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Some additive properties of sets of real numbers

by

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Abstract. Some problems concerning the additive properties of subsets of R are investigated. From a result of G. G. Lorentz in additive number theory, we show that if P is a nonempty perfect subset of R, then there is a perfect set M with Lebesgue measure zero so that $P+M=R$. In contrast to this, it is shown that (1) if S is a subset of R is concentrated about a countable set C, then $\lambda(S+R)=0$, for every closed set P with $\lambda(P)=0$; (2) there are subsets G_1 and G_2 of R both of which are subspaces of R over the field of rationals such that $G_1 \cap G_2 = \{0\}$, $G_1+G_2=R$ and $\lambda(G_1)=\lambda(G_2)=0$. Some other results are obtained under various set theoretical conditions. If $2^{\aleph_0}=\aleph_1$, then there is an uncountable subset X of R concentrated about the rationals such that if $\lambda(G)=0$, then $\lambda(G+X)=0$; if $V=L$, then X may be taken to be coanalytic.

P. Erdös and E. Straus conjectured and G. G. Lorentz proved that if $1 \leqslant a_1 < a_2 < \ldots$ is an infinite sequence of integers, then there always is an infinite sequence of integers $1 \leqslant b_1 < b_2 < \ldots$ of density zero so that all but finitely many positive integers are of the form a_i+b_j [1]. In this note we investigate the measure theoretic analogues of this result.

Throughout this paper, the real line will be denoted by R. If A and B are subsets of R, then $A+B=\{a+b:\ a\in A,\ b\in B\}$.

THEOREM 1. *Let P be a nonempty perfect subset of R. Then there is a perfect set M with Lebesgue measure zero so that $P+M=R$.*

Let us note that it suffices to prove the theorem under the additional assumption that $P\subseteq[0,1]$. Let us also note that under this assumption it suffices to prove the existence of a closed set M so that $P+M$ contains some closed interval. With this in mind, for each n and i, set $I(i,n)=[i/2^n, (i+1)/2^n]$. For each n, set

$$A_n=\{i:\ \operatorname{int}(I(i,n))\cap P\neq\emptyset\}$$

and

$$P_n=\bigcup\{I(i,n):\ i\in A_n\}.$$

Clearly, $P_1\supseteq P_2\supseteq\ldots$ and $\bigcap P_n=P$.

We will prove the following lemma.

LEMMA 2. *There is a sequence of positive integers $m_1<m_2<m_3<\ldots$ and a sequence $\{B_p\}_{p=1}^{\infty}$ of sets of nonnegative integers so that*

1) *for each p, $B_p\subseteq[1, 2^{m_p+1})$,*

2) *for each* n, $P_{m_n}+M_n\supset[1+2^{-m_1}, 2]$ *where* $M_n=\bigcup\{I(i, m_n)\colon i\in B_n\}$,

3) *for each* n, $M_{n+1}\subset(M_n-1/2^{m_n})\cup M\cup(M_n+1/2^{m_n})$,

4) *for each* n, $\lambda(M_n)<2^{-n}$.

At this point, let us note that Theorem 1 follows immediately from Lemma 2. In fact, setting

$$M=\{x\colon \exists(x_n)\to x \text{ and for each } n,\ x_n\in M_{j_n} \text{ and } j_n\to+\infty\},$$

we see that M is a closed set with Lebesgue measure zero and $P+M\supset[1+1/2^{m_1}, 2]$.

In order to prove Lemma 2, we will employ the following finite version of Lorentz's theorem.

THEOREM L. *There is a positive number c so that for any positive integers n, m, and k, if A is a set of integers, $A\subset[m, m+k)$, with $|A|\geqslant l$, there is a set B of integers, $B\subset[n, n+2k)$ so that $A+B$ contains all integers in the interval $(n+m+k, n+m+2k]$ with $|B|<c\log l/l$.*

Proof of Lemma 2. Choose m_1 so that $2c\log l_1/l_1<1/2$, where

$$l_1=\operatorname{card}(A_{m_1}\cap[1, 2^{m_1}))\,.$$

So,

$$A_{m_1}=1\leqslant a_1<a_2<\ldots<a_l<2^{m_1}.$$

By Lorentz's theorem there is a subset $B_1=1\leqslant b_1<b_2<\ldots<b_{t_1}<2^{m_1+1}$ so that $A_{m_1}+B_1$ contains all integers in $(2^{m_1}, 2^{m_1+1}]$ and such that

$$\operatorname{card}(B_1)=t_1<c2^{m_1+1}\log l_1/l_1\,.$$

$$\operatorname{set} M_1=\bigcup\{I(i, m_1)\colon i\in B_1\}\,.$$

Then $M_1\subset[0, 2]$, $\lambda(M_1)<1/2$ and $P_{m_1}+M_1\supset[1+2^{-m_1}, 2]$.

This completes stage 1.

Stage 2 will be indicated (all higher stages are similar).

Now choose $m_2=m_1+k_1$, $k_1>0$ so that for each i, $1\leqslant i\leqslant l_1$, we have

$$\frac{c\log(l(i))}{l(i)}<\frac{2^{m_1}}{8l_1t_1}$$

where

$$l(i)=\operatorname{card}\big(A_{m_2}\cap[2^{k_1}a_i, 2^{k_1}(a_i+1))\big)\,.$$

For each i, $1\leqslant i\leqslant l_1$ and j, $1\leqslant j\leqslant t_1$, we are guaranteed by Lorentz's theorem that there is a subset $B(i,j)$ of $[2^{k_1}b_1, 2^{b_1}b_1+2\cdot 2^{k_1})$ so that

$$\big(A_{m_2}\cap[2^{k_1}a_i, 2^{k_1}(a_i+1))\big)+B(i,j)\supset[2^{k_1}, 2^{k_1}(a_i+b_j)+2\cdot 2^{k_1})$$

and

$$\operatorname{card}(B(i,j))<c2^{k_1}\log l(i)/l(i)\,.$$

Let $B_2 = \bigcup \{B(i,j): 1 \leq i_1 \leq l_1, 1 \leq j \leq t_1\}$. Let $K_2(i,j) = \bigcup \{I(p, m_2): p \in B(i,j)\}$. Then

$$\lambda(K_2(i,j)) < 2^{-m_2}(2^{k_1} c \log l(i) = 2^{-m_1} c \log l(i)/l(i) < 1/8 l_1 t_1$$

and

$$P_{m_2} + K_2(i,j) \supset [a_i + b_j + 1/2^{m_1}, q_i + b_j + 2/2^{m_1}] .$$

Set $K_2 = \bigcup \{K_2(i,j): 1 \leq i \leq l_1 \text{ and } 1 \leq j \leq t_1\}$. Then

$$\lambda(K_2) < 1/3 \cdot 2^{-3} \quad \text{and} \quad K_2 \subset M_1 \cup (M_1 + 1/2^{m_1}) .$$

Set $M_2 = K_2 \cup (K_2 - 1/2^{m_1})$. Then

$$\lambda(M_2) < 2^{-2}, \quad M_2 \subset (M_1 - 1/2^{m_1}) \cup M_1 \cup (M_1 + 1/2^{m_1}) ,$$

and

$$P_{m_2} + M_2 \supset [1 + 1/2^{m_1}, 2] . \quad \text{Q.E.D.}$$

Let us remark that Theorem 1 has the following corollaries.

COROLLARY (Talagrand [2]). *Let A be an analytic subset of $\mathbf{R}$ such that if X is a closed subset of $\mathbf{R}$ of measure zero, then $A+X$ has measure zero. Then A is countable.*

Talagrand proved this result for arbitrary abelian locally compact groups. We will show later in this paper that this result cannot be extended to coanalytic sets.

We give another corollary of Theorem 1 which implies a theorem of S. J. Taylor [4].

COROLLARY. *Let P be a perfect subset of R. There is a perfect subset M of R with Lebesgue measure zero such that the linear measure of the planar set $M \times P$ is infinite.*

Proof. Let M be a perfect subset of R so that $M+P = R$ and such that $\lambda(M) = 0$. Consider the shear transformation $T: R \to R$ defined by $T((x, y)) = (x, x+y)$. Since, $\pi_2(T(M \times P))$, the projection of $T(M \times P)$ into the second coordinate, is all of R and the Lebesgue measure of $\pi_2(T(M \times P))$ is no more than the linear measure of $T(M \times P)$, $T(M \times P)$ has infinite linear measure. Noticing that if $E \subseteq R^2$, the linear measure of $T(E)$ is no more than three times the linear measure of E, it follows that the linear measure of $M \times P$ must be infinite. Q.E.D.

We note that our proof of the preceding corollary shows that if A is a subset of R such that for every subset G of R with Lebesgue measure zero, $A \times G$ has linear measure zero, then $A+E$ has Lebesgue measure zero, for every set E with Lebesgue measure zero.

QUESTION. Is the converse of this result also true?

THEOREM 3. *Let P be a nonempty perfect subset of $\mathbf{R}$. There is a subset M of $\mathbf{R}$ with Lebesgue measure 0 so that if $P = \bigcup_{i=1}^{\infty} X_i$, then there is some i so that $X_i + M = \mathbf{R}$.*

Proof. Let $\{p_n\}_{n=1}^{\infty}$ be a countable dense subset of P. For each n and m, let $M(n, m)$ be a perfect subset of $\mathbf{R}$ with Lebesgue measure zero so that

$$(P \cap [p_n - 1/m, p_n + 1/m]) + M(n, m) = \mathbf{R} .$$

Let G be a G_δ subset of R with Lebesgue measure 0 which contains $\bigcup M(n, m)$.

Suppose $P = \bigcup_{i=1}^{\infty} X_i$ and for each i, $X_i + G \neq R$.

For each i, let $r_i \in R - (X_i + G)$. Thus,

$$X_i \cap (G - r_i) = \emptyset$$

and

$$\bigcup_{i=1}^{\infty} X_i \cap \bigcap_{i=1}^{\infty} (G - r_i) = \emptyset .$$

But by construction each $G - r_i$ is a dense G_δ set with respect to P. Q.E.D.

Let us remark that Theorem 3 contrasts with several results in the opposite direction. The reminder of this paper is devoted to these contrasts.

Recall that a subset M of R is concentrated about a countable set C provided every open set which includes C contains all but countably many points of M.

THEOREM 4. *If S is a subset of R which is concentrated about a countable subset C, then $\lambda(S+P) = 0$, for every closed set P with Lebesgue measure zero.*

Proof. It is enough to prove this for compact closed sets P with $\lambda(P) = 0$. Let $C = \{x_n \colon n \in N\}$. Let $\varepsilon > 0$ and let V be an open set with $\lambda(V) < \varepsilon$ and

$$V \supset \bigcup_{n=1}^{\infty} (P + x_n) .$$

Let $T = \{x \in S \colon (P + x) \cap (R - V) \neq \emptyset\}$. It can be checked that T is closed with respect to S. Thus, $S - T$ is open with respect to S and contains C. Therefore, $S - T$ contains all but countably many points of S. This implies that $\lambda(S+P) < \varepsilon$. Q.E.D.

One may think that if S is concentrated then $\lambda(S+P) = 0$, for every set P of measure zero. However, we have the following theorems.

THEOREM 5. *There are subsets G_1 and G_2 of R both of which are subspaces of R over the field of rationals such that $G_1 \cap G_2 = \{0\}$, $G_1 + G_2 = R$ and both G_1 and G_2 have Lebesgue measure zero.*

The proof of this theorem will be based on the next lemma. Let us set some notation first. Let K_1 be the set of all x which can be expressed in the form

$$x = \sum_{i=1}^{\infty} a_i/(2i)! ,$$

where $0 \leqslant a_i < 2i$, $i = 1, 2, \ldots$

Let K_2 be the set of all x which can be expressed in the form

$$x = \sum_{i=1}^{\infty} a_i/(2i+1)! ,$$

where $0 \leqslant a_i < 2i+1$, $i = 1, 2, \ldots$

LEMMA 6. *Let H_i be the subgroup of R generated by K_i, $i = 1, 2$. Then $H_1 + H_2 = R$ and $\lambda(H_1) = \lambda(H_2) = 0$.*

Proof. Since every x in $[0, 1]$ can be written in the form

$$x = \sum_{i=1}^{\infty} a_i/i!\,,$$

where, $0 \leqslant a_i < i$, for $i = 1, 2, \ldots$, it follows that $H_1 + H_2 = R$.

The subgroup H_1 can be expressed as

$$H_1 = \bigcup_{(p_1, \ldots, p_s)} (p_1 K_1 + \ldots + p_s K_1)\,,$$

where the union is taken over all finite sequences of integers. Thus, in order to show that H_1 has measure zero, it suffices to fix $(p_1, \ldots, p_s)$ and show that $L = p_1 K_1 + \ldots + p_s K_1$ has measure zero. If $x \in L$, then x can be written

$$x = \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{s} p_i a_k^i\Big)/(2k)!$$

with $0 \leqslant a_k^i < 2k$, $i = 1, 2, \ldots, s$; $k = 1, 2, \ldots$ There is a positive integer m so that

$$x = \sum_{k=1}^{\infty} c_k/(2k)!\,,$$

where $|c_k| \leqslant m(2k)$, $k = 1, 2, \ldots$ For each k, $c_k = t_k(2k) + d$, where $|t_k| \leqslant m$ and $0 \leqslant d_k < 2k$. So, x can be expressed as

$$x = \sum_{i=1}^{\infty} b_i/i!\,,$$

where $|b_{2i-1}| \leqslant m$ and $0 \leqslant b_{2i} < 2i$, for $i = 1, 2, \ldots$ Let $E(m)$ be the set of all such x. It suffices to show that $E(m)$ has measure zero. For each tuple $(a_1, \ldots, a_{2n})$ such that $|a_1|, |a_3|, \ldots, |a_{2n-1}| \leqslant m$ and $0 \leqslant a_{2i} < 2i$, let $H(a_1, \ldots, a_{2n})$ be the closed interval with center $\sum_{i=1}^{\infty} a_i/i!$ and radius $4m/(2n-1)!$ For each n, $E(m) \subseteq \bigcup H(a_1, \ldots, a_{2n})$, where the union is taken over all appropriate $2n$-tuples. But,

$$\lambda\big(\bigcup H(a_1, \ldots, a_{2n})\big) \leqslant \sum \lambda\big(H(a_1, \ldots, a_{2n})\big) \leqslant \frac{8m}{(2n-1)!}(2m+1)^n(2\cdot 4\cdot \ldots \cdot 2n)\,.$$

Since this last expression goes to zero as n increases, $E(m)$ has measure zero.

A similar argument shows that H_2 also has measure zero.

Proof of Theorem 5. Let V_i be the subspace of R over the rationals which is generated by the additive subgroup H_i, $i = 1, 2$. Thus,

$$V_1 = \bigcup (r_1 H_1 + \ldots + r_n H_1)\,,$$

where the union is taken over all tuples of rationals. But,

$$\frac{q_1}{p_1}H_1+\ldots+\frac{q_n}{p_n}H_1=\left(\frac{1}{p_1\ldots p_n}\right)\left(\sum_{i=1}^{n} q_ip_1\ldots \hat{p}_i\ldots p_nH_1\right).$$

Therefore, each set $r_1H_1+\ldots+r_nH_1$ has measure zero. Thus, V_1 and V_2 have measure zero.

Set $G_1=V_1$. The set G_2 will be constructed by transfinite recursion.

Well order $R-G_1$: $x_0, x_1, \ldots, x_\alpha, \ldots, \alpha<\delta$. Express x_0 as $x_0=g_{10}+v_{20}$, where $g_{10}\in G_1$ and $v_{20}\in V_2$. Set $G_{20}=\{qv_{20}\colon q\in Q\}$. Then $G_{20}\subseteq V_2$. G_{20} is a subspace of R over Q, the rationals, $G_{20}\cap G_1=\{0\}$ and $x_0\in G_1+G_{20}$.

Suppose $0<\alpha<\delta$ and for every β, $0\leqslant\beta<\alpha$, subsets $G_{2\beta}$ of R have been determined so that $G_{2\beta}\subseteq V_2$, $G_{2\beta}$ is a subspace of R over Q, $G_{2\beta}\cap G_1=\{0\}$, $x_\beta\in G_1+G_{2\beta}$ and if $0\leqslant\varkappa<\tau<\alpha$, then $G_{2\varkappa}\subseteq G_{2\tau}$. Let $T_{2\alpha}=\bigcup\{G_{2\beta}\colon \beta<\alpha\}$. If $x_\alpha\in G_1+T_{2\beta}$, set $G_{2\alpha}=T_{2\alpha}$. If $x_\alpha\notin G_1+T_{2\alpha}$, write $x_\alpha=g_{1\alpha}+v_{2\alpha}$, where $g_{1\alpha}\in G_1$ and $v_{2\alpha}\in V_2$. Set $G_{2\alpha}=\{t+qv_{2\alpha}\colon t\in T_{2\alpha} \text{ and } q\in Q\}$. In either case $G_{2\alpha}$ still satisfies the defining conditions. Finally set $G_2=\bigcup\{G_{2\alpha}\colon \alpha<\delta\}$. Q.E.D.

Next, we note that under some set theoretic assumption an even stronger example along the lines of Theorem 5 can be given.

THEOREM 7. *Suppose that the union of less than continuumly many meager subsets of R is meager. There are subsets G_1 and G_2 of R both of which are subspaces of R over the field of rationals, both of which meet every meager set in a set of cardinality less than $2^{\aleph_0}$ and such that $G_1\cap G_2=\{0\}$ and $G_1+G_2=R$. (Of course, if every subset of R with cardinality $<2^{\aleph_0}$ has measure zero, then G_1 and G_2 both have measure zero. If* CH *holds, then G_1 and G_2 are both Lusin sets.)*

Proof. Let ω_c be the first ordinal with cardinality $2^{\aleph_0}$. Well-order the closed nowhere dense subsets of R into type ω_c: $F_0, F_1, \ldots, F_\alpha, \ldots, \alpha<\omega_c$. Also, well-order $R-\{0\}$ into type ω_c: $x_0, x_1, \ldots, x_\alpha, \ldots, \alpha<\omega_c$. We denote the rational numbers by Q and if $S\subseteq R$, then $\langle S\rangle_Q$ denotes the rational span of S.

It will be shown by transfinite recursion that there are elements s_α, t_α, $\alpha<\omega_c$ of R which have the following properties for each $\alpha<\omega_c$ (Notation: for $\tau<\omega_c$, $S_\tau=\{s_\beta\colon \beta\leqslant\tau\}$ and $T_\tau=\{t_\beta\colon \beta\leqslant\tau\}$).

1. $\langle S_\alpha\rangle_Q+\langle T_\alpha\rangle_Q\supseteq\{x_\gamma\colon \gamma\leqslant\alpha\}$,
2. $\langle S_\alpha\rangle_Q\cap\langle T_\alpha\rangle_Q=\{0\}$,
3. if $\gamma\leqslant\alpha$, then

$$\langle S_\alpha\rangle_Q\cap\left(\bigcup\{F_\beta\colon \beta\leqslant\gamma\}\right)\subseteq\langle S_\gamma\rangle_Q,$$

and

$$\langle T_\alpha\rangle_Q\cap\left(\bigcup\{F_\beta\colon \beta\leqslant\gamma\}\right)\subseteq\langle T_\gamma\rangle_Q.$$

Let us note that once this construction has been carried out, then the conclusion of the theorem follows immediately upon setting $G_1=\langle S_{\omega_c}\rangle_Q$ and $G_2=\langle T_{\omega_c}\rangle_Q$.

Construct s_0 and t_0 as follows. Set

$$B = \left(\bigcup \{qF_0 \colon q \in Q \text{ and } q \neq 0\}\right) \cup$$
$$\cup \left(\bigcup \{x_0 - qF_0 \colon q \in Q \text{ and } q \neq 0\}\right) \cup$$
$$\cup \left(\{qx_0 \colon q \in Q\}\right).$$

Choose s_0 to be an element of the residual set $R-B$ and set $t_0 = x_0 - s_0$.

Clearly,

$$\langle\{s_0\}\rangle_Q + \langle\{t_0\}\rangle_Q \supseteq \{x_0\},$$
$$\langle\{s_0\}\rangle_Q \cap \langle\{t_0\}\rangle_Q = \{0\},$$
$$\langle\{s_0\}\rangle_Q \cap F_0 = \{0\},$$
$$\langle\{t_0\}\rangle_Q \cap F_0 = \{0\}.$$

Suppose $0 < \tau < \omega_c$ and elements s_α, t_α, $\alpha < \tau$ have been determined so that if $\alpha < \tau$, then conditions 1, 2, and 3 all hold.

Define s_τ and t_τ as follows. First, set $S'_\tau = \{s_\alpha \colon \alpha < \tau\}$ and $T'_\tau = \{t_\alpha \colon \alpha < \tau\}$ and $W_\tau = \bigcup \{F_\alpha \colon \alpha \leqslant \tau\}$. If $x_\tau \in \langle S'_\tau\rangle_Q + \langle T'_\tau\rangle_Q$, then set $s_\tau = t_\tau = 0$. If $x_\tau \notin \langle S'_\tau\rangle_Q + \langle T'_\tau\rangle_Q$, then choose s_τ to be an element of R which is not in any of the following meager sets:

$$\bigcup \{\langle S'_\tau\rangle_Q - \langle T'_\tau\rangle_Q + rx_\tau \colon r \in Q\},$$

or

$$\bigcup \{qW_\tau - \mu \colon q \in Q \text{ and } \mu \in \langle S'_\tau\rangle_Q\}$$

or

$$\bigcup \{x_\tau - qW_\tau - v \colon q \in Q \text{ and } v \in \langle T'_\tau\rangle_Q\}.$$

Finally, set $t_\tau = x_\tau - s_\tau$.

Setting $S_\tau = S'_\tau \cup \{s_\tau\}$ and $T_\tau = T'_\tau \cup \{t_\tau\}$, we have

$$\langle S_\tau\rangle_Q + \langle T_\tau\rangle_Q \supseteq \{x_\alpha \colon \alpha \leqslant \tau\}.$$

Suppose $w \in \langle S_\tau\rangle_Q \cap \langle T_\tau\rangle_Q$. There are rationals a and b, $\mu \in \langle S'_\tau\rangle_Q$, and $v \in \langle T'_\tau\rangle_Q$ so that

$$w = \mu + as_\tau = v + bt_\tau.$$

If $x_\tau \in \langle S'_\tau\rangle_Q + \langle T'_\tau\rangle_Q$, then $s_\tau = t_\tau = 0$, and $w \in \langle S'_\tau\rangle_Q \cap \langle T'_\tau\rangle_Q = \{0\}$. If $x_\tau \notin \langle S'_\tau\rangle_Q + \langle T'_\tau\rangle_Q$, then

$$\mu + as_\tau = v + b(x_\tau - s_\tau).$$

Or,

$$(a+b)s_\tau = v - \mu + bx_\tau.$$

We consider two cases.

Case 1. $a+b = 0$. Then $bx_\tau = v - \mu$. Since $x_\tau \notin \langle S'_\tau\rangle_Q + \langle T'_\tau\rangle_Q$, $b = 0$. Then $a = 0$ and $w = \mu = v$ is an element of $\langle S'_\tau\rangle_Q \cap \langle T'_\tau\rangle$. Thus, $w = 0$.

Case 2. $a+b \neq 0$. Then

$$s_\tau = \frac{1}{a+b} v - \frac{1}{a+b}\mu + \frac{b}{a+b} x_\tau .$$

But, this is prohibited by the choice of s_τ. Thus,

$$\langle S_\tau\rangle_Q \cap \langle T_\tau\rangle_Q = \{0\}.$$

Next, suppose $\gamma<\tau$ and $w \in \langle S_\tau\rangle_Q \cap (\bigcup \{F_\beta\colon \beta\leqslant\gamma\})$. There is an element $\mu\in\langle S'_\tau\rangle_Q$ and a rational r so that $w=\mu+rs_\tau$. If $x_\tau\in\langle S'_\tau\rangle_Q+\langle T'_\tau\rangle_Q$, then $s_\tau=0$ and it follows that $w\in\langle S_\gamma\rangle_Q$. If $x_\tau\notin\langle S'_\tau\rangle_Q+\langle T'_\tau\rangle_Q$ and $r\neq 0$, then $s_\tau\in(1/r)w-\mu$. But this is prohibited by the choice of s_τ. Therefore, $r=0$, $w=\mu$ and it follows that $w\in\langle S_\gamma\rangle_Q$. Thus,

$$\langle S_\tau\rangle_Q \cap (\bigcup F_\beta\colon \beta\leqslant\gamma\})\subseteq\langle S_\gamma\rangle_Q .$$

It can be shown in a similar fashion that

$$\langle T_\tau\rangle_Q \cap (\bigcup \{F_\beta\colon \beta\leqslant\gamma\})\subseteq\langle T_\gamma\rangle_Q . \quad \text{Q.E.D.}$$

Our final goal is to show that under certain conditions there is an uncountable concentrated set such that the sum of this set with every set of Lebesgue measure zero still has measure zero. This is the content of Theorems 12 and 13. First, we prove some lemmas which hold outright.

Let us make the following conventions.

Define $\Psi\colon 2^N\to[0,1]$ by

$$\Psi(\langle z_n\rangle)=\sum_{n=1}^{\infty} z_n 2^{-n}.$$

If $S\subset N$, let

$$P_S=\{z\in 2^N\colon \forall n\notin S\ (z_n=0)\}$$

and let

$$Q_S=\Psi(P_S)\subset[0,1].$$

Notice that if $S\subset T\subset N$, then $P_S\subset P_T$ and $Q_S\subset Q_T$. Also, if S is infinite, then P_S and Q_S are perfect sets.

For each $S\subset N$, set

$$R_S=\bigcup\{Q_T\colon |S\Delta T|<\omega\}=\bigcup\{Q_T\colon T\subset^* S\}.$$

Of course, if $T\subset^* S$, then $R_T\subset R_S$.

Finally, if $f\in N^N$, P_f, Q_f, R_f denote $P_{\operatorname{ran}(f)}$, $Q_{\operatorname{ran}(f)}$, and $R_{\operatorname{ran}(f)}$, respectively.

LEMMA 6. *If I and J are subintervals of R, then $\lambda(I+J)=\lambda(I)+\lambda(J)$.*

LEMMA 7. *Let I be a subinterval of R and let $f\in N^N$ be strictly increasing. For each $n\in N$,*

$$\lambda(I+Q_f)\leqslant 2^n(\lambda(I)+2^{-f(n)}).$$

Proof. Notice that for each $n\in N$, Q_f is a subset of a union of 2^n intervals each of length $2^{-f(n)}$. Thus, this lemma follows immediately from the preceding lemma. Q.E.D.

Let us make the following convention, if $f, g\in N^N$, say $f\leqslant g$ past n provided $\forall m\geqslant n(f(m)\leqslant g(m))$ and $f\leqslant^* g$ if and only if $\exists n(f\leqslant g$ past $n)$.

LEMMA 8. *Let U be an open subset of R with $\lambda(U)<+\infty$. For each $\varepsilon>0$ and $n\in N$, there is a strictly increasing $f\in N^N$ such that if $g\in N^N$ is strictly increasing and $f\leqslant g$ past n, then*

$$\lambda(U+Q_g)\leqslant 2^n\lambda(U)+\varepsilon .$$

Proof. By blocking the components of U, there is a sequence $\{V_k\}_{k=1}$ of pairwise disjoint sets such that $U=\bigcup V_k$, each V_k is a union of finitely many open intervals and for each $k>1$,

$$\lambda(U-\bigcup_{j<k} V_j)\leqslant \varepsilon\cdot 2^{-2k-n-2} .$$

Consequently, for each $k>1$, $\lambda(V_k)\leqslant\varepsilon\cdot 2^{-2k-n-2}$ with

$$V_k=\bigcup\{I_k^i : i<r_k\},$$

where the sets I_k^i are disjoint open intervals.

Now, choose a strictly increasing f so that $2^{n-f(n)}\cdot r_1\leqslant\varepsilon\cdot 2^{-1}$ and so that for $k>1$,

$$2^{n+k-f(n+k)}\cdot r_k\leqslant\varepsilon\cdot 2^{-k-1} .$$

Assume $g\in N^N$, g is strictly increasing and $g\geqslant f$ past n. Then

$$\lambda(V_1+Q_g)\leqslant\sum_{i<r_1}\lambda(I_1^i+Q_g)\leqslant\sum_{i<r_0}2^n(\lambda(I_1^i)+2^{-g(n)})$$
$$\leqslant 2^n\lambda(U)+r_1 2^{n-g(n)}\leqslant 2^n\lambda(U)+\varepsilon\cdot 2^{-1} .$$

Also, for $k>1$,

$$\lambda(V_k+Q_g)\leqslant\sum_{i<r_k}(I_k^i+Q_g)\leqslant\sum_{i<r_k}2^{n+k}(\lambda(I_k^i)+2^{-g(n+k)})$$
$$\leqslant 2^{n+k}\lambda(V_K)+r_k\cdot 2^{n+k-g(n+k)}\leqslant\varepsilon\cdot 2^{-k-1}+\varepsilon\cdot 2^{-k-1}=\varepsilon\cdot 2^k .$$

Thus,

$$\lambda(U+Q_g)\leqslant 2^n\lambda(U)+\sum_{k=1}\varepsilon\cdot 2^{-k}=2^n\lambda(U)+\varepsilon . \quad \text{Q.E.D.}$$

LEMMA 9. *If $\lambda(G)=0$, then there is a strictly increasing element g of N^N such that $\lambda(G+Q_h)=0$, whenever h is strictly increasing and $g\leqslant^* h$.*

Proof. Let $\{U_n\}_{n=1}^\infty$ be a sequence of open sets with $U_n\supseteq U_{n+1}$, $\lambda(U_n)\leqslant 2^{-2n}$ and $G\subseteq\bigcap U_n$. For each n, let f_n be a strictly increasing element of N^N such that if h is strictly increasing and $f\leqslant h$ past n, then

$$(*)\qquad \lambda(U_n+Q_h)\leqslant 2^n\lambda(U_n)+1/n .$$

Choose a strictly increasing g so that for each n, $f_n \leqslant g$ past n. If h is strictly increasing and $g \leqslant^* h$, then (*) holds for all but finitely many n and therefore, $\lambda(G+Q_h) = 0$. Q.E.D.

LEMMA 10. *If $S \subseteq N$ and $\lambda(G+Q_S) = 0$, then $\lambda(G+R_S) = 0$.*

Proof. This follows from the fact that R_S is the union of all translates of Q_S by dyadic rationals. Q.E.D.

Let D be the set of all dyadic rationals in the interval $[0, 1)$ and let $E = \{z \in 2^N\colon z \text{ is eventually zero}\}$. Then $D = \Psi(E)$ and since $D = R_{\emptyset}$, $D \subset R_S$ for any subset S of N.

LEMMA 11. *Let U be an open set containing D. Then there is an element g of N^N such that $R_h \subset U$ whenever $h \in N^N$ is strictly increasing and*

$$|\{n\colon g(n) \leqslant h(n)\}| = \aleph_0 .$$

Proof. Let $V = \Psi^{-1}(U)$. For each $z = \langle z_n \rangle \in E$, let p_z be the least $p \in N$ such that $\forall n \geqslant p_z (z_n = 0)$. For each $p \in N$, let $k(p)$ be some integer greater than p so that

$$\forall z \in E\big(p_z \leqslant p \to \{\tau \in 2^N\colon z|p = \tau|p \text{ and } \forall m(p \leqslant m < k(p) \to \tau(m) = 0)\} \subset V\big).$$

In particular, for all p

$$\{\tau \in 2^N\colon \forall m(p \leqslant m < k(p)) \to \tau(m) = 0\} \subset V .$$

Thus, if $T \subset N$ and $\exists p\big([p, k(p)) \cap T = \emptyset\big)$, then $P_T \subset V$ and $Q_T \subset U$. Thus, whenever

$$|\{p\colon [p, k(p)) \cap T = \emptyset\}| = \aleph_0 ,$$

then $R_T \subset U$.

Suppose $h \in N^N$ is strictly increasing and there are only finitely many p such that $[p, k(p)) \cap \operatorname{ran}(h) = \emptyset$. Then for all but finitely many n,

$$[h(n)+1, k(h(n)+1))) \cap \operatorname{ran}(h) = \emptyset ,$$

or $h(n+1) \leqslant k(h(n)+1)$. Thus, for some $c \in N$, $h \leqslant^* g_c$, where g_c is defined by

$$g_c(1) = c ,$$
$$g_c(n+1) = \max\{k(p)\colon p \leqslant g_c(n)+1\} .$$

It follows that if we choose g so that $g_c \leqslant^* g$ for all $c \in N$, then for all strictly increasing h,

$$\neg(h \leqslant^* g) \to R_h \subset U . \quad \text{Q.E.D.}$$

THEOREM 12. *Assume $2^{\aleph_0} = \aleph_1$. Then there is a subset X of R such that*

(1) $|X| = \aleph_1$,

(2) $\forall G \subseteq R[\lambda(G) = 0 \Rightarrow \lambda(G+X) = 0]$,

(3) *X is concentrated on the rationals.*

Proof. Let $\{h_\alpha\colon \alpha<\omega_1\}$ be a family of strictly increasing elements of N^N such that

(a) if $\alpha<\beta$, then $h_\alpha \leqslant^* h_\beta$,

(b) if $\alpha<\beta$, then $\operatorname{ran}(h_\beta) \subset^* \operatorname{ran}(h_\alpha)$,

(c) $\forall g \in N^N, \exists \alpha (g \leqslant^* h_\alpha)$.

Choose $x_0 \in R_{h_0}$ and for each α, $0<\alpha<\omega_1$, $x_\alpha \in R_{h_\alpha} - \{x_\delta\colon \delta<\alpha\}$. Let $X = \{x_\alpha\colon \alpha<\omega_1\}$. It follows from Lemmas 9, 10, and 11 that X satisfies (1), (2) and (3) of the conclusion. Q.E.D.

Let us remark that Theorem 12 cannot be proved as it stands under MA+$\neg$CH, since MA+$\neg$CH implies that no uncountable set can be concentrated on the rationals. Of course, under MA+$\neg$CH it is true that $\lambda(G+E) = 0$ for any set G with $\lambda(G) = 0$ and $|E|<2^{\aleph_0}$. Finally, our proof of Theorem 12 can be easily modified under the assumption of MA+$\neg$CH to yield a subset X of R of size c so that if U is an open set containing the rationals then $|X-U|<c$ and such that if $\lambda(G) = 0$, then $\lambda(G+X) = 0$.

Added in proof. Friedman and Talagrand [6] have done this.

Open Question. Can one prove in ZFC that there is an X satisfying (1) and (2) of Theorem 12?

Added in proof. T. Carlson has shown that the answer is no.

Finally, we comment on where an X satisfying the conditions of Theorem 12 can lie in the projective hierarchy. From Theorem 1, X cannot contain a perfect set, so X cannot be analytic. Also, one cannot even produce a projective X in ZFC+GCH, since Solovay has shown that it is consistent with GCH that every uncountable projective set contains a perfect subset. If we assume V = L, then a standard argument, due to Gödel will produce an X which is Δ^1_2 = (PCA $\cap$ CPCA). A somewhat more careful argument yields:

Theorem 13. *If* V = L, *then there is an X satisfying the conditions of Theorem* 12 *which is coanalytic* ($=\pi^1_1$).

Proof. It is sufficient to show that there is a subset $H = \{h_\alpha\colon \alpha<\omega_1\}$ of N^N satisfying conditions (a), (b) and (c) listed in the proof of Theorem 12 so that H is coanalytic and such that $h_\alpha \neq h_\beta$, if $\alpha \neq \beta$. We may then define x_α to be $\Psi(z_\alpha)$, where $z_\alpha(n) = 1$ if and only if $n \in \operatorname{ran}(h_\alpha)$. The set $X = \{x_\alpha\colon \alpha<\omega_1\}$ will then be coanalytic, since $X = \Psi(g(H))$, where $g\colon N^N \to 2^N$ by $g(h) = \chi_{\operatorname{ran}(h)}$. The map g is Borel measurable and when restricted to the Borel set, D, of strictly increasing elements of N^N it is also one-to-one. Thus, $g|D$ is a Borel isomorphism of D onto $g(D)$. Since $H \subset D$, $g(H)$ will also be coanalytic.

To construct such an H, let

$$A = \{\varrho<\omega_1\colon L_\varrho \models \text{ZF}-P \text{ and } L_\varrho \text{ is point-definable}\}.$$

Since A is unbounded in ω_1, let $\{\varrho_\alpha\colon \alpha\leqslant\omega_1\}$ be an increasing enumeration of A. If $\varrho = \varrho_\beta$, define h_β to be the $<_L$-first $h \in N^N$ such that

(i) $\forall f \in N^N \cap L_\varrho (f <^* h)$.

(ii) $\forall \alpha < \beta (\operatorname{ran}(h) \subset^* \operatorname{ran}(h_\alpha))$,

(iii) $\forall n (|\{m\colon 2^n 3^m \in \operatorname{ran}(h)\}| = \aleph_0)$,

(iv) $\operatorname{Th}(L_\varrho)$ is recursive in h.

Notice that (i) and (ii) ensure that $\{h_\alpha\colon \alpha < \omega_1\}$ satisfies (a)–(c) of Theorem 12. Also, (iii) makes (iv) possible; $\operatorname{Th}(L_\varrho)$ can be encoded in, for example,

$$\{n\colon \mu m(2^n 3^m \in \operatorname{ran}(h)) > \mu m(2^{n+1} 3^m \in \operatorname{ran}(h))\}.$$

Finally by (iv), H is π_1^1. Q.E.D.

Let us note that Mokobodzki has demonstrated the following theorem [3].

THEOREM. *Let X and Y be compact metric spaces, let F be an analytic subset of $X \times Y$, let μ be a regular Borel measure on X and let ν be a regular Borel measure on Y. If, for every compact subset K of X with $\mu(K) = 0$,*

$$(*) \qquad \nu(\pi_Y(F \cap \pi_X^{-1}(K))) = \nu(\{y\colon \exists x((x, y) \in F \cap (K \times Y))\}) = 0,$$

then

$$\nu(\{y\colon |F^y| > \aleph_0\}) = 0.$$

Here $F^y \equiv \{x\colon (x, y) \in F\}$.

It follows from the methods of Theorem 13 that Mokobodzki's result cannot be extended to coanalytic sets.

THEOREM 14. *If $V = L$, there is a coanalytic subset F of $T \times T$, where T is the circle group such that $(*)$ holds where μ and ν are Haar measure on T and yet for every y, F^y is uncountable.*

Proof. Assume $V = L$ and construct an uncountable coanalytic subset C of T which is concentrated on the rational points of the circle and such that if $\mu(M) = 0$, then $\mu(C + M) = 0$.

Let $F = \{(x, x+c)\colon x \in T \text{ and } c \in C\}$. Clearly, F is a coanalytic subset of $T \times T$ which has the required properties. Q.E.D.

Let us note that Theorem 13 includes a partial answer to a question of A. Ostaszewski [5], namely, is there a coanalytic concentrated set?

Finally, let us note that our constructions can be slightly altered to answer a question of S. J. Taylor [4]. At the end of that paper Taylor raises the question of whether there is a subset X of R of power $2^{\aleph_0}$ such that if G is a subset of R with Lebesgue measure zero, then the planar set $X \times G$ will always have linear measure zero. We have the following theorems.

THEOREM 15. *Assume $2^{\aleph_0} = \aleph_1$. Then there is a subset X of R such that*

(1) $|X| = \aleph_1$,

(2) *$\forall G \subseteq R[\lambda(G) = 0 \Rightarrow X \times G$ has zero linear measure],*

(3) *X is concentrated on the rationals.*

THEOREM 16. *If* $V = L$, *then there is an* X *satisfying the conditions of Theorem* 15 *which is coanalytic.*

The proofs of these theorems are similar to those given for Theorems 12 and 13. These proofs use Lemma 11 as it stands and the following two lemmas which are analogous to Lemmas 9 and 10.

LEMMA 17. *If* $\lambda(G) = 0$, *then there is a strictly increasing element* g *of* N^N *such that the linear measure of the planar set* $G \times Q_h$ *is zero, whenever* h *is strictly increasing and* $g \leqslant^* h$.

We indicate how this lemma follows immediately from Theorem 1 of Taylor's paper. One only need note the following connections. Let $G \subseteq R$ with $\lambda(G) = 0$. Let $\{a_n\}$ be a sequence with $a_n > 0$ for each n such that if $\{b_n\} \prec \{a_n\}$ (Taylor's notation), then $G \times \mathscr{C}^{(y)}\{b_n\}$ (Taylor's notation) has zero linear measure. Let g be a strictly increasing element of N^N so that for each n, $2^{-g(n)} < a_n$. Now, if h is strictly increasing and $g \leqslant^* h$, then $\{0\} \times Q_h = \mathscr{C}^{(y)}\{b_n\}$, where $b_n = 2^{-h(n)}$.

LEMMA 18. *If* $S \subseteq N$ *and* $G \times Q_S$ *has zero linear measure, then* $G \times R_S$ *has zero linear measure.*

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