COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
37. FINITE AND INFINITE SETS, EGER (HUNGARY), 1981.

INVERSE EXTREMAL DIGRAPH PROBLEMS

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ABSTRACT

The authors continue their study of extremal problems of Turán type for directed graphs with multiple edges, now permitting any finite non-negative integer multiplicity. Having proved earlier (for the case of multiplicity at most 1) that *there exists,* for any family of "sample" digraphs, *a matrix which represents the structure of an "asymptotically extremal sequence"* of digraphs (containing none of the sample digraphs, and having a total number of arcs asymptotic to the maximum), they address themselves to the *inverse problem: is every matrix so realized for some finite family of sample digraphs?* They prove that this is indeed true for "dense" matrices – having certain integer entries, and such that an associated quadratic form attains its maximum for the standard simplex uniquely at an interior point.

This research was supported in part by an Operating Grant of the Natural Sciences and Engineering Council of Canada, held by the first author.

1. INTRODUCTION

In 1941, Paul Turán proved (cf. [14], [15]) the following theorem: Among (simple) graphs on n vertices not containing a complete p-graph K_p there exists exactly one having the maximum number of edges; this graph (which we shall denote by T_n), can be obtained by partitioning the n vertices into $p-1$ classes $C_1, C_2, \ldots, C_{p-1}$, whose cardinalities differ by at most 1, and joining two vertices if and only if they belong to different classes. In the following generalization, considered extensively by Erdős, Simonovits and others, one replaces K_p in the Turán problem by an arbitrary (finite or infinite) family of *prohibited* or *sample* subgraphs:

Problem 1. *Given an arbitrary family* $\mathcal{L}$ *of graphs, to determine the maximum number of edges a graph* G^n *can have without containing any* $L \in \mathcal{L}$, *and to characterize graphs attaining the maximum.*

The maximum will be denoted by $\mathrm{ex}(n, \mathcal{L})$, and called the *extremal number;* the family of graphs attaining the maximum will be denoted by $\mathrm{EX}(n, \mathcal{L})$, and its members called *extremal graphs.* Erdős and Simonovits [6] proved that for $p = \min_{L \in \mathcal{L}} \chi(L) - 1$ (if $n \to \infty$),

$$\mathrm{ex}(n, \mathcal{L}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

This implies that the extremal number is "very near" to that of K_p; it depends on $\mathcal{L}$ loosely – only on the minimum chromatic number of members of $\mathcal{L}$.

A next question to be investigated was whether $\mathrm{EX}(n, \mathcal{L})$ is also very similar to $\mathrm{EX}(n, \{K_p\})$ in some sense. The answer was in the affirmative: Erdős and Simonovits proved independently [7], [8], [12] that if S^n is an extremal graph for $\mathcal{L}$ (that is, $S^n \in \mathrm{EX}(n, \mathcal{L})$), then one can delete from and add to S^n, $o(n^2)$ edges so that the resulting graph is the Turán graph T_n defined above.

In the above results we have always excluded loops and multiple edges; for, if two vertices are joined by arbitrarily many edges, there exists no finite maximum in Problem 1. However, Brown and Harary [5] observed that if parallel edges are permitted, but with an upper bound on the number connecting any pair of vertices, results analogous to those of P. Turán hold; they also investigated digraph versions of Turán's problem. The (multi-)digraph problems appear to be more difficult than the corresponding multigraph problems: we shall restrict our attention to the former, apart from the last section, where we explain the consequences of our results to multigraph problems.

2. THE GENERAL DIGRAPH EXTREMAL PROBLEM

We are concerned in the present work with directed graphs wherein multiple arcs are permitted in either direction between pairs of vertices; these may be referred to as *digraphs* or occasionally as *graphs* in the sequel. Digraphs will always be denoted by capital Latin letters; a superscript, where used, will indicate the number of vertices.

Let q be a fixed positive integer. We consider below only digraphs in which any two vertices are joined by at most q arcs of each orientation. (Our use of the term *digraph* differs from that of Harary et al., who would permit two vertices to be joined by at most one arc in total.) Dependence on q will not normally be indicated explicitly. The general problem is

Problem 1*. *How many arcs can a digraph G^n possess without containing any prohibited subdigraphs from a given family $\mathscr{L}$? This maximum, and the family of digraphs attaining the maximum, will be denoted respectively by* $\mathrm{ex}\,(n, \mathscr{L})$ *and* $\mathrm{EX}\,(n, \mathscr{L})$; *members of the latter family will be called* extremal *digraphs.*

We make no attempt to evaluate the extremal numbers $\mathrm{ex}\,(n, \mathscr{L})$ exactly. Rather, we study their asymptotic behavior, up to terms of order $o(n^2)$. (Some explicit results are, however, known; cf. [2], [5], [9].) Accordingly, we propose the following definition.

Definition 1. A sequence $\{S^n\}$ of digraphs will be said to be *asymptotically extremal* for $\mathscr{L}$ if S^n contains no prohibited subdigraphs, and $e(S^n) = \mathrm{ex}\,(n, \mathscr{L}) + o(n^2)$ as $n \to \infty$.

Remark. Without limiting the generality of our theorems, we usually find it convenient to restrict applications of the previous definition to sequences indexed by the number n of vertices.

(Through the use of familiar techniques (cf. [10]) it can be shown that $\frac{1}{n^2}\,\mathrm{ex}\,(n, \mathscr{L})$ approaches a limit as $n \to \infty$; we call the limit the "*extremal density*" of $\mathscr{L}$. Some of our results in the present study are non-trivial only for the case where that limit is positive: in particular, whenever none of the members of the family $\mathscr{L}$ is transitive. In the corresponding problem for ordinary graphs, the extremal density is positive precisely when none of the members of the family $\mathscr{L}$ is bipartite.)

In our earlier papers [1], [2] we studied exclusively the case $q = 1$, as did the first author and Harary [5]. Certain of the proofs in [1] generalize to the present case without difficulty, and the reader will be referred to that paper for explicit details. Our main result in the present paper (Theorem 1) was announced for $q = 1$ in [1], but will be proved here in full generality. The second main result (Theorem 2) appears in Section 9; there we shall be concerned with the existence of finite subfamilies of a given family of prohibited digraphs, having extremal numbers "near" to those of the original family.

3. MATRIX DIGRAPHS, DENSITY

The basic result of [1] is that for every family $\mathscr{L}$ of prohibited subdigraphs, there exists a matrix A and an asymptotically extremal sequence $\{S^n\}$ of *optimal matrix digraphs*, obtainable from A in a very simple way. More precisely, the members of the sequence consist of digraphs whose verices are partitioned into a finite number of classes with adjacencies between classes that can be fully described by a matrix.

Definition 2. Let $A = \{a_{i,j}\}_{i,j \leqslant r}$ be an $r \times r$ matrix whose entries are constrained as follows:

(i) diagonal elements may have any non-negative integer value not exceeding $2q - 1$;

(ii) off-diagonal elements may have any non-negative even integer value not exceeding $2q$.

Let $\mathbf{b} = (b_1, b_2, \ldots, b_r)$ be an r-vector whose entries are non-negative integers such that

(iii) $b_i \leqslant a_{ii}$ for all i.

For each vector $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ having non-negative integer coordinates, we define a graph $A\langle \mathbf{b}, \mathbf{x}\rangle$ (or more briefly $A\langle \mathbf{x}\rangle$) as follows: the vertices will be divided into r distinct classes $C_1, C_2, \ldots, C_r$ containing respectively $x_1, x_2, \ldots, x_r$ members; from every vertex of the i-th class to every vertex of the j-th class there are directed precisely $\frac{1}{2} a_{ij}$ arcs (= directed edges); within the i-th class the vertices are linearly ordered, with each vertex sending exactly b_i arcs to each of its successors, and receiving $a_{ii} - b_i$ arcs from each of its predecessors. It is also convenient to define a countable digraph $A(\mathbf{b}, \infty)$ (briefly, $A(\infty)$) analogously to $A\langle \mathbf{b}, \mathbf{x}\rangle$: its vertices are the members of r countable classes $C_1, C_2, \ldots, C_r$, each labelled by the natural numbers (and so ordered), with adjacencies determined from A in the obvious way.

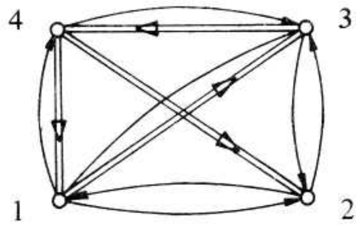
Remark. Replacing in $A\langle \mathbf{b}, \mathbf{x}\rangle$ any b_i by $a_{ii} - b_i$ yields a digraph with the same structure. Therefore we may assume that

(iii′) $b_i \leqslant \frac{1}{2} a_{ii}$.

In particular, for $q = 1$, we may and shall assume that $\mathbf{b} = \mathbf{0}$. In [1] we considered only the case $q = 1$, $\mathbf{b} = \mathbf{0}$; we used the symbol $A((\mathbf{x}))$ for $A\langle \mathbf{x}\rangle$.

Pairs $A, \mathbf{b}$ satisfying the conditions of Definition 2 may be represented in a natural way by a digraph (while, on the other hand, we use matrices to represent digraphs!): namely, by $A\langle \mathbf{b}, 2\mathbf{e}\rangle$ where $\mathbf{e} = (1, 1, \ldots, 1)$. However, it is often sufficient to take $A\langle \mathbf{e}\rangle$ and add a_{ii} loops at its i-th vertex ($i = 1, 2, \ldots, r$); while no information is given concerning the vector $\mathbf{b}$, most of our results will prove to be independent of $\mathbf{b}$. In Figure 1, we provide an example, showing how a digraph could represent a 4×4 matrix for $q = 2$. Note that A is related to, but different from the adjacency matrix of the digraph.

$$A = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 1 & 4 \\ 4 & 4 & 2 & 0 \end{bmatrix} \qquad A\langle(1, 1, 1, 1)\rangle =$$



$A\langle(2, 2, 2, 2)\rangle =$

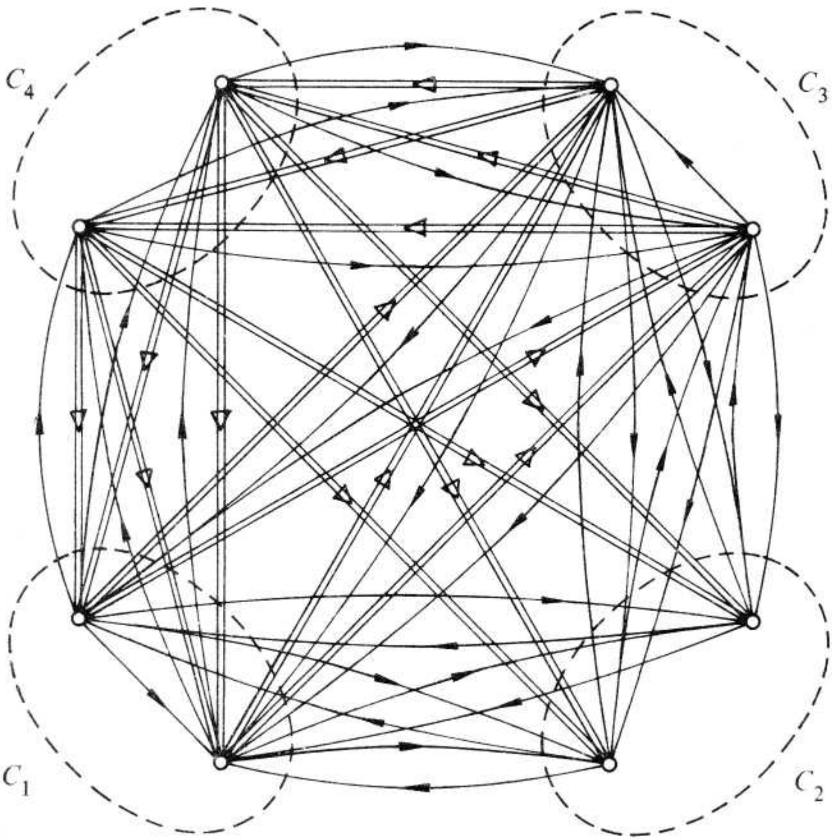


Figure 1

Example 1. A matrix and its representing digraphs $(q \geqslant 2)$

Remark. Except for several instances in Section 4 (where we recall a result of [1] and state a conjecture generalizing it), most of the restrictions of Definition 2 are inessential. The statement of none of our other results is dependent on q. The restriction (condition (ii)) of off-diagonal entries to even values leads to a useful estimate for the total number of arcs in a matrix digraph, using the quadratic form associated with the matrix; had we chosen to represent matrix digraphs by adjacency matrices with the usual definition, the diagonal entries would have had to be halved – which would have the same effect as the present. That portion of Definition 2 referring to the acyclic orientation of arcs connecting vertices within any class is not essential in this paper; the "unique" matrices used in the characterization (for $q = 1$) of Theorem A1 (cf. Section 4 below) do have this property, however.

Remark*. Later the following (technical?) observation will be of importance: Let A be an $r \times r$, B a $t \times t$ matrices, $\mathbf{e}$ and $\mathbf{e}'$ be the corresponding vectors with each coordinate equals 1. If $A\langle \mathbf{b}, \mathbf{x}\rangle$ contains $B\langle \mathbf{b}', (r+1)\mathbf{e}'\rangle$, then each $B\langle \mathbf{b}', \mathbf{y}\rangle$ is contained in some $A\langle \mathbf{b}, \mathbf{z}\rangle$. Indeed, in this case $A\langle \mathbf{b}, \mathbf{x}\rangle$ contains a $B\langle \mathbf{b}', 2\mathbf{e}'\rangle$ *canonically:* so that each class (pair of vertices) of $B\langle \mathbf{b}', 2\mathbf{e}'\rangle$ is contained wholly in one class of $A\langle \mathbf{b}, \mathbf{x}\rangle$. Now, if the i-th class of $B\langle \mathbf{b}', 2\mathbf{e}'\rangle$ is in the $p(i)$-th class of $A\langle \mathbf{b}, \mathbf{x}\rangle$, then the corresponding embedding of any $B\langle \mathbf{b}', \mathbf{y}\rangle$ into $A\langle \mathbf{b}, \mathbf{z}\rangle$ works, assumed that the coordinates of $\mathbf{z}$ are sufficiently large.

Definition 3. Optimal matrix digraphs. Restricting our attention to vectors $\mathbf{x}$, the sum of whose (non-negative integer) entries is a positive integer n, we define $A(\mathbf{b}; n)$ to be any $A\langle \mathbf{b}, \mathbf{x}\rangle$ for which the number of arcs is maximal. As $e(A(\mathbf{b}; n))$ is independent of $\mathbf{b}$, we shall usually abbreviate the symbol $A(\mathbf{b}; n)$ to $A(n)$. (When we speak of a digraph as "containing $A(n)$", or "containing an $A(n)$", we intend that it contains a subdigraph whose structure is isomorphic to *any one* of the optimal matrix digraphs $A(n)$.)

Remark. If $A\langle \mathbf{x}\rangle$ is optimal, the difference between valencies of vertices in the i-th and j-th classes can be shown not to exceed $\frac{1}{2}(a_{ij} + a_{ji})$, (cf. [1], 2(B)). Moreover, the number of optimal matrix digraphs $A(n)$

is $O(1)$ as $n \to \infty$; we shall not require these two results in the present paper.

The extremal problems we wish to consider often lead to an optimization problem of the following type: a digraph G is given, and one is permitted to replace each of its vertices by an independent set of vertices. Considering subdigraphs H^n of the resulting digraph, what is the maximum value of the ratio $\dfrac{e(H^n)}{[qv(H^n)(v(H^n)-1)]}$, i.e. the maximum proportion of the total number of edges attainable for a given number of vertices? Since G may be represented by an adjacency matrix, we are led to the following, somewhat more general, concept. Those digraphs G for which the maximum is attained for no proper subdigraph are of particular interest, cf. [11].

Definition 4. Dense matrices. For any matrix A satisfying conditions (i) through (iii) of Definition 2, we define its *density* $g(A)$ to be the maximum value of the quadratic form $\mathbf{u}A\mathbf{u}^*$ on the standard simplex in R^r:

$$g(A) = \max \left\{ \mathbf{u}A\mathbf{u}^* \colon u_i > 0, \ \sum_i u_i = 1 \right\}.$$

A will be called *dense* if, for every principal proper submatrix A' of A, $g(A) > g(A')$.

We shall be considering cases where the extremal or "almost extremal" graphs will have the structure $A(\mathbf{b}; n)$ for some fixed A and $\mathbf{b}$. As we shall see in Section 4, if A has a proper principal submatrix A' with the same density, then $e(A(n))$ and $e(A'(n))$ are asymptotically equal. Hence without loss of generality mostly we may restrict our study to dense matrices.

Example 2. Let $q = 1$, $\mathbf{b} = \mathbf{0}$. Let D_r be an $r \times r$ matrix of structure $2J - 2I$ (we follow standard usage – J is a square matrix of 1's, I is an identity matrix); $\mathbf{e}$ will be a vector of 1's. Then $D_r\langle \mathbf{x} \rangle$ is a complete r-partite digraph with x_i vertices in the i-th class, C_i: for every pair (i, j), each vertex of C_i is joined to each vertex of C_j by two arcs of opposite directions. This digraph has the maximum number of edges if $|x_i - x_j| \leqslant 1$ for every i, j satisfying $1 \leqslant i \leqslant j \leqslant r$. The matrix is dense, and $g(D_r) = 2 - \frac{1}{r}$.

Let the symmetric part of $A, \hat{A}$, be defined by

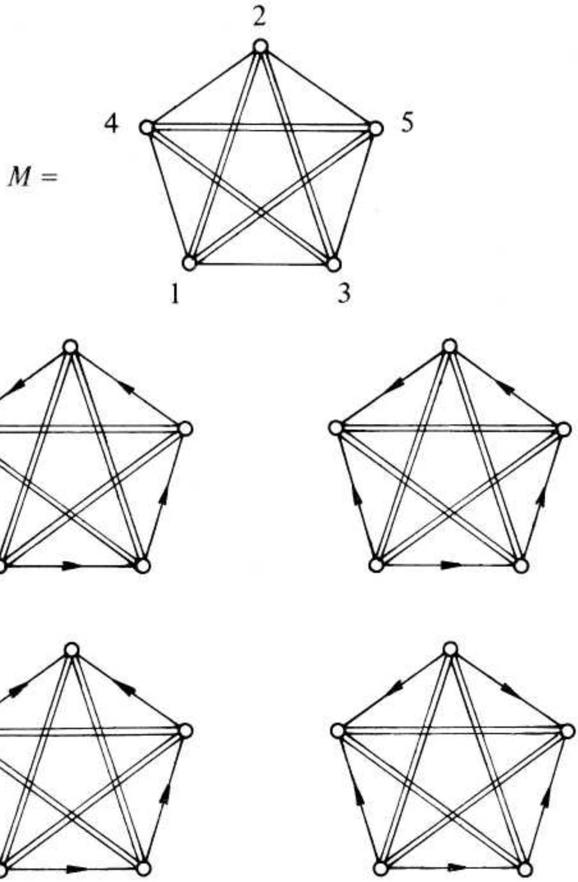
(3.1) $$\hat{A} = \frac{1}{2}(A + A^*),$$

where A^* is the transpose of A. Corresponding principal submatrices of $A, A^*, \hat{A}$ all yield the same associated quadratic form, so either each of $A, A^*, \hat{A}$ or none of them is dense. The transpose operation corresponds to reversal of orientations in the matrix digraph. It may not be possible to interpret $\hat{A}$ in terms of Definition 2, since some of its off-diagonal entries can be odd. However, is is possible to develop a theory analogous to, and simpler than the present for (undirected) multigraphs. For this purpose $\hat{A}$ would be taken to be any adjacency matrix of a multigraph of maximum multiplicity $2q$ – so the sum of symmetrically located off-diagonal entries would represent precisely the number of edges connecting vertices (cf. Section 11).

Example 3. The following four 5×5 matrices are all dense, with optimum vector $\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$ and density $\frac{6}{5}$ they correspond (for $q = 1$) to the four distinct orientations of the multigraph obtained by halving the multiplicities of the edges of one pentagon in a doubled complete 5-graph, i.e. by orienting the arcs of the multigraph M in Figure 2.

$$\begin{pmatrix} 0&2&2&0&2\\ 2&0&2&2&0\\ 0&2&0&2&2\\ 2&0&2&0&2\\ 2&2&0&2&0 \end{pmatrix} \begin{pmatrix} 0&2&2&2&2\\ 2&0&2&2&0\\ 0&2&0&2&2\\ 0&0&2&0&2\\ 2&2&0&2&0 \end{pmatrix} \begin{pmatrix} 0&2&2&2&2\\ 2&0&2&0&0\\ 0&2&0&2&2\\ 0&2&2&0&2\\ 2&2&0&2&0 \end{pmatrix} \begin{pmatrix} 0&2&2&2&2\\ 2&0&2&2&2\\ 0&2&0&2&2\\ 0&0&2&0&2\\ 2&0&0&2&0 \end{pmatrix}$$

The matrix digraphs for the partition $(1, 1, 1, 1, 1)$ are shown in Figure 2.



(The four orientations of M)

Figure 2

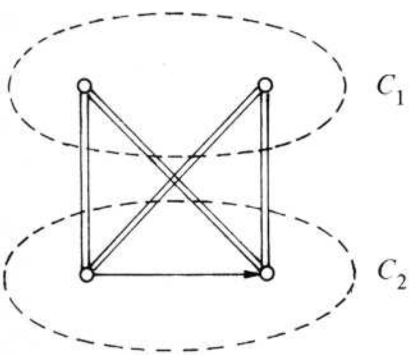
(refer to Example 3)

Example 4. Let

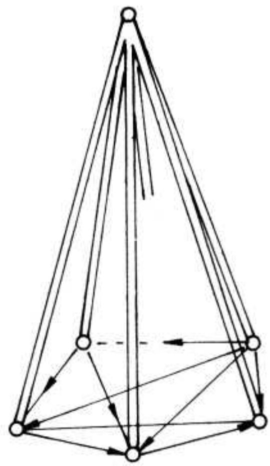
$$A = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \qquad (q = 1),$$

$$B_r = \begin{pmatrix} 0 & 2 & 2 & 2 & \dots & 2 \\ 2 & 0 & 2 & 2 & \dots & 2 \\ 2 & 0 & 0 & 2 & \dots & 2 \\ . & . & . & . & \dots & . \\ 2 & 0 & 0 & 0 & \dots & 2 \\ 2 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(B_r is an $r \times r$ matrix whose entries are all zeros and 2's, with the latter in the following locations: all positions strictly above the main diagonal; and in the first column below the first row.) The density of A is $\frac{4}{3}$, with optimum vector $\mathbf{u} = \left(\frac{1}{3}, \frac{2}{3}\right)$. The density of B_r is $\frac{4}{3} - \frac{4}{9r-6}$, with optimum vector $\mathbf{v} = \left(\frac{r}{3r-2}, \frac{2}{3r-2}, \frac{2}{3r-2}, \dots, \frac{2}{3r-2}\right)$; B_r has the structure of twice an adjacency matrix for $A\langle(1, r-1)\rangle$. The digraphs $A\langle(2, 2)\rangle$ and $B_r\langle\mathbf{e}\rangle$ are sketched in Figure 3. Observe that $B_r\langle\mathbf{e}\rangle$ has the same structure as $A\langle(1, r-1)\rangle$, but that neither realizes the optimum distribution of vertices proportional to $\mathbf{v}$.



$A\langle(2, 2)\rangle$



$B\langle\mathbf{e}\rangle$

Figure 3

(refer to Example 4)

In a forthcoming paper [4] we shall present *an algorithm for determining*, for $q = 1$ and for any family $\mathscr{L}$, *a matrix* A *for which* $\{A(n)\}$ *is asymptotically extremal* for $\mathscr{L}$; while we are unable, as yet, to completely characterize dense matrices, we do know certain necessary conditions for a matrix to be dense. One of the simplest is the following, whose proof for the case $q = 1$ follows from that of Lemma 1 of [1]. (The Lemma enunciated there was, however, weaker than the present.)

Lemma 1. *In a dense matrix* $\{a_{ij}\}$, $a_{ij} + a_{ji}$ *is never zero; moreover, if* $a_{ii} = a_{jj}$, *then* $a_{ij} + a_{ji} > 2a_{ii}$ $(i \neq j)$.

Proof. Here, as elsewhere in the paper, we shall appeal to results proved earlier for the case $q = 1$, where the original proof generalizes to meet our present needs without surprises. As observed in that paper, the essence of this lemma has appeared before in the work of Motzkin and Straus [11] and Zykov [16]. ∎

The following lemma characterizes the optimum vector of a dense matrix.

Lemma 2. *Let* $\{\mathbf{x}_n\}$ *be a sequence of optimal vectors corresponding to a sequence* $\{A(n)\}$, *where* A *is dense. Then the vectors* $\frac{1}{n}\mathbf{x}_n$ *converge in the Euclidean norm to a vector* $\mathbf{u}$ *which is uniquely determined by the system of equations*

$$(3.2) \qquad (\mathbf{u}, \mathbf{e}) = 1, \quad \hat{A}\mathbf{u} = g(A)\mathbf{e}.$$

Proof. The proof on page 84 of [1] generalizes. (3.2) admits the following graph-theoretical interpretation. Given a dense matrix A and vector $\mathbf{x}$, let $\mathbf{y} = \mathbf{x}\hat{A} = (y_1, \dots, y_r)$. Then $y_i - a_{ii}$ is precisely the valency of vertices in the i-th class of $A\langle \mathbf{x} \rangle$: a_{ii} is substracted only because the loops are excluded. Thus (3.2) asserts that an optimal $A\langle \mathbf{x} \rangle = A(n)$ is "almost regular". This was proved in [1] by showing that the difference of any two valencies in any $A(n)$ is at most $2q$. ∎

Remark. The following assertion is equivalent to Lemma 2:

Any vector $\mathbf{u}$ *for which* $\mathbf{u}A\mathbf{u}^*$ *is maximum under the condition* $(\mathbf{u}, \mathbf{e}) = 1$ *satisfies* (3.2).

Indeed, if $\mathbf{u}A\mathbf{u}^*$ $(= \mathbf{u}\hat{A}\mathbf{u}^*)$ is maximal, then its value is $g(A)$, by definition. A being dense, $\mathbf{u}$ can only be an interior point of the standard simplex; by the Lagrange method, it can be seen to satisfy (3.2). On the other hand, if $\mathbf{u}$ satisfies (3.2), then $\mathbf{u}A\mathbf{u}^* = \mathbf{u}\hat{A}\mathbf{u}^* = (\mathbf{u}, g(A)\mathbf{e}) = g(A)$.

4. MATRIX GRAPHS AND QUADRATIC FORMS. MATRIX COLOURINGS. ENUNCIATION OF THEOREM 1

If we adjoin to each vertex in the i-th class of a matrix digraph $A\langle\mathbf{x}\rangle$ exactly a_{ii} loops $(i = 1, 2, \dots, r)$, we obtain a digraph (possibly no longer simple) having exactly $\frac{1}{2}(\mathbf{x}A\mathbf{x}^*)$ arcs. It follows that if loops are excluded – which is our intention in this paper –

$$(4.1) \qquad \mathbf{x}A\mathbf{x}^* - (2q-1)\sum_i x_i \leqslant 2e(A\langle\mathbf{x}\rangle) \leqslant \mathbf{x}A\mathbf{x}^*$$

(cf. [1], (1)): herein lies the advantage of our convention that twice the multiplicity of arcs joining C_i to C_j is taken as a_{ij} $(i \neq j)$. Moreover,

$$(4.2) \qquad (g(A) - o(1))n^2 \leqslant 2e(A(n)) \leqslant g(A)n^2.$$

In fact,

$$(4.2^*) \qquad g(A)n^2 - O(n) \leqslant 2e(A(n)) \leqslant g(A)n^2 \quad \text{as} \quad n \to \infty$$

(cf. [1], (3), (4)). Statement (4.2*) follows from (4.1) using the easily proven fact that $g(A)$ can be realized by a vector with rational coordinates.

The preceeding inequalities permit the following version of Lemma 2, which we state as a corollary:

Corollary to Lemma 2. *Let A be a dense matrix. Subject to the constraint that $\mathbf{u}\mathbf{e} = 1$, the maximum of each of the quadratic forms $\mathbf{u}A\mathbf{u}^*$, $\mathbf{u}A^*\mathbf{u}^*$ and $\mathbf{u}\hat{A}\mathbf{u}^*$ is attained uniquely for a vector $\mathbf{u}$ satisfying*

$$\hat{A}\mathbf{u}^* = 2g(A)\mathbf{e}.$$

Moreover, if $\{\mathbf{x}_n\}$ is a sequence in R^r for which $A\langle\mathbf{x}_n\rangle$ is optimal for n sufficiently large, then, for this unique vector $\mathbf{u}$, $\frac{1}{n}\mathbf{x}_n \to \infty$, as $n \to \infty$.

In [1] the following theorem was proved:

Theorem A1 (Theorem 1 of [1]). *Let $q = 1$. For any finite or infinite family L of simple digraphs there exists a dense matrix A for which any sequence $\{A(n)\}$ is asymptotically extremal. (By* (iii') *of Definition* 2, $\mathbf{b}$ *may be taken to be* $\mathbf{0}$ *when* $q = 1$.*)*

We are not yet able to prove the following

Conjecture 1. *Theorem* A1 *holds without restrictions on q, with $\{A(n, \mathbf{b})\}$ as the asymptotically extremal sequence, for appropriate (dense) A and* $\mathbf{b}$.

Remark. The following theorem is not difficult to prove for any maximum multiplicity q: For every finite or infinite family $\mathscr{L}$ of sample digraphs, and for every $\epsilon > 0$, there exists a dense matrix $A = A(\mathscr{L}, \epsilon)$ such that $A(n)$ contains no prohibited subdigraphs $L \in \mathscr{L}$ and

$$\mathrm{ex}\,(n, \mathscr{L}) \leqslant e(A(n)) + \epsilon n^2 .$$

In this sense we can approximate the solution to any extremal digraph problem with a sequence $\{A(n)\}$; but, as $\epsilon \to 0$, the size of A need not remain bounded. The truth of Conjecture 1 would imply that such an approximation exists within $o(n^2)$ rather than ϵn^2 (see [5*]).

In our first paper [1] we also announced the case $q = 1$ of the following theorem, which solves the "inverse" problem to Theorem A1.

Theorem 1. *For every dense matrix A and vector* $\mathbf{b}$ *there exists a finite family $\mathscr{L}$ of prohibited subdigraphs such that*

(i) *optimal matrix digraphs $A(n)$ are extremal digraphs for $\mathscr{L}$, and there are no other extremal digraphs for $\mathscr{L}$;*

(ii) *any asymptotically extremal sequence $\{G^n\}$ for $\mathscr{L}$ can be obtained from a sequence $\{A(n)\}$ by deleting and/or adjoining $o(n^2)$ arcs;*

(iii) *if $\{B(n)\}$ is an asymptotically extremal sequence of optimal* *matrix digraphs, where B is dense, then B and A are identical up to the same permutation of rows and columns.*

Theorem 1 shows that Theorem A1 is sharp. (Part (iii) of Theorem 1 was not included in the formulation in [1]. It will be seen to be a simple consequence of Part (ii), cf. Lemma 10 below.)

Our aim in the present paper is to prove Theorem 1 for arbitrary q and $\mathbf{b}$. The main innovation is the finiteness of $\mathscr{L}$. We shall first provide, for motivational purposes, a proof which permits that $\mathscr{L}$ may be infinite. Before that we indicate similarities with the chromatic theory of graphs.

Definition 5. Let $A = \{a_{ij}\}_{i,j \leqslant r}$ be a dense matrix, and $\mathbf{b} = (b_1, b_2, \ldots, b_r)$ an associated vector satisfying the conditions of Definition 1. A digraph G is $(A, \mathbf{b})$*-coloured* (or simply A*-coloured*), when its vertices are partitioned into classes $C_1, C_2, \ldots, C_r$ corresponding to the rows of A, and when the vertices of each of these classes are so ordered that

(i) any vertex in class C_i is connected to a successor in that class by at most b_i arcs, and to a predecessor in that class by at most $a_{ii} - b_i$ arcs;

(ii) any vertex in class C_i is connected to any vertex in a distinct class C_j by *at most* $\frac{1}{2} a_{ij}$ arcs.

Thus an A-colouring of G is an embedding in $A(\infty)$; G is A*-colourable* if such as embedding exists. All subdigraphs of an A-colourable digraph are A-colourable. Since the sizes of classes of optimal matrix digraphs $A(n)$ tend to ∞ with n, a digraph is A-colourable if and only if it can be embedded in some $A(n)$. The family of digraphs on at most m vertices having no A-colouring will be denoted by $\mathscr{A}_m$. (The preceding is a true generalization of vertex-colouring of ordinary graphs. We could define for any directed graph G, the *chromatic number* χ to be

$$\frac{1}{1 - \min\{g(A)\colon\ G \text{ is } A\text{-colourable}\}}.$$

This number is not altered if orientations of any arcs of G are changed;

in particular, it is constant for all orientations of an ordinary graph, and equal to the usual chromatic number.)

First we sketch a proof of a weakening of the first parts of Theorem 1, where $\mathscr{L}$ is permitted to be infinite. Let A be a dense matrix, and $\mathbf{b}$ a vector, satisfying the conditions of Definition 2, and let $\mathscr{A}_\infty$ denote the family of all non-$(A, \mathbf{b})$-colourable digraphs. We show that the weakened version of Theorem 1 holds with $\mathscr{L} = \mathscr{A}_\infty$. Indeed, a digraph G^n contains no prohibited subdigraphs if and only if it is A-colourable. (The subdigraphs of an A-colourable digraph are A-colourable.) Therefore, the extremal digraphs for $\mathscr{A}_\infty$ must be A-colourable with the maximum number of arcs, i.e. optimal matrix digraphs $A(n)$: this proves (i). Let now $\{G^n\}$ be any asymptotically extremal sequence. Each G^n must be A-colourable, i.e. must be a subdigraph of some $A\langle \mathbf{y} \rangle$, where the sum of the entries of $\mathbf{y}$ is n. (We require a y_n for each G^n, but shall not always show the dependence on n explicitly in our notation.) Since

$$e(G^n) = e(A(n)) - o(n^2) \geqslant e(A\langle \mathbf{y} \rangle) - o(n^2) \qquad \text{as } n \to \infty,$$

G^n can be obtained from the maximal A-coloured digraph $A\langle \mathbf{y} \rangle$ through deletion of $o(n^2)$ arcs. Let $\mathbf{v}$ be a limit point of the vectors $\frac{1}{n}\mathbf{y}_n$.

$$(\mathbf{v}A\mathbf{v}^* + o(1))n^2 = 2e(A\langle \mathbf{y} \rangle) = \mathbf{y}A\mathbf{y}^* + O(n) = \qquad \text{by (4.1)}$$

$$= 2e(A(n)) + o(n^2) = g(A)n^2 + o(n^2), \qquad \text{by (4.2)}$$

so $\mathbf{v}A\mathbf{v}^* = g(A)$. As $\mathbf{v}$ realizes the maximum of the form in the standard simplex, and A is dense, $\mathbf{v}$ satisfies equations (3.2), and is the unique vector with that property. But then, if $\mathbf{x}\,(= \mathbf{x}_n)$ is an optimum vector for some $A(n)$, the sequences $\left\{\frac{1}{n}\mathbf{x}_n\right\}$ and $\left\{\frac{1}{n}\mathbf{y}_n\right\}$ both approach $\mathbf{v}$ as $n \to \infty$; it follows that $|\mathbf{x} - \mathbf{y}| = o(n)$, so the number of edges that need to be changed to convert $A\langle \mathbf{x} \rangle$ into $A\langle \mathbf{y} \rangle$ is $o(n)2qn = o(n^2)$.

Remark. We shall prove that if m is sufficiently large, Theorem 1 holds with $\mathscr{A}_m$ for $\mathscr{L}$. It is natural to try to prove this, since conclusion (i) of the Theorem implies that $\mathscr{L}$ must consist solely of non-A-colourable digraphs; and any finite collection of such digraphs is a subcollection of some $\mathscr{A}_m$. If $\mathscr{L} = \mathscr{A}_m$ satisfies Theorem 1 for some m, any $\mathscr{L} = \mathscr{A}_{m'}$ with $m' > m$ will also do. Indeed,

$$\mathrm{ex}\,(n,\ \mathscr{A}_{m'}) \geqslant g(A)\,\frac{n^2}{2} + o(n^2) \quad \text{as} \quad n \to \infty$$

since no $A(n)$ contains subgraphs from $\mathscr{A}_{m'}$; and if for some m the reverse inequality holds (up to $o(n^2)$), the inclusion $\mathscr{A}_m \subset \mathscr{A}_{m'}$ implies its truth for m' as well.

A basic result needed for our proof of Theorem 1 is

Lemma 3. *For every positive integer s and dense matrix A there exist an integer $m = m(s, A)$ and a positive constant $c_1 = c_1(s, A)$ such that any digraph G^n containing neither an optimal matrix digraph $A(s)$ nor subdigraphs from $\mathscr{A}_m$ has fewer than $\dfrac{(g(A) - c_1)n^2}{2}$ arcs.*

We shall prove Lemma 4 below, which will be seen to be equivalent to the preceding.

Lemma 4. *For every positive integer s and dense matrix A there exists an integer $m = m(s, A)$ such that for any sequence $\{G^n\}$ of digraphs which contain no subdigraph from $\mathscr{A}_m$, and which satisfy*

$$e(G^n) \geqslant (g(A) - o(1))\,\frac{n^2}{2} \quad \text{as} \quad n \to \infty \tag{4.3}$$

each G^n contains, for sufficiently large n, a subdigraph of structure $A(s)$.

Proof of the equivalence of Lemma 3 and Lemma 4. Evidently Lemma 3 implies Lemma 4. Assume Lemma 4, and fix an m provided by it. Should Lemma 3 fail for this m, it would follow that there exist for every $k > 0$ an integer $n > k$ and a digraph G^{n_k} having at least $\frac{1}{2}\,g(A)n^2 - \frac{1}{k}\,n^2$ arcs containing neither an $A(s)$ nor members of $\mathscr{A}_m$, contradicting Lemma 4. ∎

We first prove a weakened form of Lemma 3, where we permit m to be infinite.

Lemma 5. *For every positive integer* s *and dense matrix* A *there exists a constant* $c_2 = c_2(s, A) > 0$ *such that if* G^n *is A-colourable and does not contain an* $A(s)$, *then*

$$e(G^n) \leqslant (g(A) - c_3)\frac{n^2}{2}.$$

Proof of Lemma 5. Since A is dense, there exists a positive constant c_3 such that all principal proper submatrices of A have density less than $g(A) - 2c_3$. By Lemma 2 the optimum vector of A lies in the interior of the standard simplex. Continuity of the quadratic form ensures that there exists a constant $c_4 > 0$ such that if $u_1 + u_2 + \ldots + u_r = 1$ and some $u_i \leqslant c_4$, then

(4.4) $\quad \mathbf{u}A\mathbf{u}^* \leqslant g(A) - c_3.$

Thus, by (4.1), when $x_1 + x_2 + \ldots + x_r = n$,

(4.5) $\quad e(A\langle \mathbf{x}\rangle) < (g(A) - c_3)\frac{n^2}{2}$

if, for some i, $x_i \leqslant c_4 n$.

An A-coloured digraph G^n has the structure of a subdigraph of $A\langle \mathbf{x}\rangle$, where $\sum_i x_i = n$. The lemma will hold with $c_2 = c_3$ unless all x_i exceed $c_4 n$, which we now assume. The remainder of the proof is "probabilistic". Then $A\langle \mathbf{x}\rangle$ contains at least

(4.6) $\quad \binom{x_1}{s_1}\binom{x_2}{s_2}\ldots\binom{x_r}{s_r} > c_5 n^s$

copies of $A(s)$, where $s_1, s_2, \ldots, s_r$ are the sizes of the classes of any fixed $A(s)$. Each edge of $A\langle \mathbf{x}\rangle$ not belonging to G^n is contained in at most $c_6 n^{s-2}$ copies of $A(s) \subset A\langle \mathbf{x}\rangle$, hence we would have to delete at least $\frac{c_5}{c_6}n^2$ arcs of $A\langle \mathbf{x}\rangle$ to eliminate all $A(s)$'s, and

$$(4.7) \qquad e(G^n) \leqslant e(A\langle \mathbf{x}\rangle) - \frac{c_5}{c_6} n^2.$$

It suffices to choose c_2 as the smaller of c_3 and $2\dfrac{c_5}{c_6}$. ∎

The proof of Lemma 4 appears in Section 7.

5. UNIQUE A-COLOURINGS. PSEUDO-A-COLOURINGS

Can a matrix digraph be recoloured in an essentially different way? We claim not – provided that all classes of the original colouring are sufficiently large. The following result will be required in our proof of Theorem 1, to extend a colouring of a "small" matrix subdigraph to a larger one.

Lemma 6. *Let A be a dense $r \times r$ matrix, and $A\langle \mathbf{x}\rangle$ a matrix digraph, for which $x_i > r$ $(i = 1, 2, \ldots, r)$. Then all A-colourings of $A\langle \mathbf{x}\rangle$ yield the same partition of vertices (into unordered classes).*

Remark. Lemma 6 generalizes properties of complete multipartite graphs. It is completely trivial when A has zero diagonal; the vertices of any class are independent, but, by Lemma 1, $a_{ij} + a_{ji}$ is always positive; thus vertices of different classes are always joined by at least one arc, and the "colour" classes of vertices are simply the maximal independent sets.

In our proof of Lemma 6 we shall consider the effect of a temporary *suppression of orientation*. More precisely, a partition of the vertices of a digraph is a *pseudo-A-colouring* if either

(i) it is an A-colouring; or

(ii) it can be transformed to an A-colouring through reversal of the orientations of some arcs.

For a matrix A, pseudo-A-colourability depends, like the property of being dense (cf. (3.1)), only on "symmetric part" of A, i.e. $\frac{1}{2}(A + A^*) = \hat{A}$.

Proof of Lemma 6.

(A) Let $A\langle \mathbf{x}\rangle$ satisfy the stated conditions, with colour classes $C_1, C_2, \ldots, C_r$. The second colouring results from a second embedding into $A(\infty)$; the classes of this $A(\infty)$ will be denoted by $C_1^*, C_2^*, \ldots, C_r^*$: Since $|C_i| > r$, there exists for each i an integer $p(i)$ such that $C^*_{p(i)}$ contains two or more vertices of C_i $(i = 1, 2, \ldots, r)$.

(B) We propose to show that the integers $p(1), p(2), \ldots, p(r)$ are distinct. By assuming the contrary we shall argue that the colouring may be transformed into a pseudo-A-colouring for which the classes C_i will be each contained entirely in one of the classes C_j^*, and in at least one case two C_i's will together belong to the same C_j^*. This will imply that a submatrix of A has density equal to $g(A)$, a contradiction.

(C) We first observe that if a vertex of C_j is contained in $C^*_{p(i)}$, then there exists another pseudo-A-colouring in which all of $C_i \cup C_j$ is concentrated in one colour class. For the following inequalities must surely hold:

$$a_{ii} \leqslant a_{p(i),p(i)}$$

$$\frac{1}{2}(a_{ij} + a_{ji}) < a_{p(i),p(i)}$$

$$a_{jj} \leqslant a_{p(j),p(j)}$$

and one of

$$a_{jj} \leqslant a_{p(i),p(i)} \tag{5.1}$$

$$a_{jj} > a_{p(i),p(i)}. \tag{5.2}$$

When (5.1) holds (in particular, when $i = j$), we may absorb all of $C_i \cup C_j$ into $C^*_{p(i)}$. Otherwise (5.2) entails that

$$a_{ii} < a_{p(j),p(j)}$$

$$\frac{1}{2}(a_{ij} + a_{ji}) < a_{p(j),p(j)}$$

and so the union may be absorbed into $C^*_{p(j)}$.

(D) Through repetitions of the procedure of paragraph (C), we obtain a pseudo-A-colouring in which each $C^*_{p(i)}$ contains all vertices of C_i. Moreover, the integers $p(i)$ are not distinct. Consequently, some class C^*_s contains no vertices of $A\langle \mathbf{x}\rangle$. If A' is the principal submatrix of A obtained through deletion of a corresponding row and column, then every $A(n)$ admits a pseudo-A'-colouring. A pseudo-A'-colouring of the digraph G^n cannot have more than $\frac{1}{2}g(A')n^2$ arcs. Thus

$$g(A) = \lim_{n\to\infty} \frac{2e(A(n))}{n^2} \leqslant g(A')$$

contradicting the assumption that A is dense. (Note that, while every $A(n)$ admits a pseudo-A'-colouring, we have not proved that it admits an A'-colouring!) ∎

6. AUGMENTATION OF MATRICES

To prove Lemma 4 we require the notion of *augmentation of matrices,* developed in [1]. Lemma 7 and Lemma 9 below are generalized restatements of Lemmas 2 and 4 of that paper, to which the reader is referred for proofs for the case $q = 1$ (which generalize to meet present needs without difficulty.)

Let A be an $r \times r$ dense matrix, m an integer and $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ an r-vector for which some optimum $A(m) = A\langle\mathbf{x}\rangle$. We construct a new graph by taking x_{r+1} new vertices to form a new class C_{r+1}, and then joining every vertex of the new class to every vertex of the i-th class in exactly the same way (i.e. the same numbers of arcs in each direction $(i = 1, 2, \ldots, r)$; the valencies of the vertices in the new class will have to be "sufficiently large". Then we change the proportions of vertices allotted to the various $r+1$ classes so that the digraph obtained is an optimum digraph $A(m + x_{r+1}))$: even the relative proportions of the numbers of vertices in the original classes may change under this operation. This construction motivates the following definition:

Definition 6. Let $B = \{a_{ij}\}$ be an $(r+1)\times(r+1)$ matrix and A the submatrix obtained by deleting the last row and column. Suppose, moreover, that A is dense, with the optimum vector $\mathbf{u}$. If γ is any constant such that

$$(6.1) \qquad \frac{1}{2}\Big(\sum_j a_{r+1,j}u_j + \sum_j a_{j,r+1}u_j\Big) \geqslant \gamma > g(A)$$

we say that B is a *γ-augmentation* of A, obtained from A by *augmentation by* γ. The entry $a_{r+1,r+1}$ is not restricted. (In this paper, however, it suffices to take it to be zero in all cases.)

Remark. In the graph construction given to motivate Definition 6, the inequality

$$\frac{1}{2}\Big(\sum_j a_{r+1,j}x_j + \sum_j a_{j,r+1}x_j\Big) \geqslant \gamma n > g(A)n$$

means that each of the new vertices are joined to $A(m)$ by more arcs than the average valency of the vertices of $A(m)$.

Lemma 7. *If B is obtained from A by augmentation by γ, then*

$$(6.2) \qquad g(B) - g(A) > \frac{(\gamma - g(A))^2}{2\gamma - g(A)} > 0.$$

Proof. See the proof of Lemma 2 of [1]. ∎

The symbol $D(B)$ will be used to denote any fixed minimal principal submatrix of B of density $g(B)$. Clearly $D(B)$ is dense; it may coincide with B.

Given a dense matrix A, and a constant $\gamma > g(A)$, we can find first all the possible augmentations of A by this constant. Then, whenever an augmentation B is not dense, we replace it by some $D(B)$. We then augment all the matrices obtained with densities less than γ again by γ, taking $D(B)$ where necessary. In this way we obtain a directed graph which, in our forthcoming papers, will be called an *augmentation scheme* of A by γ. The vertices of this scheme are the dense matrices obtained through this procedure: we direct an edge from A to B whenever $B = D(B^{\#})$, where $B^{\#}$ is an augmentation of A by γ. By imposing additional conditions, we can specialize an *augmentation arborescence* in this scheme.

Remark. Whenever we speak of *iterative augmentation* we always permit that each augmentation may be followed by restriction to a maximal dense submatrix. As this restriction may involve the deletion of rows and columns, it may well happen that not all (or indeed any) of the original rows and columns are present after sufficiently many iterations (cf. however Lemma 3 of [1]).

Example 5. Let A be the matrix $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$.

(i) Its optimum vector is $\left(\frac{2}{7}, \frac{3}{7}, \frac{2}{7}\right)$, giving density $\frac{8}{7}$. This matrix admits (up to isomorphism) the following dense augmentations (under augmentation by any $\gamma > \frac{8}{7}$):

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ \hline 2 & 2 & 2 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ \hline 2 & 0 & 2 & 0 \end{array}\right] \quad \text{having densities respectively } \frac{7}{5} \text{ and } \frac{5}{4};$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ \hline 2 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 2 & 2 & 2 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ \hline 2 & 2 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ \hline 2 & 2 & 2 & 0 \end{array}\right]$$

all of density $\frac{7}{6}$,

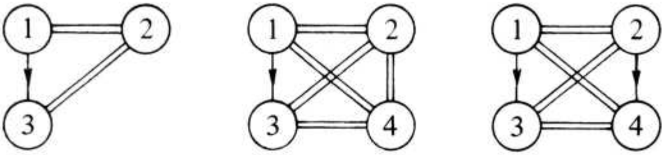
$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ \hline 0 & 2 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ \hline 2 & 2 & 0 & 0 \end{array}\right] \quad \text{both of density } \frac{6}{5}.$$

All other augmentations are not dense, but have density $= \frac{4}{3}$ and contain a 3×3 matrix of structure $2(J - I)$.

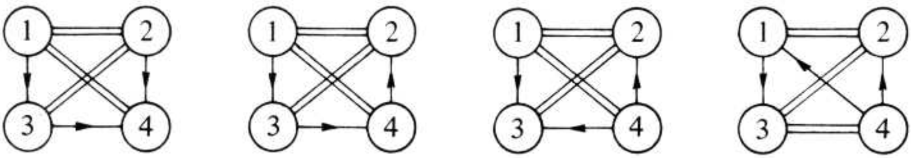
Cases (ii) are examples where restriction to a maximal dense submatrix entails the deletion of rows and columns. For example, in the augmentation

$$\left(\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & 0 \\ \hline 2 & 2 & 0 & 0 \end{array}\right)$$ density is not changed if the third row and third column are suppressed, although the optimum vector then has equal entries.

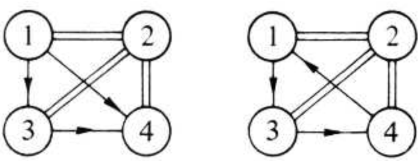
Sketches of the complements (for $q = 1$) of the associated digraphs are provided in Figure 4.



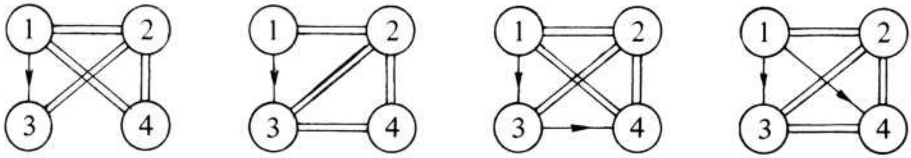
$A\langle(1, 1, 1)\rangle$ augmentation of density $\frac{7}{5}$ augmentation of density $\frac{5}{4}$



four augmentations of density $\frac{7}{6}$



two augmentations of density $\frac{6}{5}$



four augmentations of density $\frac{4}{3}$

Figure 4

(cf. Example 5)

(Note that the figure shows complements.)

The following lemma is not stated explicitly in [1]; however, it is an immediate consequence of Lemma 2 of that paper, or of Lemma 7 of the present.

Lemma 8. *For every* $\gamma > 0$ *and* $\epsilon > 0$, *there exists an integer* $K = K(\gamma, \epsilon)$ *such that if* B *is obtained from* A *by a sequence of at least* K *iterative augmentations by* γ, *then*

(6.3) $$g(B) > \gamma - \epsilon.$$

Proof. Define a real function f by

(6.4) $$f(x) = x + \frac{(\gamma - x)^2}{2\gamma - x} = \frac{\gamma^2}{2\gamma - x}.$$

We define a sequence $\{x_k\}$ recursively by

(6.5) $$x_0 = 0, \quad x_k = f(x_{k-1}) \qquad (k > 0).$$

The sequence is monotonely increasing, and converges to γ; let K be the first integer k for which x_k is within ϵ of γ. Lemma 7 ensures that $g(B) \geqslant x_K > \gamma - \epsilon$. ∎

The theorems we wish to prove are concerned with ensuring the existence of subdigraphs in digraphs having sufficiently many arcs. The following lemma, numbered 4 in [1], describes a connection between (abstract) augmentation of matrices (considered until now) and the subdigraphs of a digraph having more than $\frac{1}{2}g(A)n^2$ arcs.

Lemma 9. *For every dense matrix A, positive integer m, and $\epsilon > 0$, there exists an integer $m_1 = m_1(A, \epsilon, m)$ such that the conditions*

(i) *G^n contains $A(m_1)$;*

(ii) *all (total) valencies in G^n exceed $(g(A) + \epsilon)n$*

ensure, for n sufficiently large, that there exists a matrix B obtained from A by augmentation, containing a maximal dense submatrix $A^{\#} = D(B)$ such that G^n contains $A^{\#}(m)$.

Proof. The proof of Lemma 4 of [1] generalizes without significant change, beyond the replacement of several constants. ∎

7. PROOF OF LEMMA 4

Let A be an $r \times r$ dense matrix, s be a positive integer, and set $\epsilon = \frac{1}{3}c_2$, where $c_2 = c_2(s, A)$ is the constant of Lemma 5; let $K = K(g(A) - \epsilon, \epsilon)$ is the constant of Lemma 8 and $m = (r+1)K$. Let $\{G^n\}$ be a sequence of digraphs each containing no member of $\mathscr{A}_m$, and which satisfy (4.3). Define the graphs $\widetilde{G}^l$ successively by $\widetilde{G}^n = G^n$ and by deleting a vertex of valence $\leqslant (g(A) - \epsilon)l$ from $\widetilde{G}^l$ if $\widetilde{G}^l$ has such a vertex. Stop if such a vertices do not exist. Clearly,

$$e(G^n) - (g(A) - \epsilon) \sum_{i=l+1}^{n} i \leqslant e(\widetilde{G}^l) < \frac{1}{2} q l^2$$

which, together with (4.3), implies that the process will terminate for $l > c_7 n$, $c_7 > 0$. We propose to apply Lemma 9 iteratively to $\widetilde{G}^l$ $K = K(g(A) - \epsilon, \epsilon)$ of fewer times: in the first application we take the matrix (0) and $m(1) = l$. In the k-th application we have already defined a dense A_k and know that $G^n \supseteq \widetilde{G}^l \supseteq A_k(m_k)$ for an arbitrarily fixed m_k, if l (that is, n) is sufficiently large. Using Lemma 9 we infer that $\widetilde{G}^n \supseteq \widetilde{G}^l \supseteq A_{k+1}(m_{k+1}))$ for an arbitrary m_{k+1} and some $A_{k+1} = D(A')$, where A' is an augmentation of A_k by $\geqslant g(A) - \epsilon$. By Lemma 8 for the dense matrix $B := A_K$

$$(7.1) \qquad g(B) > (g(A) - \epsilon) - \epsilon > g(A) - \frac{2}{3} c_2 .$$

This B may depend on n, but, since its size is K, and each of its entry can be chosen in at most $(q + 1)$ ways, there are only $O(1)$ possibilities for this B. For each possible B, the optimum vector $\mathbf{u}$ of Lemma 2 has strictly positive coordinates. Since $\widetilde{m} = m_K$ can be chosen arbitrarily, we may assume that it tends to ∞, as $n \to \infty$. For n sufficiently large $G^n \supseteq \widetilde{G}^l \supseteq B(\widetilde{m}) \supseteq B\langle (r + 1)\mathbf{e} \rangle$. We have assumed that G^n contains no subgraphs from $\mathscr{A}_m$. In other words, all the subdigraphs of G^n of at most $m = (r + 1)K$ vertices are A-colourable. Thus $B\langle (r + 1)\mathbf{e} \rangle$ is also A-colourable. By Remark* (preceding Definition 3) every $B(\widetilde{m})$ is A-colourable (cf. Remarks, preceding Example 1). Now, by (4.2) and (7.1),

$$e(B(\widetilde{m})) = g(B) \frac{\widetilde{m}^2}{2} + o(\widetilde{m}^2) > \left(g(A) - \frac{2}{3} c_2 - o(1)\right) \frac{\widetilde{m}^2}{2} .$$

For sufficiently large $\widetilde{m}$ this implies that

$$(7.2) \qquad e(B(\widetilde{m})) > (g(A) - c_2) \frac{\widetilde{m}^2}{2},$$

from which Lemma 5 enables us to conclude that $B(\widetilde{m})$ contains an $A(s)$. On two occasions above we have required $\widetilde{m}$ to be "sufficiently large": Lemma 9 will ensure this if n is sufficiently large, i.e. if we confine ourselves to sufficiently advanced members of the sequence $\{G^n\}$. ∎

8. PROOF OF THEOREM 1

Let $\mathbf{u} = (u_1, u_2, \ldots, u_r)$ be the optimum vector of A, i.e. satisfying (3.2); $\mathbf{u}$ must have rational positive coordinates. We choose an integer t such that $t\mathbf{u}$ has integer coordinates, each exceeding r. Lemma 2 ensures the existence of an integer s such that, for every $n \geqslant s$, any $A(n)$ contains $A\langle t\mathbf{u} \rangle$. Fix an integer $m \geqslant t + 3$ for which Lemma 4 holds for s. We shall prove that $\mathscr{L} = \mathscr{A}_m$ has the desired properties.

(A) First we show that the digraphs $A(n)$ (not uniquely determined by n), are the only extremal digraphs for this $\mathscr{A}_m$. The method applied here will be one we call *"progressive induction"*, introduced by the third author in [12]. (This method can be applied to prove statements which are seem to be hereditary for large n, but where – perhaps the statement may not even be valid for small values of n – one cannot use induction. We forego a detailed description of the technique, but describe below its application to our problem in such a way that the method should be understandable in itself.)

(B) Let $\{S^n\}$ be a sequence of extremal digraphs for $\mathscr{A}_m$. Since no $A(n)$ contains prohibited subdigraphs, $e(A(n))$ is a lower bound for $\mathrm{ex}\,(n, \mathscr{A}_m) = e(S^n)$. Define $d(n)$ to be the *excess* of $e(S^n)$ over $e(A(n))$:

$$d(n) = e(S^n) - e(A(n)).$$

We recall that this last number is independent of the particular optimum digraph chosen. We wish to prove that $d(n) = 0$ for n sufficiently large. For n sufficiently large Lemma 4 and (4.2) ensure that S^n must contain an $A(s)$, and hence a digraph H of structure $A\langle t\mathbf{u}\rangle$; by our definition of s, any $A(n)$ must also contain a subdigraph of the same structure $A\langle t\mathbf{u}\rangle$. We denote this also by H. Let H', H'' respectively denote the digraphs obtained from some fixed $A(n)$ and from S^n by deleting all vertices in H and all incident arcs – i.e., $H' = A(n) - H$, $H'' = S^n - H$; H' has the structure of a matrix digraph.

(C) If H is contained in a matrix graph $A\langle \mathbf{x}\rangle$ (which, by Lemma 6, can happen only in the obvious way if $\sum_i x_i$ is sufficiently large!) then a vertex p of $A\langle \mathbf{x}\rangle - H$ belonging, say, to the i-th class of $A\langle \mathbf{x}\rangle$, is adjacent with vertices of H by exactly

$$\frac{1}{2} \sum_{\substack{j \\ j \neq i}} (a_{ij} + a_{ji}) t u_j + a_{ii} t u_i = tg(A)$$

by (3.2). More generally, if H is contained in any A-colourable digraph G, and p is a vertex of $G - H$, then p is adjacent with H by at most $tg(A)$ arcs.

(D) Thus any vertex of H' is adjacent to or from vertices of H by a total of exactly $tg(A)$ arcs. It follows that the partition of vertices into classes in H' is optimal – otherwise a rearrangement of its vertices could increase the number of arcs in $A(n)$, without altering the numbers of arcs connecting it to H; i.e. H' is of structure $A(n-t)$. To simplify the following calculations, we shall make comparisons between $A(n)$ and S^n. The difference

$$
\begin{aligned}
& d(n) - d(n-t) = \\
& \quad = \{e(S^n) - e(A(n))\} - \{e(S^{n-t}) - e(A(n-t))\} = \\
& \quad = \{e(S^n) - e(S^{n-t})\} - \{e(A(n)) - e(H')\} \leqslant \\
(8.2) \qquad & \quad \leqslant (e(S^n) - e(H'')) - g(A)t(n-t) - e(A\langle t\mathbf{u}\rangle) \\
& \qquad\qquad \text{since } S^{n-t} \text{ is extremal} = \\
& \quad = \{e(S^n) - e(H'') - e(H)\} - g(A)t(n-t)\}.
\end{aligned}
$$

To estimate the first expression we observe that any vertex p in H'' spans together with H an A-colourable digraph, since $t+1 \leqslant m$. By the argument of paragraph (C) above, p is joined to H by at most $g(A)t$ arcs, thus $e(S^n) - e(H'') - e(H)$ – which is precisely the number of arcs between H and H'' in S^n – is bounded above by $g(A)t(n-t)$; from which follows that

$$d(n) \leqslant d(n-t).$$

Of course, we cannot claim that this holds for small n: earlier we applied Lemma 2, and required that $n > s$. Nevertheless, it follows that there exists within each congruence class of integers modulo t an integer beyond which all integers n satisfy

$$(8.3) \qquad d(n) = d(n-t);$$

the number of such congruence classes being finite, (8.3) must hold for all sufficiently large n. Returning to (8.2), we see that this implies that $e(S^n) - e(H'') - e(H) = g(A)t(n-t)$, so the average number of arcs connecting a vertex of H'' to H is $g(A)t$. Again by the argument of paragraph (C), this number can never be exceeded, and all vertices of H'' are joined to H by exactly $g(A)t$ arcs. Moreover, the digraph spanned by H and each vertex of H'' is A-colourable. The strict lower bound of r imposed earlier on the size of the classes of H (of structure $A\langle t\mathbf{u}\rangle$) ensures that Lemma 6 is applicable. Thus the vertices of H'' are partitioned into classes $C_1'', C_2'', \ldots, C_r''$ respectively associated with the colour-classes $C_1, C_2, \ldots, C_r$ of H. We prove now that this partition is an A-colouring, using the assumption that all "small" subdigraphs of S^n are A-colourable. All subdigraphs consisting of H and 2 vertices of H'' in distinct classes C_i'', C_j'' are surely A-colourable – so there can be no violations to A-colourability caused by arcs between classes; also, subdigraphs spanned by H and up to 3 vertices from the same class C_i'' are A-colourable. This last observation ensures that an ordering of all vertices in the class of H'' in question, together with those of the associated class C_i of H, induced by the out-valencies (which must all be within the bounds imposed by the diagonal elements of the matrix A and the associated vector $\mathbf{b}$) within the united classes, must be transitive. Since S^n is an A-colourable extremal digraph, it must be maximal with respect to A-colouring; indeed, it must be an $A(n)$.

(E) Let $\mathbf{u}, t, n$ be defined as above, and $\{S^n\}$ now be an asymptotically extremal sequence for $\mathscr{A}_m$; let $g = g(A)$. We shall prove a statement slightly stronger than (ii): namely, that deletion of $o(n)$ vertices from S^n renders the digraph A-colourable. Fix $\epsilon > 0$, and define $h = h(n, \epsilon)$ to be the number of vertices of valency at most $(g-\epsilon)n$. By deleting h' of those h vertices and their incident arcs (at most $h'(g-\epsilon)n$ in number), we obtain a subdigraph having at most $\operatorname{ex}(n-h', \mathscr{A}_m)$ arcs; but S^n has $\operatorname{ex}(n, \mathscr{A}_m) + o(n^2)$ arcs; h' will be further restricted below. Combining, and applying (4.2), we obtain the inequality

$$(8.4) \qquad \frac{1}{2} gn^2 - (g-\epsilon)nh' < \frac{1}{2} g(n-h')^2 + o(n^2)$$

implying

$$\epsilon nh' \leqslant \frac{1}{2} gh'^2 + o(n^2)$$

and

$$(8.5) \qquad \left(\frac{\epsilon n}{g}\right)^2 \leqslant \left(h' - \frac{\epsilon n}{g}\right)^2 + o(n^2).$$

Suppose that $h(n, \epsilon) > \epsilon n$ for infinitely many n; for sufficiently large n, we could choose h' to be $\left[\frac{\epsilon n}{g}\right]$, and then (8.5) would yield

$$\left(\frac{\epsilon n}{g}\right)^2 \leqslant O(1) + o(n^2) \quad \text{as} \quad n \to \infty,$$

a contradiction. We conclude that $h(n, \epsilon) \leqslant \epsilon n$ for $n > N(\epsilon)$. Denote by $G' = G'(n, \epsilon)$ the digraph on $n - h$ vertices obtaind by deleting from S^n all h vertices of valency less than $(g - \epsilon)n$: this deletion reduces the valencies of some remaining vertices, but only by at most $2q\epsilon n$ each: thus no valency is less than $(g - \epsilon(1 + 2q))n \geqslant g(n - h) - \epsilon O(n - h)$ as $n \to \infty$. These subdigraphs G'^{n-h} of S^n contain no member of $\mathscr{A}_m$. For each n define $l(n) = \operatorname{Max}\{j\colon n > N(2^{-j})\}$; the sequence $\{G'(n, 2^{-l(n)})\}_n$ satisfies (4.3). Our initial selection of s and m is now rewarded – Lemma 4 provides that each contains a subdigraph of structure $A(s)$, which, in turn, must contain a subdigraph H of structure $A\langle t\mathbf{u}\rangle (= A(t)$, since in this case the optimal digraph is unique); and all valencies are at least $(g - o(1))(n - h)$. Define H'' as in (B) to be $S^n - H$. This last fact implies that H'' is connected to H by at least $(g - o(1))(n - h)t$ arcs. The choice of $m > t$ ensures that no outside vertex is connected with H by more than gt arcs (cf. paragraph (C) above). The number of vertices of H'' connected by fewer than $gt - 1$ arcs is $o(n)$, by a simple computation. Deleting those vertices, we obtain an A-colourable subdigraph on $n - o(n)$ vertices, all valencies of which are at least $(g - o(1))n$. We have thus proved slightly more than claimed in the theorem: the $o(n^2)$ arcs to be deleted can all be taken incident with a fixed $o(n)$ vertices!

(F) Part (iii) of the theorem follows immediately from Part (ii) via the following lemma.

Lemma 10. *Let A, B be dense matrices such that, for infinitely many n, some $A(n)$ can be transformed into some $B(n)$ through addition and deletion of $o(n^2)$ arcs. Then A can be transformed into B through like permutations of rows and columns.*

Proof. By (4.2), $g(A) = g(B)$. By Lemma 2, we may choose an integer s so large that every class of every $A(s)$ and of every $B(s)$ contains at least $r + 1$ vertices, where r denotes the maximum number of rows of A and of B. If an $A(n)$ can be transformed into some $B(n)$ through the addition and deletion of $o(n^2)$ arcs, then the deletion of $o(n^2)$ arcs (without additions) yields a digraph G^n which is both A- and B-colourable. For infinitely many n, (4.2) and Lemma 5 ensure that each $A(n)$ contains a $B(s)$, and hence it also contains a $B\langle 2\mathbf{e}\rangle$ such that each class of $B\langle 2\mathbf{e}\rangle$ is contained in once class of $A(n)$. This implies for arbitrary n that $B(n)$ is contained in $A(\infty)$ *canonically:* so that each class of $B(n)$ meets at most one class of $A(\infty)$. Further, each class of $A(\infty)$ must contain some class of $B(n)$, otherwise for some principal submatrix A' of A $A'(\infty)$ would also contain $B(n)$. This would imply that $g(B) \leqslant g(A') < g(A)$, a contradiction. Thus we obtained a permutation p mapping the classes of $B(n)$ onto the classes of $A(\infty)$ and mapping the rows of B onto the rows of A so that $b_{ij} \leqslant a_{p(i),p(j)}$. Here p must be a 1–1 mapping, that is, a permutation, indeed, since (exchanging A and B we get that) $A(n)$ can also be embedded canonically into $B(\infty)$, and each class of $B(\infty)$ must be used: A and B have the same size. Finally, if at least once we have strict inequality in $b_{ij} = a_{p(i),p(j)}$, then we have $g(B) < g(A)$, a contradiction, again. This proves the lemma. ∎

9. THE SET OF ATTAINED DENSITIES. A CONTINUITY THEOREM

Our early research (unpublished, [3]) into multigraph extremal problems included an extensive cataloguing for $q = 1$ of matrices of density less than $\frac{3}{2}$. The attained densities not exceeding $\frac{4}{3}$ were found to be

$$0, \frac{1}{2}, \frac{2}{3}, \ldots, 1 - \frac{1}{r}, \ldots, 1, \frac{8}{7}, \frac{6}{5}, \ldots, \frac{4r}{3r+1}, \ldots, \frac{4}{3}.$$

From this, and more abundant evidence in the interval from $\frac{4}{3}$ to $\frac{3}{2}$, we were led to formulate the following conjecture.

Conjecture 2.

(a) *The set* $D_q = \{g(A)\colon A$ *dense*$\}$ *is well ordered for all* q.

(b) *No density is realized by an infinite number of dense matrices.*

The truth of this conjecture would imply that of

Conjecture 2*. *For every indinite family $\mathscr{L}$ of prohibited digraphs there exists a finite family $\mathscr{L}^* \subset \mathscr{L}$ such that* $\mathrm{ex}\,(n, \mathscr{L}) = \mathrm{ex}\,(n, \mathscr{L}^*) - o(n^2)$ *as* $n \to \infty$, *and for which two families the dense matrices* A *for which* $A(n)$ *is asymptotically extremal are precisely the same.*

We have provided in Section 4 a simple proof of a weakening of Theorem 1 – permitting an infinite family $\mathscr{L}$. Thus Theorem 1 would also be a consequence of either conjecture. For the case $q = 1$, we are able to prove Conjecture 2. The proof will be included in our forthcoming paper [4] where we deal primarily with $q = 1$. In Section 10 of the present paper we shall prove a somewhat weaker result, for general q:

Theorem 2 (Continuity theorem). *For every family* $\mathscr{L}$ *of prohibited digraphs and every* $\epsilon > 0$, *there exists a finite subfamily* $\mathscr{L}^* \subset \mathscr{L}$ *for which*

$$\mathrm{ex}\,(n, \mathscr{L}) \leqslant \mathrm{ex}\,(n, \mathscr{L}^*) \leqslant \mathrm{ex}\,(n, \mathscr{L}) + \epsilon n^2 \tag{9.1}$$

for n *sufficiently large.*

10. PROOF OF THE CONTINUITY THEOREM

(A) The left inequality of (9.1) is trivial, since deletions from a family of forbidden subdigraphs cannot decrease the extremal number.

(B) Suppose that the right inequality fails for some $\epsilon > 0$. Then, for every finite subfamily $\mathscr{L}^*$ of $\mathscr{L}$, there exist infinitely many n such that

$$\mathrm{ex}\,(n, \mathscr{L}^*) > \mathrm{ex}\,(n, \mathscr{L}) + \epsilon n^2. \tag{10.1}$$

Let

$$\gamma = 2 \limsup \frac{\mathrm{ex}\,(n, \mathscr{L})}{n^2} \quad \text{as } n \to \infty.$$

We shall eventually derive a contradiction from (10.1), via Lemma 9. In preparation for this we study the effect of iterative augmentations by

$\gamma + \frac{\epsilon}{2}$ of the 1×1 matrix (0) (cf. Section 6).

(C) For non-negative integers k we define recursively a family $\mathcal{M}_k$ of dense matrices. $\mathcal{M}_0 = \{(0)\}$. Each member of $\mathcal{M}_k$ is obtained from $\mathcal{M}_{k-1}$ through augmentation of a matrix of the latter by at least $\gamma + \frac{\epsilon}{2}$ to obtain a matrix B, then taking $A = D(B)$, a dense principal submatrix of B having density $g(B)$; this construction to be carried out in all possible ways. However, we do not include in $\mathcal{M}_k$ any A for which $A(\infty)$ contains a prohibited digraph L in $\mathcal{L}$. Define $\mathcal{M} = \bigcup \mathcal{M}_k$. No matrix A in $\mathcal{M}$ can have density exceeding γ: for that would imply, by (4.2), that $e(A(n))$ exceeds $\operatorname{ex}(n, \mathcal{L})$ infinitely often, and so $A(\infty)$ contains a member of $\mathcal{L}$. Consequently, Lemma 8 implies that, as each matrix has only finitely many $\left(\gamma + \frac{\epsilon}{2}\right)$-augmentations, we are considering only a finite set of dense matrices in $\mathcal{M}$. We define a finite subfamily $\mathcal{L}^*$ of $\mathcal{L}$ as follows: wherever a matrix in $\mathcal{M}$ possesses a dense augmentation B for which $B(\infty)$ contains as a subdigraph a member L of $\mathcal{L}$, select one such prohibited subdigraph L; $\mathcal{L}^*$ is to be the set of prohibited digraphs so selected, and satisfies (10.1) for infinitely many n; hence

$$2 \limsup \frac{\operatorname{ex}(n, \mathcal{L}^*)}{n^2} \geqslant \gamma + \epsilon \quad \text{as} \quad n \to \infty.$$

Let now $\{S^n\}$ be any asymptotically extremal sequence for $\mathcal{L}^*$. Through the use of familiar methods (cf. above, either Section B(E), or the proof of Lemma 4 in Section 7) we can show that there exists an infinite sequence $\{n'\}$ of integers and $\{G'^{n'}\}$ of digraphs such that $G'^{n'} \subset S^n$ and

(i) $n' > n - o(n)$;

(ii) all valencies exceed $\left(\gamma + \frac{\epsilon}{2}\right) n'$.

Every abstract augmentation A contained in $\mathcal{M}$ has associated with it a sequence of integers $\{h(n, A)\}$ defined by $h(n, A) = \max\{m' \colon A(m') \subset \subset G'^{n'}\}$. Among those matrices A for which the sequence is unbounded – the matrix (0) is certainly one matrix in $\mathcal{M}$ having this property – select one having maximum density. This matrix A has only finitely many

$\left(\gamma + \frac{\epsilon}{2}\right)$-augmentations. By Lemma 9, one of these, B, has a maximal dense submatrix A^* of density greater than $g(A)$, having the following property: that there exists an optimal matrix graph of structure $A^*(m)$ in infinitely many S^n where m is unbounded as $n \to \infty$. Since A was chosen to have maximum density with precisely this property, A^* must be one of the dense matrices excluded from $\mathscr{M}$, because $A^*(\infty)$ contains a prohibited subdigraph L selected from $\mathscr{L}^*$. Thus S^n contains, for sufficiently large n, a prohibited subdigraphs – a contradiction.

Added in proof. The above proof is included here partly for the sake of completeness, partly, to illustrate the methods used here. We describe in [4] some schemes of dense matrices called *augmentation schemes.* As a matter of fact, above we have built up one of these augmentation schemes, see [4] for the details. Later we have found a simpler proof of the Continuity Theorem using only the methods of [10] and yielding a much more general result, see Brown and Simonovits [5*].

11. INVERSE EXTREMAL MULTIGRAPH PROBLEMS

In this section we explore consequences and analogues of our digraph theorems for multigraphs whose edges are of multiplicity not exceeding a fixed positive integer q; loops are excluded.

Definition 7. Let $\hat{A} = \{\hat{a}_{ij}\}$ be a fixed non-negative *symmetric* $r \times r$ matrix, with integer entries: off-diagonal entries not exceeding q, main diagonal entries not exceeding $q-1$. Let $\mathbf{x}$ be a vector with non-negative integer coordinates. A *matrix multigraph* $\hat{A}\langle\mathbf{x}\rangle$ is defined as follows (cf. Definition 2): the vertices are divided into r distinct classes, $C_1, C_2, \dots, C_r$ containing respectively $x_1, x_2, \dots, x_r$ members; connecting each vertex of C_i with each vertex of C_j there are exactly $\hat{a}_{ij} = \hat{a}_{ji}$ parallel (undirected) edges. Pairs of vertices of the i-th class are connected by edges of multiplicity $\hat{a}_{ii}$.

Remark. When the off-diagonal entries of $\hat{A}$ are all even, $\hat{A}\langle\mathbf{x}\rangle$ may be viewed as the result of suppression of all orientations in a matrix digraph $A\langle\mathbf{x}\rangle$. More generally, we may certainly find even integers a_{ij}

such that $\frac{1}{2}(a_{ij} + a_{ji}) = \hat{a}_{ij}$.

Analogous to (4.1) we have

$$(11.1) \qquad 2e(\hat{A}\langle \mathbf{x} \rangle) = \mathbf{x}\hat{A}\mathbf{x}^* + O\left(\sum_i x_i\right).$$

We may define optimal matrix graphs (we prefer to suppress the prefix "multi"), dense matrices (now symmetric), the extremal (multi-)graph problem, and functions $\widehat{\mathrm{ex}}(n, \mathscr{L})$, $\widehat{\mathrm{EX}}(n, \mathscr{L})$ analogously: the theory carries over without surprises. We use a circumflex to denote the multigraph analogue. For example, the following theorem holds:

Theorem 3. *For every dense (symmetric) matrix $\hat{A}$ there exists a finite family $\hat{\mathscr{L}}$ of prohibited multigraphs such that*

(i) *each optimal matrix graph $\hat{A}(n)$ is extremal for $\hat{\mathscr{L}}$, and there are no other extremal multigraphs for $\hat{\mathscr{L}}$;*

(ii) *any asymptotically extremal sequence $\{\hat{G}^n\}$ for $\hat{\mathscr{L}}$ can be obtained from a sequence $\{\hat{A}(n)\}$ by deleting and/or adjoining $o(n^2)$ edges;*

(iii) *If $\{\hat{B}(n)\}$ is an asymptotically extremal sequence of optimal matrix graphs, where $\hat{B}$ is dense, then $\hat{B}$ and $\hat{A}$ are identical up to the same permutation of rows and columns.*

Sketch of proof. (Parts (i) and (ii) were announced as Theorem 2 of [1].) A proof can be constructed by reproving successively all the lemmas leading up to Theorem 1, with orientations suppressed. ∎

Remark. Lest the reader jump to conclusions concerning the suppression of orientations, we hasten to observe that this operation can change the value of the extremal number, i.e. it is not necessary that $\widehat{\mathrm{ex}}(n, \hat{\mathscr{L}}) = \mathrm{ex}(n, \mathscr{L})$. As an example, we propose, for $q = 1$, the single digraph obtained from the digraph $D\langle(1, 1, 1, 1)\rangle$ of Example 2 by deleting the three edges of a cyclic triangle (shown in Figure 4, with density $\frac{6}{5}$). The extremal numbers for this digraph are $\left(\frac{3}{4} + o(1)\right)n^2$ as $n \to \infty$, while those of the multigraph obtained by suppressing all orientation are

$\left(\frac{3}{4} + o(1)\right)n^2$ as $n \to \infty$; the other orientation of this multigraph (also shown in Figure 4) does indeed have extremal numbers $\left(\frac{2}{3} + o(1)\right)n^2$.

Theorem 2 also admits a multigraph generalization.

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