geqslant 3$ or $f \geqslant 2$ and $a \in \mathbf{N}^2$. Since $k < k_0$ we infer that $n < n_0$, a constant depending only on a and δ. So both n and k are bounded and there can be only finitely many sets $\{n_1, \ldots, n_f\} \subset [n, n+k]$ for which $\prod_{i=1}^{f} n_i^{m_i} \in a\mathbf{N}^m$ for some $m, m_1, \ldots, m_f \in \mathbf{N}$ with $m \geqslant 2$ and $\gcd(m_i, m) = 1$ for $i = 1, \ldots, f$.

Corollary 6.1. *For every $t \in \mathbf{N}_0$ and every $a \in \mathbf{N}$ there exists a minimal $k_a(t) \in \mathbf{N}$ with the following property. Let $n_1, \ldots, n_f$ be integers obtained by deleting t integers from $k_a(t)$ or more consecutive positive integers. Then*

$$\prod_{i=1}^{f} n_i^{m_i} \notin a\mathbf{N}^m$$

for any $m, m_1, \ldots, m_f \in \boldsymbol{N}$ with $m \geqslant 2$ and $\gcd(m_i, m) = 1$ for $i = 1, \ldots, f$.
Moreover,

(1) *$k_a(t) < ct \log t$ for any $c > 2$ and all $t > t_a(c)$, a constant depending only on a and c.*

(2) *$k_1(t) > t \log t$ for infinitely many $t \in \boldsymbol{N}$.*

Proof. Let $t \geqslant 0$ and $a \in \boldsymbol{N}$ and $0 < \delta < 1/2$ be given. Let k satisfy $\delta k/\log k \geqslant t$. If $n_1, \ldots, n_f$ are obtained by deleting t integers from $n+1, \ldots, n+k$ and $\prod_{i=1} n_i^{m_i} \in a\boldsymbol{N}^m$ for certain $m, m_1, \ldots, m_f$, then, by Theorem 6.1, $k < k_0(a, \delta)$, a constant depending only on a and δ. So if $k_a(t)$ satisfies $k_a(t)/\log k_a(t) \geqslant \delta^{-1} t$ and $k_a(t) > k_0(a, \delta)$ for some $0 < \delta < 1/2$ then it satisfies the property defined in Corollary 1. This proves the existence of $k_a(t)$ and also (1). To prove (2) we argue as follows. For every $k \in \boldsymbol{N}$ there exists a $t \leqslant \pi(k)$ such that there exists some way to delete t integers from $1, 2, \ldots, k$ such that the remaining integers have a perfect square as their product (by Lemma 6.2). Since certainly the primes in $(k/2, k]$ have to be deleted we have

$$\pi(k) - \pi(k/2) \leqslant t \leqslant \pi(k),$$

so there exist infinite sequences $k_1 < k_2 < \ldots$ and $t_1 < t_2 < \ldots$ with

$$t_i \leqslant \pi(k_i) \quad \text{and} \quad (k_i)!/n_1 \ldots n_{t_i} \in \boldsymbol{N}^2$$

for certain distinct $n_1, \ldots, n_{t_i} \in \{1, \ldots, k_i\}$. So $k_1(t_i) \geqslant k_i + 1 \geqslant p_{t_i} + 1 > t_i \log t_i$ (p_t denotes tth prime number).

Note that $k_1(0) = 2$ (if we change the definition of $k_a(t)$ somewhat by taking $m_i = 1$ for all i) by the Erdös–Selfridge theorem and that $k_1(1) \geqslant 11$, $k_7(0) \geqslant 11$ since $10! \in 7\boldsymbol{N}^2$.

Lemma 6.2. *Let $n_1, \ldots, n_f$ be distinct positive integers and let $m \in \boldsymbol{N}$ with $m \geqslant 2$. There exists a subset $\{n_i\colon i \in I\}$ of $\{n_1, \ldots, n_f\}$ with at least $f - \omega(n_1 \cdot \ldots \cdot n_f)$ elements such that*

$$\prod_{i \in I} n_i^{m_i} \in \boldsymbol{N}^m \quad \textit{for certain } m_i \in \{1, \ldots, m-1\},\ i \in I.$$

Proof. We may assume $f > \omega(n_1 \cdot \ldots \cdot n_f)$ (otherwise take $I = \emptyset$). Let $J \subset \{1, \ldots, f\}$ with $|J| = 1 + \omega(n_1 \cdot \ldots \cdot n_f)$. Then n_j, $j \in J$ are composed of less than $|J|$ primes, hence multiplicatively dependent: $\prod_{j \in J} n_j^{a_j} = 1$ for certain $a_j \in \boldsymbol{Z}$, not all zero. In fact we may assume that not all a_j are divisible by m, since the only root of unity in $\boldsymbol{N}$ is 1. Reduce all m_j modulo m, then we obtain a nonempty $J_0 \subset J$ with $\prod_{j \in J_0} n_j^{m_j} \in \boldsymbol{N}^m$, where $m_j \in \{1, \ldots, m-1\}$ for $j \in J_0$.

Now remove the n_j with $j \in J_0$ from $\{n_1, \ldots, n_f\}$. Choose another set J with $1 + \omega(n_1 \cdot \ldots \cdot n_f)$ elements from the remaining integers and repeat the above procedure. We obtain disjoint sets $J_0, J_0^{(1)}, J_0^{(2)}, \ldots, J_0^{(v)}, \ldots$, with $\prod_{j \in J_0^{(v)}} n_j^{m_j} \in N^m$ for certain $m_j \in \{1, \ldots, m-1\}$. Take $I = \bigcup_v J_0^{(v)}$, then $\prod_{j \in I} n_j^{m_j} \in N^m$ and $|I| \geqslant f - \omega(n_1 \cdot \ldots \cdot n_f)$.

THEOREM 6.3. *For $m \in N$ with $m \geqslant 2$ and $n \in N$ we define*

$$k^{(m)}(n) = \min\{k \in N\colon [n, n+k] \text{ contains two or more distinct integers, } n_1, \ldots, n_f, \text{ say, for which } \prod_{i=1}^{f} n_i^{m_i} \in N^m \text{ for certain } m_1, \ldots, m_f \in N \text{ with } m \nmid m_i \text{ for } i = 1, \ldots, f\}$$

and

$$k(n) = \min\{k \in N\colon [n, n+k] \text{ contains two or more distinct integers the product of which is a perfect power}\}.$$

We have, for certain positive absolute constants c_0, c_1, c_2,

(1) $k^{(m)}(n) > c(m) \log\log n$ *for all $n \in N$ with $n \geqslant 3$, where $c(m) = c_0 m^{-10}$,*

(1)′ $k(n) > c_1 \log\log\log n$ *for all $n \in N$ with $n \geqslant 15$.*

For every $\varepsilon > 0$ there exists an infinite set N_1 of positive integers with

(2) $k^{(m)}(n) > \exp\left((1/\sqrt{2} - \varepsilon)(\log n \log\log n)^{1/2}\right)$ *for $n \in N_1$ and all $m \geqslant 2$,*

(2)′ $k(n) > \exp\left((1/\sqrt{2} - \varepsilon)(\log n \log\log n)^{1/2}\right)$ *for $n \in N_1$.*

For every $\varepsilon > 0$ there exists an infinite set N_2 of positive integers with

(3) $k^{(m)}(n) < \exp\left((\sqrt{2} + \varepsilon)(\log n \log\log n)^{1/2}\right)$ *for $n \in N_2$ and all $m \geqslant 2$,*

(3)′ $k(n) < \exp\left((\sqrt{2} + \varepsilon)(\log n \log\log n)^{1/2}\right)$ *for $n \in N_2$,*

(4) $k^{(m)}(n) < c_2 n^{0.496}$ *for all $n \in N$ and all $m \geqslant 2$,*

(4)′ $k(n) < c_2 n^{0.496}$ *for all $n \in N$.*

Proof. Suppose $n_1, \ldots, n_f$ are two or more distinct integers in $[n, n+k]$ with $\prod_{i=1}^{f} n_i^{m_i} \in N^m$ for certain $m, m_1, \ldots, m_f \in N$ with $m \nmid m_i$ for $i = 1, \ldots, f$. Put $m_i^* = m/\gcd(m_i, m)$ and write $n_i = a_i x_i^{m_i^*}$ with $a_i \in N$ m_i^*-free $(i = 1, \ldots, f)$. Suppose $p | a_i$ for some i. Since $\prod_{i=1}^{f} a_i^{m_i} \in N^m$ and a_i is m_i^*-free we infer that $p | a_j$ for some $j \neq i$. Hence $p | \gcd(a_i, a_j) | \gcd(n_i, n_j) | |n_i - n_j| \in \{1, \ldots, k\}$. Hence $a_i \leqslant \prod_{p \leqslant k} p^{m_i^* - 1} < 3^{km}$ for $i = 1, \ldots, f$.

Case 1: $m_i^* \geqslant 3$ for some i. Choose $j \neq i$. We have

$$F(x_j) := a_j x_j^{m_j^*} - d = a_i x_i^{m_i^*}$$

for some d with $0 < |d| \leqslant k$, where $m_i^* \geqslant 3$ and $m_j^* \geqslant 2$. We now use an explicit version of the estimates of Sprindžuk for the solutions $x, y \in Z$ of the

Diophantine equation $F(x) = Ay^m$ (see [17]). Using that $a_i, a_j \leqslant 3^{km}$ we obtain that $(n \leqslant)\ a_j x_j^{m_j^*} \leqslant \exp(C^{m^{10}k})$ for some absolute constant C. This implies (1), for this case.

Case 2: $m_i^* = 2$ for all i. Then $\prod_{i=1}^{f} n_i \in \boldsymbol{N}^2$. In [14] it is proven that this implies that $k \gg (\log\log n)^2 (\log\log\log n)^{-1}$ so (1) also follows in this case. This proves (1). For the proof of (1′) we refer to [14]. We note that a lower bound for $\min_{m \geqslant 2} k^{(m)}(n)$ seems unattainable in the present state of mathematics. That it is possible to prove the lower bound (1)′ for $k(n)$ is due to the requirement in the definition of $k(n)$ that all multiplicities m_i are 1. (Actually it would be sufficient to require only that $\gcd(m_i, m) = \gcd(m_j, m)$ for some $i \neq j$).

To prove (2) we use Lemma 2.6: let $n_1, \ldots, n_f$ be any distinct integers in $[n, n+k^*(n)]$ and let $p|n_1$, $p^2 \nmid n_1$, $p > k^*(n)$. Then $p \nmid n_j$ for $j \neq i$ hence $v_p(\prod_{i=1}^{f} n_i^{m_i}) = m_1$, in particular $\prod_{i=1}^{f} n_i^{m_i} \notin \boldsymbol{N}^m$ for any $m, m_1, \ldots, m_f \in \boldsymbol{N}$ with $m \nmid m_i$ for $i = 1, \ldots, f$. Since clearly $k(n) \geqslant \min_{m \geqslant 2} k^{(m)}(n)$, we obtain (2)′ immediately from (2).

The inequality (3) follows from Lemma 2.7 and Lemma 6.2. Since clearly $k(n) \leqslant k^{(2)}(n)$ we also have (3)′.

To prove (4) we note that, by Lemma 2.4, we have

$$f(n, k) > ck \geqslant \pi(k) + 2 \quad \text{for} \quad k \geqslant n^{0.496} \text{ and } n \geqslant n_1,$$

where c and n_1 are positive constants. Now use Lemma 6.2 to obtain (4). Again by $k(n) \leqslant k^{(2)}(n)$, the inequality (4)′ follows immediately.

In the next two theorems we give some results about sets $\{n_1, \ldots, n_f\}$ of integers in short intervals $[n, n+k(n)]$ with the property that $\prod_{i=1}^{f} n_i$ is a perfect power where the number f of elements is restricted.

THEOREM 6.4. *Let $n, k \in \boldsymbol{N}$ be arbitrary and suppose $\prod_{i=1}^{f} n_i$ is a perfect power for distinct ($f \geqslant 2$) integers $n_1, \ldots, n_f$ in $(n, n+k]$. Then*

$$f \leqslant k - \delta_0 k/\log k,$$

where δ_0 is a positive absolute constant.

On the other hand, for all $n, k \in \boldsymbol{N}$ with $k \geqslant n$ there exist distinct $n_1, \ldots, n_f \in (n, n+k]$ with $\prod_{i=1}^{f} n_i$ is a perfect power and

$$f \geqslant k - 4k/\log k.$$

For every α *with* $1/2 \leqslant \alpha < 1$ *there exists a* $c_\alpha < 1$ *such that if* $n_1, \ldots, n_f$ *are distinct* ($f \geqslant 2$) *integers in* $(n, n+k]$, *where* $k = n^\alpha$, *with* $\prod_{i=1}^{f} n_i$ *is a perfect power then*

$$f \leqslant c_\alpha k.$$

On the other hand, for every $\alpha \geqslant \alpha_0$ *there exists a* $c_\alpha^* > 0$ *such that for all* n *there exist distinct integers,* $n_1, \ldots, n_f$, *say, in* $(n, n+k]$, *where* $k = n^\alpha$, *with* $\prod_{i=1}^{f} n_i$ *is a perfect power and*

$$f > c_\alpha^* k.$$

Proof. To prove the first assertion we use Theorem 6.1: we obtain $f \leqslant k - \frac{1}{3}k/\log k$ provided that $k \geqslant k_0$, an absolute constant. Now choose $0 < \delta_0$ $(\leqslant \frac{1}{3})$ such that $\delta_0 k/\log k \leqslant 1$ for $2 \leqslant k < k_0$, then $f \leqslant k-1 \leqslant k - \delta_0 k/\log k$ also holds when $2 \leqslant k < k_0$ by the Erdös–Selfridge theorem.

To prove the second assertion we argue as follows: for $k \geqslant n$ we have $\omega\big((n+1)\ldots(n+k)\big) = \pi(n+k)$. By Lemma 6.2 there exist, therefore, $n_1, \ldots, n_f \in (n, n+k]$ with $f \geqslant k - \pi(n+k)$ for which $\prod_{i=1}^{f} n_i$ is a perfect square. Furthermore we have $\pi(n+k) \leqslant \pi(2k) < 4k/\log k$.

To prove the third assertion, assume $\prod_{i=1}^{f} n_i$ is a perfect power, where $n_1, \ldots, n_f$ are distinct ($f \geqslant 2$) integers in $(n, n+k]$, $k = n^\alpha \geqslant n^{1/2}$. Then $P(n_i) \leqslant k$ for $i = 1, \ldots, f$ (a prime $p > k$ cannot divide two distinct integers in $(n, n+k]$ and p^2 cannot divide an integer in $(n, n+k]$ either, since $(k+1)^2 > n+k$), so $f \leqslant f(n, k)$. Now use Lemma 2.1.

The last assertion follows from Lemma 2.4 and Lemma 6.2.

THEOREM 6.5. *For* m *and* $f \in \boldsymbol{N}$ *with* $m \geqslant 2$ *and* $f \geqslant 2$ *there exist* $\varepsilon_1 = \varepsilon_1(m, f) > 0$ *and* $\varepsilon_2 = \varepsilon_2(m, f) > 0$ *such that if* $[n, n+k]$ *contains* f *distinct integers with a perfect m-th power as their product then* $k > \varepsilon_1 (\log n)^2$.

For $m \in \boldsymbol{N}$ *with* $m \geqslant 2$ *and* $\varepsilon \in \boldsymbol{R}$ *with* $0 < \varepsilon \leqslant 1$ *there exist* $\delta_1 = \delta_1(m, \varepsilon) > 0$ *and* $\delta_2 = \delta_2(m, \varepsilon) > 0$ *such that if* $[n, n+k]$ *contains* f *distinct integers with a perfect m-th power as their product and* $f \geqslant \varepsilon k$ *then* $k > \delta_1 (\log n)^{\delta_2}$.

Proof. This has been proven in [14]. Similar assertions, though with different numbers $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2$, hold for the property

$$\prod_{i=1}^{f} n_i^{m_i} \in \boldsymbol{N}^m \quad \text{for certain } m_i \in \boldsymbol{N} \text{ not divisible by } m,$$

see the first part of the proof of Theorem 6.3 and the proof of Corollary 4 in [14].

Suppose m and f are given integers, $m \geqslant 2$, $f \geqslant 2$. How far do we have to go from n to obtain f distinct integers which have a perfect mth power as their product? Trivially, the first f mth powers larger than or equal to n have a perfect mth power as their product, so we do not have to go further than $n+Cn^{1-1/m}$, $C = C(m, f)$. We are not able to find a better upper bound than $Cn^{1-1/m}$, valid for all n (it does not exist when $f = m = 2$). One method to try and find one is to search for f distinct neighbouring integers n_i of the form $n_i = a_i x_i^m$, where the $a_1, \ldots, a_f$ are pre-chosen (m-free) integers with $\prod_{i=1}^{f} a_i \in \mathbf{N}^m$, for example $a_1 \cdot \ldots \cdot a_{f-1}$ arbitrary and $a_f = (a_1 \cdot \ldots \cdot a_{f-1})^{m-1}$. One can show (see [15]) that this gives an upper bound $Cn^{1-1/m-1/m(f-1)}$, $C = C(m, f)$ valid for infinitely many $n \in \mathbf{N}$ ($(m, f) \neq (2, 2)$). In particular, *for every m, f with $m \geqslant 2$, $f \geqslant 2$, except $(m, f) = (2, 2)$, there exist infinitely many $n \in \mathbf{N}$ such that between n^m and $(n+1)^m$ there exist f distinct integers whose product is a perfect m-th power.*

This method (with pre-chosen $a_1, \ldots, a_f$) is certainly not able to produce upper bounds Cn^σ with $\sigma < 1-1/m-1/m(f-1)$, as was proven in [15]. In particular, if $[n, n+k]$ contains $2x_1^2$, $3x_2^2$, $6x_3^2$, then $k > c(\varepsilon) n^{1/4-\varepsilon}$ for any $\varepsilon > 0$. An interesting example of three distinct integers whose product is a perfect square is 10082, 10086, 10092 ($= 2x_1^2, 6x_3^2, 3x_2^2$), found by Selfridge.

7. Generalizations and problems.

7.1. Integral values of a polynomial. Let $F \in Z[X]$, where we assume, for simplicity, that F is irreducible. We shall consider the integers $F(t)$, $t \in Z$. We are interested in the following properties of $F(n_1), \ldots, F(n_f)$, where $n_1, \ldots, n_f$ are distinct integers:

(1) $\omega\left(\prod_{i=1}^{f} F(n_i)\right) < f$.

(2) $F(n_1), \ldots, F(n_f)$ are multiplicatively dependent.

(3) $\prod_{n \in N_1} F(n) = \prod_{n \in N_2} F(n)$ for distinct subsets N_1, N_2 of $\{n_1, \ldots, n_f\}$.

(4) $\prod_{i=1}^{f} F(n_i)$ is a perfect power.

In the preceeding sections we have shown that when $F(X) = X$ these properties

(A) never occur when $n_1, \ldots, n_f$ are any distinct ($f \geqslant 2$) integers in any "short" interval,

(B) always occur for some distinct ($f \geqslant 2$) integers $n_1, \ldots, n_f$ in any "large" interval.

We can prove the (A)-theorems also for the general case: there exist positive constants c_1, c_2, c_3, c_4, c_5, depending only on F, such that for all $n \geqslant 15$ we have

(1A) *For all distinct* $(f \geqslant 2)$ *integers* $n_1, \ldots, n_f$ *in* $[n, n+c_1 \times \times (\log n)^3/(\log\log n)^{c_2}]$ *we have* $\omega\left(\prod_{i=1}^{f} F(n_i)\right) \geqslant f$.

(2A) *For all distinct* $(f \geqslant 2)$ *integers* $n_1, \ldots, n_f$ *in* $[n, n+c_3 \log n/\log\log n]$ *the integers* $F(n_1), \ldots, F(n_f)$ *are multiplicatively independent.*

(3A) *For all subsets* $N_1 \neq N_2$ *of integers in* $[n, n+c_3 \log n/\log\log n]$ *we have* $\prod_{n \in N_1} F(n) \neq \prod_{n \in N_2} F(n)$.

(4A) *For all distinct* $(f \geqslant 2)$ *integers* $n_1, \ldots, n_f$ *in* $[n, n+c_4 \times \times (\log\log\log n)^{c_5}]$ *the product* $\prod_{i=1}^{f} F(n_i)$ *is not a perfect power.*

These results can be proven like in the special case $F(X) = X$, using the following lemma.

LEMMA ([16]). *Let* $F \in Z[X]$ *be irreducible. Then for any distinct integers* x, y *we have*

$$\gcd(F(x), F(y)) \leqslant c_6 |x-y|^{c_7},$$

where c_6 *and* c_7 *are constants depending only on* F.

The first problem we propose is

P1: *Prove* (3A) *for intervals larger than in* (2A) *also when the degree of* F *exceeds* 1 (see Theorems 4.1. (1) and 5.1. (1)).

We are only able to prove (**B**)-theorems when the degree of F equals (one or) two, and the intervals are actually "very large":

Let $F \in Z[X]$ *be of degree* 2. *There exists a number* n_0, *depending only on* F, *such that for all* $n > n_0$ *the interval* $(n/\log n, n)$ *contains*

(1**B**) *a set of integers* S_1 *with* $\omega\left(\prod_{s \in S_1} F(s)\right) < |S_1|$,

(2**B**) *a set of integers* S_1 *such that* $F(s)$, $s \in S_1$ *are multiplicatively dependent,*

(3**B**) *two distinct sets* S_2, S_3 *of integers with* $\prod_{s \in S_2} F(s) = \prod_{s \in S_3} F(s)$,

(4**B**) *for every* $m \in \mathbf{N}$, $m \geqslant 2$, *a set* S_m *of integers with* $\prod_{s \in S_m} F(s)^{m(s)} \in N^m$ *for certain* $m(s) \in \{1, \ldots, m-1\}$, $s \in S_m$.

Proof. It follows from Lemma 4 and Lemma 5 in [2] that, if F is irreducible of degree 2, for all $n > n_0$ the interval $(n/\log n, n)$ contains at least $\varepsilon_0 n(\log n)^{-1} \log\log n \log\log\log n$ integers v with $P(F(v)) \leqslant n$. This clearly holds, too, when F is reducible and of degree 2. Let S_1 be the set of these v, then (1**B**) holds (we take n_0 sufficiently large) and (2**B**) follows immediately. To prove (3**B**) we invoke Lemma 5.1. The set S_1 does not necessarily fulfill the conditions of Lemma 5.1; let S_1^* be the subset of S_1 obtained by deleting all v with $P(F(v)) > n/\log n$. The number of deletions is at most

$$\sum_{n/\log n < p \leqslant n} \varrho(p)\left(\left[\frac{n}{p}\right]+1\right) \leqslant \frac{n}{\log n}\left(\log\log n + O(1)\right).$$

Here $\varrho(p)$ denotes the number of $x \in \{0, 1, \ldots, p-1\}$ with $F(x) \equiv 0 \bmod p$. Hence $|S_1^*| > \frac{1}{2}|S_1|$, if n_0 is sufficiently large. We apply Lemma 5.1 to $\{F(s), s \in S_1^*\}$ to obtain (4B). To prove (4B) we apply Lemma 6.2 to the set $\{F(s), s \in S_1\}$.

Corollary. *Let $F = X^2+bX+c \in Z[X]$. Then there exist infinitely many finite sets $S \subset Z$ with $\prod_{s\in S} F(s) \in N^2$ and infinitely many finite sets $T \subset Z$ with $\prod_{t\in T} F(t) \in N^3$.*

Proof. We obtain the sets $S \subset N$ from (4B) with $m = 2$. From (4B) with $m = 3$ we obtain infinitely $T' \subset N$ with $\prod_{t\in T'} F(t)^{m(t)} \in N^3$ with $m(t) \in \{1, 2\}$. Since $F(t)^2 = F(t)F(-t-b)$ and $t \neq -t-b$ for $t \neq b/2$ this gives the sets T.

Note that if $F = X^2+bX+c \in Z[X]$ then, for certain $\sigma \in N$, there exist infinitely many $x \in N$ such that $F(x) \in \sigma N^2$ (e.g. for any $\sigma = F(t)$ with t such that $F(t) \in N - N^2$). Hence there exist infinitely many sets S of two distinct integers with $\prod_{s\in S} F(s) \in N^2$.

We propose for consideration:

P2: *Let $F \in Z[X]$ be of degree at least three (and irreducible). Do there exist infinitely many sets $\{n_1, \ldots, n_f\}$ of integers with property* (4)?, (3)?, (2)?, (1)?

We finally mention that we can prove the following results on the values of a polynomial taken at integers from a short interval (see [10] and [6] for the case $F(X) = X$).

Let $F \in Z[X]$ be irreducible. There exist positive numbers c_8, c_9, c_{10}, c_{11}, depending only on F, such that for any $n \geqslant 3$ we have

(5) if n_1, n_2 are distinct integers in $[n, n+c_8(\log n)^{c_9}]$ then $F(n_1)$ and $F(n_2)$ do not have the same set of distinct prime divisors.

(6) if n_1, n_2 are distinct integers in $[n, n+c_{10}(\log\log n)^{c_{11}}]$ then $F(n_1)$ and $F(n_2)$ do not have the same greatest prime divisor.

7.2. Some more problems. In Section 6 we considered the property $\prod_{i=1}^{f} n_i^{m_i} \in N^m$, where $m, m_1, \ldots, m_f \in N$ with $m \geqslant 2$ and $m \nmid m_i$ for $i = 1, \ldots, f$ and $n_1, \ldots, n_f$ are two or more distinct integers in an interval $[n, n+k]$, with $n, k \in N$. We noted that it is a difficult matter to prove a lower bound for k when there is no (further) restriction on the multiplicities m_i (we only have k

$\geqslant 2$ for n larger than an absolute constant by Tijdeman's result [11] on the Catalan equation), but that we can prove $k \gg \log\log\log n$ when (e.g.) $m_i = 1$ for $i = 1, \ldots, f$. On the other hand, it is more difficult to prove the occurrence of the property in an interval $[n, n+k]$ when there are restrictions on the m_i.

P_3: *Let $m \in \boldsymbol{N}$ with $m \geqslant 3$. For $n \in \boldsymbol{N}$ we define*

$k_*^{(m)}(n) = \min\{k \in \boldsymbol{N}$: $[n, n+k]$ *contains two or more distinct integers whose product is a perfect m-th power*$\}$.

Find upper bounds for $k_*^{(m)}(n)$ valid for (1) *all $n \in \boldsymbol{N}$* (2) *infinitely many $n \in \boldsymbol{N}$.*

Let $f \in \boldsymbol{N}$ be fixed and let P be some property of sets of integers. For $n \in \boldsymbol{N}$ define $k_{P,f}(n) = \min\{k \in \boldsymbol{N}$: $[n, n+k]$ contains f distinct integers having property P$\}$. Find upper bounds for $k_{P,f}(n)$ for the properties P occurring in this paper. For example:

P_4: *Given $n \in \boldsymbol{N}$ find an upper bound for the minimal $k \in \boldsymbol{N}$ for which there exist three distinct integers in $[n, n+k]$ whose product is a perfect square.*

Another complication in a search for integers in an interval with a certain property would be to insist that one of them is fixed. For example:

For $n \in \boldsymbol{N}$ let $k(n)$ be the least integer such that there exist $n = a_1 < \ldots < a_f = k(n)$ with $\prod_{i=1}^{f} a_i \in \boldsymbol{N}^2$.

So $k(1) = 1$, $k(2) = 6$, $k(3) = 8$, $k(4) = 4$, $k(5) = 10$, $k(6) = 12$, $k(7) = 14$, $k(8) = 15$, $k(9) = 9$, $k(10) = 20$, $\ldots$

Clearly $k(n) \leqslant 2n$ for $n \geqslant 10$: let x^2 be a perfect square in $(n/2, n)$, then $n \cdot 2x^2 \cdot 2n \in \boldsymbol{N}^2$. On the other hand, clearly $k(n) \geqslant n + P_*(n)$, where $P_*(n) = 0$ for $n \in \boldsymbol{N}^2$ and $P_*(n)$ is the largest prime p with $v_p(n)$ odd for $n \in \boldsymbol{N} - \boldsymbol{N}^2$. It follows that $k(p) = 2p$ for primes $p \geqslant 5$. We show that $k(n) \leqslant n + 3(P_*(n)\,n)^{1/2}$: We may suppose that $n \notin \boldsymbol{N}^2$. Let p be a prime with $v_p(n)$ odd. Let $t_p \in \boldsymbol{N}$ be minimal with $n + pt_p \in p\boldsymbol{N}^2$. Then $n + pt_p \leqslant n + 2\sqrt{np} + p$ and $n \cdot \prod (n + pt_p) \in \boldsymbol{N}^2$, where the product is over the primes p with $v_p(n)$ odd.

Since the $n + pt_p$ are distinct we obtain

$$k(n) \leqslant n + 2\sqrt{nP_*(n)} + P_*(n) \leqslant n + 3\sqrt{P_*(n)\,n}.$$

P_5: *Can the bounds for $k(n)$ be improved?*

We observe that k is 1-to-1: Suppose $m < n$ and $k(m) = k(n)$. Then there exist $m = a_1 < \ldots < a_f = k(m)$ and $n = b_1 < \ldots < b_g = k(n)$ with

$$\prod_{i=1}^{f} a_i \in \boldsymbol{N}^2 \quad \text{and} \quad \prod_{j=1}^{g} b_j \in \boldsymbol{N}^2.$$

Hence

$$\prod_{i=1}^{f} a_i \prod_{j=1}^{g} b_j \in \boldsymbol{N}^2$$

and, since $a_f = b_g$, also

$$\prod_{i=1}^{f-1} a_i \prod_{j=1}^{g-1} b_j \in \boldsymbol{N}^2.$$

Cancelling any other integers that occur twice we obtain a set of integers from m to at most $\max\{a_{f-1}, b_{g-1}\}$ whose product is a square, contradicting the definition of $k(m)$.

It may be possible to prove that *distinct sets of neighbouring integers have distinct products*, i.e. there exists a function $k\colon \boldsymbol{N} \to \boldsymbol{N}$ with $\lim_{n\to\infty} k(n) = \infty$ such that if S_1 and S_2 are distinct sets of integers from intervals $[n_i, n_i + k(n_i)]$, $i = 1, 2$, where n_1, n_2 are arbitrary integers > 1, then $\prod_{s\in S_1} s \neq \prod_{s\in S_2} s$.

Note that $k(5)$ would have to be 1 in view of $5\cdot 6\cdot 7 = 14\cdot 15$ and that $k(n) < 3 \log n$ for infinitely many n in view of [7]:

$$2^k(2^k+1)\dots(2^k+k) = (2^{k+1}+2)(2^{k+1}+4)\dots(2^{k+1}+2k).$$

We certainly do not see how to obtain such a function k explicitly. Note that for the restricted problem with $n_1 = n_2$ we can take $k(n) = [c(\log n/\log\log n)^2]$ for sufficiently large n, by Theorem 5.1. (1).

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Received on 14. 3. 1983 (1345)