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Some asymptotic formulas on generalized divisor functions, III

by

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1. Throughout this paper, we use the following notation:

$c_1, c_2, \ldots, X_0, X_1, \ldots$ denote positive absolute constants. We denote the number of elements of the finite set S by $|S|$. We write $e^x = \exp(x)$. We denote the least prime factor of n by $p(n)$, while the greatest prime factor of n is denoted by $P(n)$. We write $p^a \| n$ if $p^a | n$ but $p^{a+1} \nmid n$. $\omega(n)$ denotes the number of all the prime factors of n so that $\omega(n) = \sum_{p^a \| n} a$ and we write

$$\omega(n, x, y) = \sum_{\substack{p^a \| n \\ x < p \leqslant y}} a.$$

The divisor function is denoted by $d(n)$:

$$d(n) = \sum_{d|n} 1.$$

Let A be a finite or infinite sequence of positive integers $a_1 < a_2 < \ldots$ Then we write

$$N_A(x) = \sum_{\substack{a \in A \\ a \leqslant x}} 1, \quad f_A(x) = \sum_{\substack{a \in A \\ a \leqslant x}} \frac{1}{a}, \quad d_A(n) = \sum_{\substack{a \in A \\ a | n}} 1$$

(in other words, $d_A(n)$ denotes the number of divisors amongst the a_i's) and

$$D_A(x) = \max_{1 \leqslant n \leqslant x} d_A(x).$$

The aim of this series is to investigate the function $D_A(x)$. (See [1] and [2]; see also Hall [4].) Clearly,

$$\sum_{1 \leqslant n \leqslant x} d_A(n) = x f_A(x) + O(x).$$

Thus if $f_A(x)$ is large then we have $D_A(x)/f_A(x) \gg 1$. In Part II of this series (see [2]), we showed that $f_A(x) \to +\infty$ implies that

$$\limsup_{x\to+\infty} D_A(x)/f_A(x) = +\infty,$$

in fact, we have

$$\limsup_{x\to+\infty} D_A(x)\exp\left(c_1(\log f_A(x))^2\right) = +\infty.$$

We proved this by showing that if $f_A(x)$ is large (in fact, it is sufficient to assume that $N_A(x)$ is large) then there exists an integer y such that $x \leqslant y \leqslant \exp\left((\log x)^2\right)$ and $D_A(y)$ is large. In this paper, our aim is to prove that *if we have more information about $f_A(x)$ then $D_A(x)/f_A(x)$ must be large for the same x.* In fact we prove that

THEOREM 1. *For all $\omega > 0$ and for $x > X_0(\omega)$,*

$$f_A(x) > (\log\log x)^{20} \tag{1}$$

implies that

$$D_A(x) > \omega f_A(x). \tag{2}$$

(Note that by Theorem 1 in [1], the lower bound $(\log\log x)^{20}$ in (1) cannot be replaced by $\log\log x$.)

Sections 2 and 3 are devoted to the proof of this theorem while in Sections 4 and 5 we discuss some other related results.

2. In order to prove Theorem 1, we need some lemmas.

LEMMA 1. *There exists an absolute constant c_2 such that for all $u \geqslant 0$ and $y \geqslant 1$ we have*

$$\sum_{\substack{u<n\leqslant uy\\ p(n)>y}} \frac{1}{n} < c_2. \tag{3}$$

Proof. Lemma 1 can be proved easily by using Brun's sieve. In fact, (3) is trivial for $u \leqslant 1$ (since in this case, the left-hand side is equal to 0), while for $u > 1$, (3) is a consequence of [7], p. 53, Theorem 4.10.

LEMMA 2. *Let us write*

$$Q(x) = x - (1+x)\log(1+x). \tag{4}$$

Then for $1 \leqslant y$, $2y < z \leqslant v$, $0 \leqslant a \leqslant 1$ we have

$$\sum_{\substack{n\leqslant v\\ \omega(n,y,z)\leqslant(1-a)\sum_{y<p\leqslant z} 1/p}} 1 < c_3 v \exp\left(Q(-a)\log\frac{\log z}{\log y}\right). \tag{5}$$

Proof. Let $1 \leqslant v$, $0 \leqslant a \leqslant 1$, and let E be an arbitrary nonempty set of prime numbers not exceeding v. Put $E(v) = \sum_{p\in E} 1/p$. K. K. Norton proved (see [5], Theorem (5.9); see also Halász [3]) that

$$\sum_{\substack{n\leqslant v\\ \sum\limits_{p^{\beta}\| n,\, p\in E}\beta\leqslant(1-a)E(v)}} 1 < c_4 v \exp\big(Q(-a)E(v)\big).$$

By using this theorem with $E = \{p\colon\ y < p \leqslant z\}$ (note that $E \neq \emptyset$ by $2y < z$), and with respect to the well-known formula

$$\sum_{p\leqslant x} 1/p = \log\log x + c_5 + o(1), \tag{6}$$

we obtain (5).

Lemma 3. *For* $1 \leqslant y,\ 2y < z \leqslant v,\ 0 < a \leqslant \beta < 1$ *we have*

$$\sum_{\substack{n\leqslant v\\ \omega(n,y,z)\geqslant(1+a)\sum\limits_{y<p\leqslant z}1/p}} 1 < c_6(\beta)\,a^{-1} v \Big(\sum_{y<p\leqslant z}\frac{1}{p}\Big)^{-1/2} \exp\Big(Q(a)\log\frac{\log z}{\log y}\Big) \tag{7}$$

(*where* $Q(x)$ *is defined by* (4)).

Proof. Let $0 < a \leqslant \beta < 1$, and let E be an arbitrary nonempty set of prime numbers not exceeding v. Put $E(v) = \sum\limits_{p\in E} 1/p$. K. K. Norton proved (see [5], Theorem (5.12); see also Halász [3]) that

$$\sum_{\substack{n\leqslant v\\ \sum\limits_{p^{\gamma}\| n,\, p\in E}\gamma\geqslant(1+a)E(v)}} 1 < c_7(\beta)\,a^{-1} v \big(E(v)\big)^{-1/2}\exp\big(Q(a)E(v)\big).$$

By using this theorem with $E = \{p\colon\ y < p \leqslant z\}$ (again, $E \neq \emptyset$ by $2y < z$), and with respect to (6), we obtain (7).

3. In this section, we complete the proof of Theorem 1. Define the positive integer R by

$$x^{1-1/2^{R-1}} \leqslant x e^{-f_A(x)/3} < x^{1-1/2^{R}}, \tag{8}$$

i.e.,

$$2^{R-1} < \frac{3\log x}{f_A(x)} \leqslant 2^R, \qquad R-1 < \frac{1}{\log 2}\log\frac{3\log x}{f_A(x)} \leqslant R.$$

Then for large x, we have

$$R < \frac{1}{\log 2}\log\frac{3\log x}{f_A(x)} + 1 < 2\log\log x. \tag{9}$$

For $i = 0, 1, \ldots, R$, let

$$x_i = x^{1-1/2^{i}},$$

and for $i = 1, 2, \ldots, R$, put

$$A_i = A \cap [x_{i-1}, x_i).$$

Then by (1) and (8), for large x we have

$$(10)\quad \sum_{i=1}^{R}\Big(\sum_{a\in A_i}\frac{1}{a}\Big)=\sum_{\substack{a\in A\\ a<x_R}}\frac{1}{a}=\sum_{\substack{a\in A\\ a\leqslant x}}\frac{1}{a}-\sum_{\substack{a\in A\\ x_R\leqslant a\leqslant x}}\frac{1}{a}$$

$$\geqslant f_A(x)-\sum_{x_R\leqslant n\leqslant x}\frac{1}{n}\geqslant f_A(x)-\sum_{xe^{-f_A(x)/3}\leqslant n\leqslant x}\frac{1}{n}$$

$$=f_A(x)-\big(1+o(1)\big)\log e^{f_A(x)/3}>f_A(x)-\frac{f_A(x)}{2}=\frac{f_A(x)}{2}.$$

Obviously, there exists an integer j such that $1\leqslant j\leqslant R$ and

$$\sum_{a\in A_j}\frac{1}{a}\geqslant\frac{1}{R}\sum_{i=1}^{R}\Big(\sum_{a\in A_i}\frac{1}{a}\Big)$$

hence with respect to (9) and (10),

$$(11)\quad f_{A_j}(x)=\sum_{a\in A_j}\frac{1}{a}>\frac{1}{R}\frac{f_A(x)}{2}>\frac{f_A(x)}{4\log\log x}.$$

Let us fix an integer j $(1\leqslant j\leqslant R)$ satisfying (11), and write A_j in the form

$$(12)\quad A_j=A_j'\cup A_j''$$

where A_j' consists of the integers a such that $a\in A_j$ and there exists an integer d satisfying

$$(13)\quad (\log x)^3<d\leqslant x^{1/2^{j+1}\omega f_A(x)}$$

and $d\mid a$, while A_j'' consists of the integers a such that $a\in A_j$ and $d\nmid a$ for all d satisfying (13). (For $x^{1/2^{j+1}\omega f_A(x)}\leqslant(\log x)^3$, we have $A_j'=\emptyset$.) We have to distinguish two cases.

Case 1. Assume first that

$$f_{A_j'}(x)=\sum_{a\in A_j'}\frac{1}{a}>\frac{1}{2}f_{A_j}(x).$$

Then by (11), we have

$$(14)\quad f_{A_j'}(x)=\sum_{a\in A_j'}\frac{1}{a}>\frac{1}{2}f_{A_j}(x)>\frac{f_A(x)}{8\log\log x}.$$

For $a\in A_j'$, write a in the form

$$a=d^*(a)b(a),$$

where $d^*(a)$ denotes the least integer d such that d satisfies (13) and $d\,|\,a$. Then for $a \in A'_j$ we have $b(a) \leqslant a < x_j = x^{1-1/2^j}$ and $(\log x)^3 < d^*(a)$ so that

$$\sum_{a\in A'_j}\frac{1}{a} = \sum_{a\in A'_j}\frac{1}{d^*(a)b(a)} = \sum_{b\leqslant x^{1-1/2^j}}\frac{1}{b}\sum_{\substack{a\in A'_j\\ b^*(a)=b}}\frac{1}{d^*(a)} \tag{15}$$

$$< \sum_{b\leqslant x^{1-1/2^j}}\frac{1}{b}\sum_{\substack{a\in A'_j\\ b^*(a)=b}}\frac{1}{(\log x)^3} = \frac{1}{(\log x)^3}\sum_{b\leqslant x^{1-1/2^j}}\frac{1}{b}\sum_{\substack{a\in A'_j\\ b^*(a)=b}}1$$

$$\leqslant \frac{1}{(\log x)^3}\Big(\max_{1\leqslant b\leqslant x^{1-1/2^j}}\sum_{\substack{a\in A'_j\\ b^*(a)=b}}1\Big)\sum_{b<x}\frac{1}{b}$$

$$< \frac{1}{(\log x)^3}\Big(\max_{1\leqslant b\leqslant x^{1-1/2^j}}\sum_{\substack{a\in A'_j\\ b^*(a)=b}}1\Big)2\log x = \frac{2}{(\log x)^2}\Big(\max_{1\leqslant b\leqslant x^{1-1/2^j}}\sum_{\substack{a\in A'_j\\ b^*(a)=b}}1\Big).$$

If x is large enough (in terms of ω) then (14) and (15) yield that

$$\max_{1\leqslant b\leqslant x^{1-1/2^j}}\sum_{\substack{a\in A'_j\\ b^*(a)=b}}1 > \frac{(\log x)^2}{2}\sum_{a\in A'_j}\frac{1}{a} > \frac{(\log x)^2}{16\log\log x}f_A(x) > \omega f_A(x)+1$$

so that there exists an integer b_0 for which

$$1 \leqslant b_0 \leqslant x^{1-1/2^j} \tag{16}$$

and

$$\sum_{\substack{a\in A'_j\\ b^*(a)=b_0}}1 > \omega f_A(x)+1. \tag{17}$$

Put $s = [\omega f_A(x)]+1$. Then by (17), there exist distinct integers $a_1, a_2, \ldots, a_s$ such that a_i can be written in the form

$$a_i = b_0 d^*(a_i) = b_0 d_i$$

where

$$\big((\log x)^3 <\big)\, d_i \leqslant x^{1/2^{j+1}\omega f_A(x)}. \tag{18}$$

Let

$$u = b_0 d_1 d_2 \ldots d_s.$$

Then by (16) and (18), we have

$$\begin{aligned} u = b_0 d_1 d_2 \ldots d_s &\leqslant x^{1-1/2^j}\big(x^{1/2^{j+1}\omega f_A(x)}\big)^s \\ &< x^{1-1/2^j}\big(x^{1/2^{j+1}\omega f_A(x)}\big)^{2\omega f_A(x)} = x, \end{aligned} \tag{19}$$

and obviously, $a_i = b_0 d_i / u$ and $a_i = b_0 d_i \in A$ so that

$$d_A(u) \geqslant s = [\omega f_A(x)] + 1 > \omega f_A(x). \tag{20}$$

(19) and (20) yield (2) and this completes the proof of Theorem 1 in this case.

Case 2. Assume now that

$$f_{A_j'}(x) = \sum_{a \in A_j'} \frac{1}{a} \leqslant \frac{1}{2} f_{A_j}(x) = \frac{1}{2} \sum_{a \in A_j} \frac{1}{a}. \tag{21}$$

Then (12) and (21) yield that

$$\begin{aligned} f_{A_j''}(x) &= \sum_{a \in A_j''} \frac{1}{a} \geqslant \sum_{a \in A_j} \frac{1}{a} - \sum_{a \in A_j'} \frac{1}{a} \\ &\geqslant f_{A_j}(x) - \frac{1}{2} f_{A_j}(x) = \frac{1}{2} f_{A_j}(x) > \frac{f_A(x)}{8 \log\log x}. \end{aligned} \tag{22}$$

For $u \geqslant 1$, let

$$g(u) = \left(\frac{3 \log f_A(x)}{u} \right)^u.$$

Then for $1 \leqslant u < \frac{3}{e} \log f_A(x)$, the function $g(u)$ is increasing since

$$g'(u) = g(u) \log \frac{3 \log f_A(x)}{eu} > 0$$

and for large x, we have

$$g(1) = 3 \log f_A(x) < \frac{f_A(x)}{(\log\log x)^2}$$

and

$$g\left(\frac{3}{e} \log f_A(x) \right) = \left(f_A(x) \right)^{3/e} > \frac{f_A(x)}{(\log\log x)^2}.$$

Thus there exists a uniquely determined real number t such that

$$1 < t < \frac{3}{e} \log f_A(x) \tag{23}$$

and

$$g(t) = \left(\frac{3 \log f_A(x)}{t} \right)^t = \frac{f_A(x)}{(\log\log x)^2}. \tag{24}$$

We need a lower bound for this number t. By (1), we have

$$g\left(\tfrac{1}{2}\log f_A(x)\right) = (6^{1/2})^{\log f_A(x)} = \left(f_A(x)\right)^{(1/2)\log 6} \tag{25}$$

$$< \left(f_A(x)\right)^{9/10} = \frac{f_A(x)}{\left(f_A(x)\right)^{1/10}} < \frac{f_A(x)}{(\log\log x)^2}.$$

(23), (24) and (25) imply that

$$\tfrac{1}{2}\log f_A(x) < t \tag{26}$$

(since $g(u)$ is increasing for $1 < u < \dfrac{3}{e}\log f_A(x)$).

Let us write

$$z_j = \max\{x^{1/2^{j+1}\omega f_A(x)},\ (\log x)^3\}.$$

Let A_j^* denote the set of the integers a such that $a \in A_j''$ and

$$\omega(a, z_j, x^{1/2^j}) > t.$$

Now we are going to give an upper estimate for

$$\sum_{\substack{a\in A_j''\\ a\notin A_j^*}} \frac{1}{a} = \sum_{\substack{a\in A_j''\\ \omega(a,z_j,x^{1/2^j})\leqslant t}} \frac{1}{a}.$$

If $a \in A_j''$ and $\omega(a, z_j, x^{1/2^j}) \leqslant t$ then by the definition of A_j'', we have

$$x^{1-1/2^{j-1}} = x_{j-1} \leqslant a < x_j = x^{1-1/2^j}$$

and a can be written in the form

$$a = up_1^{\alpha_1}\ldots p_m^{\alpha_m} v$$

where $P(u) \leqslant (\log x)^3$, $z_j < p_1 < \ldots < p_m \leqslant x^{1/2^j}$, $m \leqslant \omega(a, z_j, x^{1/2^j}) \leqslant t$ and $p(v) > x^{1/2^j}$.

Thus by Lemma 1, we have

$$\sum_{\substack{a\in A_j''\\ a\notin A_j^*}} \frac{1}{a} \leqslant \sum_{P(u)\leqslant(\log x)^3} \frac{1}{u}\Bigg\{\sum_{0\leqslant m\leqslant t}\ \sum_{z_j<p_1<\ldots<p_m\leqslant x^{1/2^j}} \frac{1}{\prod_{i=0}^{m} p_i^{\alpha_i}} \times \tag{27}$$

$$\times\Bigg(\sum_{\substack{p(v)>x^{1/2^j}\\ x_{j-1}<up_1^{\alpha_1}\ldots p_m^{\alpha_m}v<x_j=x_{j-1}x^{1/2^j}}} \frac{1}{v}\Bigg)\Bigg\}$$

$$< c_2 \left(\sum_{P(u) \leqslant (\log x)^3} \frac{1}{u} \right) \left(1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left(\sum_{z_j < p \leqslant x^{1/2^j}} \sum_{\alpha=1}^{+\infty} \frac{1}{p^\alpha} \right)^m \right)$$
$$= c_2 \left(\prod_{p \leqslant (\log x)^3} \sum_{\alpha=0}^{+\infty} \frac{1}{p^\alpha} \right) \left(1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left(\sum_{z_j < p \leqslant x^{1/2^j}} \sum_{\alpha=1}^{+\infty} \frac{1}{p^\alpha} \right)^m \right)$$
$$= c_2 \left(\prod_{p \leqslant (\log x)^3} \frac{1}{1-1/p} \right) \left(1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left(\sum_{z_j < p \leqslant x^{1/2^j}} \sum_{\alpha=1}^{+\infty} \frac{1}{p^\alpha} \right)^m \right).$$

It is well-known that

$$\prod_{p \leqslant y} \frac{1}{1-1/p} < c_8 \log y$$

and

$$\sum_{p \leqslant y} \sum_{\alpha=1}^{+\infty} \frac{1}{p^\alpha} = \log\log y + c_9 + o(1).$$

Thus with respect to (1), (23), (24) and (26), and by using the Stirling-formula, we obtain from (27) that for $x > X_1(\omega)$,

(28) $$\sum_{\substack{a \in A_j'' \\ a \notin A_j^*}} \frac{1}{a} < c_{10} \log\left((\log x)^3\right) \left(1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left(\log\log x^{1/2^j} - \log\log z_j + c_{11}\right)^m \right)$$
$$= c_{10} \log\log x \left(1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left(\log \frac{\log x^{1/2^j}}{\log z_j} + c_{11} \right)^m \right)$$
$$\leqslant c_{10} \log\log x \left(1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left(\log \frac{\log x^{1/2^j}}{\log x^{1/2^{j+1} \omega f_A(x)}} + c_{11} \right)^m \right)$$
$$= c_{10} \log\log x \left(1 + \sum_{m=1}^{[t]} \frac{1}{m!} \left(\log 2\omega f_A(x) + c_{11}\right)^m \right)$$
$$< c_{12} t \log\log x \frac{\left(\log \omega f_A(x) + c_{13}\right)^t}{t!}$$
$$< c_{14} t^{1/2} \log\log x \left(\frac{e\left(\log \omega f_A(x) + c_{13}\right)}{t} \right)^t$$
$$< \frac{1}{16} \log\log x \left(\frac{3 \log f_A(x)}{t} \right)^t = \frac{1}{16} \log\log x \frac{f_A(x)}{(\log\log x)^2}$$
$$= \frac{1}{16} \frac{f_A(x)}{\log\log x}.$$

(22) and (28) yield that

$$(29)\qquad f_{A_j}^*(x) = \sum_{a\in A_j^*}\frac{1}{a} = \sum_{a\in A_j''}\frac{1}{a} - \sum_{\substack{a\in A_j'' \\ a\notin A_j^*}}\frac{1}{a}$$

$$> \frac{f_A(x)}{8\log\log x} - \frac{f_A(x)}{16\log\log x} = \frac{f_A(x)}{16\log\log x}.$$

Let S denote the set of the integers n such that $n \leqslant x$ and n can be written in the form

$$(30)\qquad n = au \quad \text{where} \quad a\in A_j^* \text{ and } \omega(u, z_j, x^{1/2^j}) > \frac{18}{19}\log f_A(x).$$

For fixed $n \in S$, let $\varphi(n)$ denote the number of representations of n in the form (30).

Then we have

$$(31)\qquad \sum_{n\leqslant x}\varphi(n) = \sum_{a\in A_j^*}\sum_{\substack{au\leqslant x \\ \omega(u,z_j,x^{1/2^j})>\frac{18}{19}\log f_A(x)}} 1$$

$$= \sum_{a\in A_j^*}\Big(\sum_{u\leqslant x/a}1 - \sum_{\substack{u\leqslant x/a \\ \omega(u,z_j,x^{1/2^j})\leqslant \frac{18}{19}\log f_A(x)}} 1\Big).$$

In order to estimate the last sum, we use Lemma 2 with z_j, $x^{1/2^j}$, x/a and $1/400$ in place of y, z, v and α, respectively. Then $1\leqslant y$ and $2y < z$ hold trivially by the definition of z_j (and by $j\leqslant R$), and also $z\leqslant v$ holds by

$$z = x^{1/2^j} = \frac{x}{x^{1-1/2^j}} = \frac{x}{x_j} < \frac{x}{a} = v$$

(since we have $a\in A_j^*$ and thus $a < x_j$). Thus Lemma 2 can be applied, and we obtain with respect to (1) and the definition of z_j that for large x

$$(32)\qquad \sum_{\substack{n\leqslant x/a \\ \omega(u,z_j,x^{1/2^j})\leqslant \frac{399}{400}\sum_{z_j<p\leqslant x^{1/2^j}}\frac{1}{p}}} 1 < c_3\frac{x}{a}\exp\left(Q\left(-\frac{1}{400}\right)\log\frac{\log x^{1/2^j}}{\log z_j}\right)$$

$$< c_3\frac{x}{a}\exp\left(-3\cdot 10^{-6}\log\frac{\log x^{1/2^j}}{\log x^{1/2^{j+1}\omega f_A(x)}}\right)$$

$$= c_3\frac{x}{a}\exp\big(-3\cdot 10^{-6}\log 2\omega f_A(x)\big) < \frac{1}{3}\,\frac{x}{a}.$$

Furthermore, by (6) we have

$$\frac{399}{400}\sum_{z_j<p\leqslant x^{1/2^j}}\frac{1}{p}>\frac{399}{400}\left(\log\frac{\log x^{1/2^j}}{\log z_j}-c_{15}\right). \tag{33}$$

By (8), we have

$$\frac{f_A(x)}{6}\leqslant \log x^{1/2^R}\leqslant \log x^{1/2^j}. \tag{34}$$

We obtain from (1) and (34) that

$$\begin{aligned}\log z_j &= \log\max\{(\log x)^3,\, x^{1/2^{j+1}\omega f_A(x)}\}\\ &= \max\left\{3\log\log x,\ \frac{\log x^{1/2^j}}{2\omega f_A(x)}\right\}\\ &= \max\left\{\log x^{1/2^j}\frac{3\log\log x}{\log x^{1/2^j}},\frac{\log x^{1/2^j}}{2\omega f_A(x)}\right\}\\ &\leqslant \max\left\{\log x^{1/2^j}\frac{3\log\log x}{f_A(x)/6},\frac{\log x^{1/2^j}}{2\omega f_A(x)}\right\}\\ &\leqslant \max\left\{\log x^{1/2^j}\frac{3\left(f_A(x)\right)^{1/20}}{f_A(x)/6},\frac{\log x^{1/2^j}}{2\omega f_A(x)}\right\}\\ &= \frac{18\log x^{1/2^j}}{\left(f_A(x)\right)^{19/20}}.\end{aligned} \tag{35}$$

(33) and (35) yield for large x that

$$\begin{aligned}\frac{399}{400}\sum_{z_j<p\leqslant x^{1/2^j}}\frac{1}{p}&>\frac{399}{400}\left(\log\frac{\log x^{1/2^j}}{\log z_j}-c_{15}\right)\\ &\geqslant \frac{399}{400}\left(\log\frac{\left(f_A(x)\right)^{19/20}}{18}-c_{15}\right)>\frac{18}{19}\log f_A(x)\end{aligned} \tag{36}$$

hence

$$\sum_{\substack{u\leqslant x/a\\ \omega(u,z_j,x^{1/2^j})<\frac{18}{19}\log f_A(x)}} 1\leqslant \sum_{\substack{u\leqslant x/a\\ \omega(u,z_j,x^{1/2^j})\leqslant\frac{399}{400}\sum_{z_j<p\leqslant x^{1/2^j}}\frac{1}{p}}} 1. \tag{37}$$

(32) and (37) yield that

$$\sum_{\substack{u\leqslant x/a\\ \omega(u,z_j,x^{1/2^j})>\frac{18}{19}\log f_A(x)}} 1<\frac{1}{3}\frac{x}{a}.$$

Thus we obtain from (29) and (31) that

$$\sum_{n\leqslant x}\varphi(n)\geqslant\sum_{a\in A_j^*}\left(\left[\frac{x}{a}\right]-\frac{1}{3}\frac{x}{a}\right)>\sum_{a\in A_j^*}\frac{x}{2a} \tag{38}$$

$$=\frac{x}{2}f_{A_j}^*(x)>\frac{xf_A(x)}{32\log\log x}.$$

Now we are going to give an upper estimate for $\sum_{n\leqslant x}\varphi(n)$. Obviously, for $n\leqslant x$ we have $\varphi(n)\leqslant d_A(n)\leqslant D_A(x)$ hence

$$\sum_{n\leqslant x}\varphi(n)=\sum_{n\in S}\varphi(n)\leqslant\sum_{n\in S}D_A(x)=|S|D_A(x). \tag{39}$$

Thus in order to obtain an upper bound for $\sum_{n\leqslant x}\varphi(n)$, we have to estimate $|S|$.

If $n\in S$ then by (26), (30) and the definition of the set A_j^*, we have

$$\omega(n,z_j,x^{1/2^j})=\omega(au,z_j,x^{1/2^j})=\omega(a,z_j,x^{1/2^j})+\omega(u,z_j,x^{1/2^j})$$

$$>t+\frac{18}{19}\log f_A(x)>\frac{1}{2}\log f_A(x)+\frac{18}{19}\log f_A(x)=\frac{55}{38}\log f_A(x)$$

hence

$$|S|\leqslant\sum_{\substack{n\leqslant x\\ \omega(n,z_j,x^{1/2^j})>\frac{55}{38}\log f_A(x)}}1. \tag{40}$$

In order to estimate this sum, we use Lemma 3 with z_j, $x^{1/2^j}$, x, $17/37$ and $9/10$ in place of y, z, v, α and β, respectively. ($1\leqslant y$, $2y<z\leqslant v$ and $0<\alpha\leqslant\beta<1$ hold trivially with respect to the definition of z_j.) We obtain with respect to (8) and the definition of z_j that for $x>X_2(\omega)$,

$$\sum_{\substack{n\leqslant x\\ \omega(n,z_j,x^{1/2^j})>\frac{54}{37}\sum_{z_j<p\leqslant x^{1/2^j}}\frac{1}{p}}}1 \tag{41}$$

$$<c_6x\left(\sum_{z_j<p\leqslant x^{1/2^j}}\frac{1}{p}\right)^{-1/2}\exp\left(Q\left(\frac{17}{37}\right)\log\frac{\log x^{1/2^j}}{\log z_j}\right)$$

$$<c_6x\exp\left(-\frac{91}{1000}\log\frac{\log x^{1/2^j}}{\log x^{1/2^{j+1}\omega f_A(x)}}\right)$$

$$<c_6x\exp\left(-\frac{91}{1000}\log\omega f_A(x)\right)<x\exp\left(-\frac{9}{100}\log f_A(x)\right)=x\left(f_A(x)\right)^{-9/100}.$$

Furthermore, with respect to (6) and the definition of z_j we have

$$\frac{54}{37}\sum_{z_j<p\leqslant x^{1/2^j}}\frac{1}{p}\leqslant\frac{54}{37}\sum_{x^{1/2^{j+1}}\omega f_A(x)<p\leqslant x^{1/2^j}}\frac{1}{p}$$

$$=\frac{54}{37}\Big(\sum_{p\leqslant x^{1/2^j}}\frac{1}{p}-\sum_{p\leqslant x^{1/2^{j+1}}\omega f_A(x)}\frac{1}{p}\Big)$$

$$<\frac{54}{37}\,(\log\log x^{1/2^j}-\log\log x^{1/2^{j+1}\omega f_A(x)}+c_{16})$$

$$=\frac{54}{37}\big(\log 2\omega f_A(x)+c_{16}\big)<\frac{55}{38}\log f_A(x)$$

and thus

$$\sum_{\substack{n\leqslant x\\ \omega(n,z_j,x^{1/2^j})>\frac{55}{38}\log f_A(x)}} 1\leqslant\sum_{\substack{n\leqslant x\\ \omega(n,z_j,x^{1/2^j})>\frac{54}{37}\sum_{z_j<p\leqslant x^{1/2^j}}\frac{1}{p}}} 1. \tag{42}$$

(40), (41) and (42) yield that

$$|S|<x\big(f_A(x)\big)^{-9/100}. \tag{43}$$

Finally, we obtain from (38), (39) and (43) that

$$\frac{xf_A(x)}{32\log\log x}<\sum_{n\leqslant x}\varphi(n)\leqslant|S|D_A(x)<x\big(f_A(x)\big)^{-9/100}D_A(x)$$

hence with respect to (1),

$$D_A(x)>f_A(x)\frac{\big(f_A(x)\big)^{9/100}}{32\log\log x}>f_A(x)\,\frac{\big(f_A(x)\big)^{9/100}}{32\big(f_A(x)\big)^{1/20}}$$

$$=f_A(x)\cdot\frac{1}{32}\big(f_A(x)\big)^{1/25}>\omega f_A(x)$$

for $x>X_3(\omega)$. Thus (2) holds also in Case 2 and this completes the proof of Theorem 1.

4. By using the same method, we can show that Theorem 1 is true also with $(\log\log x)^{2+\varepsilon}$ in place of $(\log\log x)^{20}$ on the right-hand side of (1). In fact, in order to prove this, the only non-trivial modifications are that t must be defined as $t=\eta\log f_A(x)$ where $\eta=\eta(\varepsilon)$ (>0) is sufficiently small in terms of ε, and in (30), the condition $\omega(u,z_j,x^{1/2^j})>\frac{18}{19}\log f_A(x)$ must be replaced by $\omega(u,z_j,x^{1/2^j})>K\log f_A(x)$ where $K=K(\varepsilon)$ is sufficiently large in terms of ε. Furthermore, then Lemmas 2 and 3 must be replaced by lower and upper estimates for

$$\sum_{\substack{n\leqslant v \\ \omega(n,y,z)\geqslant L\sum\limits_{y<p\leqslant z} 1/p}} 1$$

where L is arbitrary large but fixed. Such estimates could be deduced by the methods used by K. K. Norton in [6]. (Norton's estimates cannot be used in the original form since the error terms in his lower and upper estimates depend implicitly on the set E of the prime numbers whose multiples we investigate. Thus in our case, these results would yield lower and upper bounds depending implicitly on $\{p\colon y<p\leqslant z\}$, i.e., on y and z, instead of the explicit estimates needed by us.)

On the other hand, we guess that also the exponent $2+\varepsilon$ could be improved, and, perhaps, Theorem 1 is true also with $(\log\log x)^{1+\varepsilon}$ or even $c_{17}(\omega)\log\log x$ on the right-hand side of (1). This is the reason of that that we preferred to work out the slightly weaker estimate given in Theorem 1 whose proof is much simpler.

5. One may expect that if we know that $f_A(y)$ is large for *all* $y\leqslant x$ then Theorem 1 can be sharpened in the sense that the lower bound given for $f_A(x)$ in (1) (for fixed x) can be replaced by a much smaller lower bound for $f_A(y)$ (for all y). In fact, we show in this section that

THEOREM 2. *For all* $\omega>0$, *there exists a real number* $X_4=X_4(\omega)$ *such that if* $x>X_4$ *and writing* $y=\exp\left(\frac{\log x}{(\log\log x)^{21}}\right)$, *we have*

$$f_A(y)>22\log\log\log y, \tag{44}$$

then

$$D_A(x)>\omega f_A(x). \tag{45}$$

Furthermore, we show that Theorem 2 is best possible except the value of the constant factor on the right of (44):

THEOREM 3. *There exist positive constants* c_{18}, c_{19}, X_5 *and an infinite sequence* A *such that*

$$f_A(x)>c_{18}\log\log\log x \quad \textit{for all } x>X_5 \tag{46}$$

and

$$\liminf_{x\to+\infty}\frac{D_A(x)}{f_A(x)}<c_{19}. \tag{47}$$

In order to prove Theorem 2, we need the following lemma:

LEMMA 4. *If* $x>1$, $t\geqslant 1$ *and* A *is an arbitrary sequence of positive integers such that*

$$D_A(x)\leqslant t \tag{48}$$

then we have

$$N_A(x^{1/(t+1)}) \leqslant t.$$

Proof of Lemma 4. Assume indirectly that

$$N_A(x^{1/(t+1)}) > t,$$

i.e.,

$$N_A(x^{1/(t+1)}) \geqslant [t]+1.$$

Then there exist integers $a_1, a_2, \ldots, a_{[t]+1}$ such that $a_1 \in A$, $a_2 \in A, \ldots$ $\ldots, a_{[t]+1} \in A$ and

$$a_1 < a_2 < \ldots < a_{[t]+1} \leqslant x^{1/(t+1)}. \tag{49}$$

Put $u = a_1 a_2 \ldots a_{[t]+1}$. Then $a_i | u$ for $1 \leqslant u \leqslant [t]+1$ and thus

$$d_A(u) \geqslant [t]+1 > t. \tag{50}$$

On the other hand, by (49) we have

$$u = a_1 a_2 \ldots a_{[t]+1} \leqslant (x^{1/(t+1)})^{[t]+1} \leqslant (x^{1/(t+1)})^{t+1} = x. \tag{51}$$

(50) and (51) imply that

$$D_A(x) > t$$

in contradiction with (48) which completes the proof of Lemma 4.

Proof of Theorem 2. We have to distinguish two cases.

Case 1. Let

$$f_A(x) > (\log\log x)^{20}.$$

Then for $x > X_6(\omega)$, (45) holds by Theorem 1.

Case 2. Let

$$f_A(x) \leqslant (\log\log x)^{20}. \tag{52}$$

Assume indirectly that

$$D_A(x) \leqslant \omega f_A(x). \tag{53}$$

Then by using Lemma 4 with $t = \omega f_A(x)$, we obtain that

$$N_A(x^{1/(t+1)}) = N_A(x^{1/(\omega f_A(x)+1)}) \leqslant t = \omega f_A(x). \tag{54}$$

Put $M = N_A(x^{1/(\omega f_A(x)+1)})$ and let $a_1 < a_2 < \ldots < a_M$ denote the a's not exceeding $x^{1/(\omega f_A(x)+1)}$. Then by (52) and (54), we have

$$\begin{aligned} f_A(x^{1/(\omega f_A(x)+1)}) &= \sum_{i=1}^{M} \frac{1}{a_i} \leqslant \sum_{i=1}^{M} \frac{1}{i} < \log M + c_{20} \\ &\leqslant \log \omega f_A(x) + c_{20} \leqslant \log \omega (\log\log x)^{20} + c_{20} \\ &< 21 \log\log\log x. \end{aligned} \tag{55}$$

On the other hand, by (51) we have

$$x^{1/(\omega f_A(x)+1)} = \exp\left(\frac{\log x}{\omega f_A(x)+1}\right) \geqslant \exp\left(\frac{\log x}{\omega(\log\log x)^{20}+1}\right)$$
$$\geqslant \exp\left(\frac{\log x}{(\log\log x)^{21}}\right) = y.$$

Thus (44) yields that

$$f_A(x^{1/(\omega f_A(x)+1)}) \geqslant f_A(y) > 22\log\log\log y$$
$$= 22\log\log\frac{\log x}{(\log\log x)^{21}} > 21\log\log\log x$$

in contradiction with (55) which completes the proof of Theorem 2.

Proof of Theorem 3. In the proof of Theorem 1 in [1], for $x \geqslant X_7$ we constructed a sequence $B(x)$ such that

$$f_{B(x)}(x) > c_{21}\log\log x \tag{56}$$

and

$$D_{B(x)}(x) < 2\log\log x. \tag{57}$$

Let us define the infinite sequence $x_1 < x_2 < \dots$ by the following recursion: let

$$x_1 = X_7 \quad \text{and} \quad x_k = \exp\left(\exp\left(\exp(x_{k-1})\right)\right).$$

For $x > 1$, let

$$E(x) = \{n:\ \sqrt{x} < n \leqslant x\}.$$

Finally, let

$$A = \bigcup_{k=1}^{+\infty} B(x_k) \cup E(\log\log x_k).$$

We are going to show that this sequence A satisfies both (46) and (47).

First we prove (46). Assume that $x > X_7$. Then there exists a uniquely determined positive integer k ($\geqslant 2$) such that $x_{k-1} < x \leqslant x_k$. Then either

$$x_{k-1} < x \leqslant \exp(x_{k-1}) = \log\log x_k \tag{58}$$

or

$$\exp(x_{k-1}) = \log\log x_k < x \leqslant x_k \tag{59}$$

holds. If (58) holds, then by (56) we have

$$f_A(x) \geqslant f_A(x_{k-1}) \geqslant f_{B(x_{k-1})}(x_{k-1}) > c_{21}\log\log x_{k-1} \geqslant c_{21}\log\log\log x$$

while if (59) holds then

$$f_A(x) \geqslant E(\log\log x_k) = \sum_{(\log\log x_k)^{1/2} < n \leqslant \log\log x_k} \frac{1}{n}$$

$$> \tfrac{1}{3} \log\log\log x_k \geqslant \tfrac{1}{3} \log\log\log x.$$

Thus in fact, (46) holds in both cases.

In order to prove that also (47) holds, it is sufficient to show that for $k = 1, 2, \ldots,$ we have

$$\frac{D_A(x_k)}{f_A(x_k)} < c_{22}. \tag{60}$$

If $u \leqslant x_k$ then by (57) we have

$$d_A(u) = \sum_{\substack{a|u \\ a \in A}} 1 = \sum_{\substack{a \leqslant \log\log x_k \\ a|u, a \in A}} 1 + \sum_{\substack{\log\log x_k < a \\ a|u, a \in A}} 1$$

$$= \sum_{\substack{a \leqslant \log\log x_k \\ a|u, a \in A}} 1 + \sum_{\substack{a|u \\ a \in B(x_k)}} 1 = \sum_{\substack{a \leqslant \log\log x_k \\ a|u, a \in A}} 1 + d_{B(x_k)}(u)$$

$$\leqslant \log\log x_k + D_{B(x_k)}(u) < 3 \log\log x_k$$

hence

$$D_A(x_k) < 3 \log\log x_k. \tag{61}$$

Furthermore, by (56), we have

$$f_A(x_k) = \sum_{\substack{a \leqslant x_k \\ a \in A}} \frac{1}{a} \geqslant \sum_{a \in B(x_k)} \frac{1}{a} = f_{B(x_k)}(x_k) > c_{21} \log\log x_k. \tag{62}$$

(61) and (62) yield (60) and the proof of Theorem 3 is completed.

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