Algebra Universalis, 12 (1981) 81–92 0002-5240/81/001081-12\$01.50+0.20/0

Sets of natural numbers of positive density and cylindric set algebras of dimension 2

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A (diagonal-free) *cylindric algebra of sets* (of dimension 2) is a Boolean algebra of subsets of the cartesian product $X \times Y$ of two sets (called the *axes*) which is closed under the operations c_x and c_y of *cylindrification parallel to an axis:*

$$c_x A = \{u \in X \times Y \mid u_y = v_y \text{ for some } v \in A\},$$
$$c_y A = \{u \in X \times Y \mid u_x = v_x \text{ for some } v \in A\}.$$

Cylindric algebras of higher dimensions are defined analogously (see [7, p. 164]). Examples of cylindric algebras of sets are the *projective algebras* of subsets of the plane: classes of sets situated in the Euclidean plane, closed under Boolean operations and under projection onto either axis, and containing the direct product $A \times B$ whenever A and B are situated on the x-axis and y-axis respectively.

In [11, p. 12], Ulam has asked some fundamental questions about projective algebras of sets in the plane (and higher dimensional Euclidean spaces). It is the purpose of this paper to settle some of these questions. In §1 certain questions concerning sets of natural numbers of positive density are discussed which arise from computations involved in the construction in §2 of a countable collection of sets in the plane which is not contained in a finitely generated cylindric algebra of sets in the plane. In §3 we summarize the status of each of the other problems on projective algebras mentioned by Ulam in [11]. The first author is responsible for §1 while the results in §2 and §3 are due to the other two authors.

We use the following notation and terminology. Each ordinal is identified with the set of ordinals smaller than it. Each initial ordinal is identified with its cardinality. The first infinite ordinal is ω. Often we call a cylindric algebra of sets, a *cylindric set algebra*. We should also point out that our notation and terminology

Presented by J. D. Monk. Received December 16, 1977. Accepted for publication in final form August 2, 1979.

concerning cyclindric algebras of dimension 2 does not follow precisely that given in [7]. First, we use c_x and c_y for the cylindrifications they call c_0 and c_1. Second, and more important, we are mainly interested in what they call diagonal-free cylindric algebras, so we omit the term *diagonal-free* from our definitions. However, many of our cyclindric set algebras of dimension 2 whose axes are the same set X do, in fact, include the diagonal, $\{(x, x) \mid x \in X\}$, and so are truly cylindric algebras of sets in terminology of [7]. Thus one must be careful to understand that when we count the generators of a cylindric set algebra, the diagonal is treated just like any other set. Finally, if we wish to include the diagonal as a distinguished element, we refer to a *cylindric algebra with diagonal.*

§1. A set $I \subseteq \omega$ has *upper density* $\bar{\rho}(I) = \limsup |I \cap k|/k$ and *lower density* $\underline{\rho}(I) = \liminf |I \cap k|/k$. If $\bar{\rho}(I) = \underline{\rho}(I)$, we say I has *density* $\rho(I) = \bar{\rho}(I)$. Notice that if $I = I_1 \cup I_2$, then $\bar{\rho}(I) \leq \bar{\rho}(I_1) + \bar{\rho}(I_2)$.

LEMMA 1. *If I has positive upper density, then for some $J \subseteq I$ and for some $d > 0$, J has positive upper density and $J + d = \{j + d \mid j \in J\} \subseteq I$.*

Proof. Choose i so large that $\bar{\rho}(I) \cdot i > 2$. Then for infinitely many natural numbers k, I has at least two elements in the interval $[ki, (k+1)i)$. Any two elements in an intérval of length i have a difference l with $0 < l < i$. Let I' be the set of all natural numbers which are the largest element of I in some interval $[ki, (k+1)i)$. Then $|I' \cap ki| \leq k$ for every $k \in \omega$, so

$$\bar{\rho}(I') = \limsup \frac{|I' \cap k|}{k} \leq \limsup \frac{|I' \cap ki|}{ki} \leq \limsup \frac{k}{ki} = \frac{1}{i} < \frac{2}{i} < \bar{\rho}(I)$$
$$\leq \bar{\rho}(I') + \bar{\rho}(I \setminus I'),$$

so $\bar{\rho}(I \setminus I') > 0$. For l with $0 < l < i$, let $I_l = \{x \in I - I' \mid x + l \in I\}$. Since $I - I' = I_1 \cup \cdots \cup I_{i-1}$, we can find d such that $I_d = J$ has positive upper density. This completes the proof of the lemma.

An integer $d > 0$ *belongs* to a set $I \subseteq \omega$ if d occurs as a difference of elements of I infinitely many times, that is, $(I + d) \cap I$ is infinite.

THEOREM 2. *Suppose ω is partitioned into two infinite sets $A = \{a_i\}$ and $B = \{b_i\}$ listed in their natural order. If both* $\liminf (a_{i+1} - a_i)$ *and* $\liminf (b_{i+1} - b_i)$ *are finite, then there is a d which belongs to both A and B.*

Proof. Since $\liminf (a_{i+1} - a_i)$ and $\liminf (b_{i+1} - b_i)$ are finite, then there is a d_1 which belongs to A and a d_2 which belongs to B. Assume there is no common d

which belongs to both A and B. Then d_2 does not belong to A and d_1 does not belong to B. It follows that there is some integer $N \in \omega$ such that $b_n + d_1 \in A$ and $a_n + d_2 \in B$ for every $n > N$. So $b_n + d_1 + d_2 \in B$ and $a_n + d_2 + d_1 \in A$ whenever $n > N$. Consequently, $d_1 + d_2$ belongs to both A and B.

Remark 3. Erdos conjectured that if A and B are two sequences of positive upper density, then there is a d which belongs to both of them. Stewart and Tijdeman [10] and independently, Prikry [9] have shown that if $A_1, A_2, \ldots, A_k$ are sequences with positive upper density, then the set A of integers which belong to each of the A_i's has positive lower density. Stewart and Tijdeman give an explicit lower bound for the lower density of A.

§2. The main results in this section can be summarized in the following statement.
There is a countable cyclindric set algebra $\mathscr{C}$ of dimension 2 with the properties

(Theorem 14) *$\mathscr{C}$ is not isomorphic to a subalgebra of any finitely generated cylindric set algebra of dimension 2,*
(Theorem 15) *every finitely generated subalgebra of $\mathscr{C}$ is contained in a 2-generated cyclindric set algebra on $\omega \times \omega$.*

Before proving this statement, we prove that certain families of rectangles are subsets of finitely generated cylindric set algebras of dimension 2.

THEOREM 4. *Let U and V be infinite sets. Let $\mathscr{F} = \{X_i \times Y_i \mid i < \omega\}$ be a family of (rectangular) subsets of $U \times V$ and let $p = (u, v) \in U \times V$. Then there exists a set R such that $\mathscr{F}$ is a subset of the cylindric set algebra generated by $\{\{p\}, R\}$ in $U \times V$.*

Proof. Let $U = \dot{\bigcup}_{i<\omega} U_i$, $V = \dot{\bigcup}_{i<\omega} V_i$ where $U_0 = \{u\}$, $V_0 = \{v\}$ and $|U_j| = |V_j| = 1$ for $j = 1, 2, 3, 4$. We shall show below that $\mathscr{A}$, the cylindric set algebra generated by $\{\{p\}, R\}$ contains the sets $U_j \times V_j$, $j < 5$, and $(X_i \setminus \bigcup_{j<5} U_j) \times (Y_i \setminus \bigcup_{j<5} V_j)$. (Note that $\{p\} = U_0 \times V_0$.) Once this is accomplished, it is not hard to show that $\mathscr{A}$ contains $X_i \times Y_i$, so without loss of generality we may assume for all $i < \omega$ that $X_i \cap (\bigcup_{j<5} U_j) = \varnothing$ and $Y_i \cap (\bigcup_{j<5} V_j) = \varnothing$. We break the proof into a series of lemmas.

LEMMA 5. [7; p. 253] *If N is countable and well-ordered as $\{a_i \mid i < \omega\}$, then the cyclindric set algebra generated by the upper triangular set $\{(a_i, a_j) \mid i < j\}$ includes all singletons.*

Proof. Let $\Delta = \{(a_i, a_j) \mid i < j\}$. Let $\mathscr{A}$ be the cyclindric set algebra generated by Δ. For each $i \in \omega$, let $N_i = \{a_j \mid j \in \omega \setminus i\}$ and let $\Delta_i = \Delta \cap (N_i \times N_i)$. We claim that assuming that Δ_i, $N_i \times N_i$ are elements of $\mathcal{A}$, then $\{(a_{i+1}, a_{i+1})\}$, Δ_{i+1}, $N_{i+1} \times N_{i+1}$ are all elements of $\mathcal{A}$. It is enough to show that $\{(a_0, a_0)\}$, Δ_0, $N_0 \times N_0$ are all elements of $\mathcal{A}$ since the claim is exactly this statement relativized to $N_i \times N_i$. Note that $X = (c_x \Delta)^c = \{(a_i\ a_0) \mid i \in \omega\}$ and $Y = (c_y(\Delta^c \setminus X))^c = \{(a_0, a_i) \mid i \in \omega\}$. Thus $\{(a_0, a_0)\} = X \cap Y$ and $N_0 \times N_0 = (X \cup Y)^c$. This proves the claim. Finally, $\{(a_i, a_j)\} = c_y\{(a_i\ a_i)\} \cap c_x\{(a_j, a_j)\}$ for all $i, j \in \omega$.

Let $E_1 = \bigcup_{i \text{ even}} U_i$, $E_2 = \bigcup_{i \text{ even}} V_i$, $O_1 = U \setminus E_1$ and $O_2 = V \setminus E_2$. Let $\Delta E = \cup\{U_i \times V_j \mid i, j \text{ even } i < j\}$ and $\Delta O = \cup\{U_i \times V_j \mid i, j \text{ odd}, i < j\}$. Let $C_k = \cup\{U_i \mid i \equiv k \pmod 4 \text{ and } i > 4\}$ and $D_k = \cup\{V_i \mid i \equiv k \pmod 4 \text{ and } i > 4\}$ for $k = 1, 2, 3, 4$. Let $\mathscr{C} = \cup\{U_i \times D_{5-i} \mid 1 \le i \le 4\} \cup \{C_{5-i} \times V_i \mid 1 \le i \le 4\}$. Let $\mathscr{X}_j = \cup\{(X_i \cap C_j) \times V_{4i+9-j} \mid i < \omega\}$ and $\mathscr{Y}_j = \cup\{U_{4i+11-j} \times (Y_i \cap D_j) \mid i < \omega\}$ for $j = 1, 2, 3, 4$. Finally, define $R = \{p\} \cup (O_1 \times V_0) \cup \Delta E \cup \Delta O \cup \mathscr{C} \cup (\cup\{\mathscr{X}_j \cup \mathscr{Y}_j \mid 1 \le j \le 4\})$ and let $\mathcal{A}$ be the cyclindric set algebra generated by $\{\{p\}, R\}$.

LEMMA 6. *The sets* $\{p\}$; $O_1 \times V_0$; $(E_1 \times E_2) \setminus \{p\}$; $O_1 \times O_2$; $U_j \times D_{5-j}$, $1 \le j \le 4$; $C_{5-j} \times V_j$, $1 \le j \le 4$; $C_j \times V_{4i+9-j}$, $i \in \omega$, $i \le j \le 4$; *and* $U_{4i+11-j} \times D_j$, $i \in \omega$, $1 \le j \le 4$, *are mutually disjoint and cover* R.

Proof. Details of this proof are left to the reader. Note that j has parity opposite to $5-j$, $4i+9-j$ and $4i+11-j$.

LEMMA 7. *The sets* $E_1 \times E_2$, ΔE, $O_1 \times O_2$, ΔO *and* $U_i \times V_j$ *for all* $i, j \in \omega$ *are members of* $\mathcal{A}$.

Proof. Since $(U \times V_0) \cap R = (O_1 \times V_0) \cup \{p\}$, we have $O_1 \times V_0 = (c_x\{p\}) \cap (R \setminus \{p\}) \in \mathcal{A}$, and $E_1 \times V_0 = (c_x\{p\}) \setminus (O_1 \times V_0) \in \mathcal{A}$. Since $(U_0 \times V) \cap R = (U_0 \times V) \cap (\Delta E \cup \{p\})$, we have $U_0 \times E_2 = (c_y\{p\}) \cap R \in \mathcal{A}$, and $U_0 \times O_2 = (c_y\{p\}) \setminus (U_0 \times E_2) \in \mathcal{A}$. Thus $O_1 \times O_2 = c_y(O_1 \times V_0) \cap c_x(U_0 \times O_2)$ and $E_1 \times E_2 = c_y(E_1 \times V_0) \cap c_x(U_0 \times E_2) \in \mathcal{A}$. Hence $\Delta O = (O_1 \times O_2) \cap R \in \mathcal{A}$ and $\Delta E = (E_1 \times E_2) \cap R \in \mathcal{A}$. Finally, the proof of Lemma 7 relativized to $O_1 \times O_2$ and $E_1 \times E_2$ shows that $U_i \times V_j \in \mathcal{A}$ when i and j have the same parity; so if i and j have opposite parity $U_i \times V_j = c_x(U_{i+1} \times V_j) \cap c_y(U_i \times V_{j+1}) \in \mathcal{A}$.

LEMMA 8. *For each* $i \in \omega$ *and* $1 \le j \le 4$, $C_j \times V_{4i+9-j} \in \mathcal{A}$ *and* $U_{4i+11-j} \times D_j \in \mathcal{A}$.

Proof. For $i \in \omega$, $C_j \times V_{4i+9-j} = (C_j \times D_{5-j}) \cap c_x(U_0 \times V_{4i+9-j})$ for $1 \le j \le 4$, $U_{4i+11-j} \times D_j = (C_{3-j} \times D_j) \cap c_y(U_{4i+11-j} \times V_0)$ for $j = 1, 2$, and $U_{4i+11-j} \times D_j = (C_{7-j} \times D_j) \cap c_y(U_{4i+11-j} \times V_0)$ for $j = 3, 4$. Hence it is sufficient to show that $C_i \times D_j \in \mathcal{A}$ when $1 \le i, j \le 4$. Let $S = R \setminus \Delta E \setminus \Delta O$. By Lemma 7, $S \in \mathcal{A}$. Since $(U_j \times V) \cap S \subseteq \mathscr{C}$, $1 \le j \le 4$, then $U_j \times D_{5-j} = (U_j \times V) \cap \mathscr{C} = (U_j \times V) \cap S = c_y(U_j \times V_j) \cap S \in \mathcal{A}$, $1 \le j \le 4$, by Lemma 7. Since $(U \times V_j) \cap S \subseteq (U \times V_j) \cap \mathscr{C}$,

$1 \le j \le 4$, then $C_{5-j} \times V_j = (U \times V_j) \cap \mathscr{C} = (U \times V_j) \cap S = c_x(U_j \times V_j) \cap S \in \mathscr{A}$, $1 \le j \le 4$, by Lemma 7. Finally, $C_i \times D_j = (c_y(C_i \times V_{5-i})) \cap (c_x(U_{5-j} \times D_j)) \in \mathscr{A}$ for $1 \le i, j \le 4$.

LEMMA 9. *For each $i \in \omega$, $X_i \times Y_i \in \mathscr{A}$.*

Proof. Since we have assumed $X_i \cap U_j = \varnothing$, then $X_i = X_i \cap U = \bigcup \{X_i \cap C_k \mid 1 \le k \le 4\}$. Also $R \cap (C_j \times V_{4i+9-j}) \subseteq \mathscr{X}_i$, so $(X_i \cap C_j) \times V_{4i+9-j} = \mathscr{X}_i \cap (C_j \times V_{4i+9-j}) = R \cap (C_j \times V_{4i+9-j}) \in \mathscr{A}$ by Lemma 8. Similarly, $U_{4i+11-j} \times (Y_i \cap D_j) \in \mathscr{A}$. Then $X_i \times V = c_y(\bigcup \{R \cap (C_j \times V_{4i+9-j}) \mid 1 \le j \le 4\} \in \mathscr{A}$ and $U \times Y_i = c_x(\bigcup \{R \cap (U_{4i+11-j} \times D_j) \mid 1 \le j \le 4\}) \in \mathscr{A}$. Finally, $X_i \times Y_i = (X_i \times V) \cap (U \times Y_i) \in \mathscr{A}$. This completes the proof of Theorem 4.

Remark 10. As the referee has pointed out, the assumption that U and V are both infinite is necessary, since the theorem fails in case one of the sets U, V is finite while the other is infinite. For example, if $U = \{u\}$, then the algebra generated by any finite collection of subsets of $U \times V$ is finite.

Remark 11. If $U = V$, $u = v$ and $\dot\bigcup U_i = \dot\bigcup V_i$, then the intersection of R with the diagonal, $\{(u, u) \mid u \in U\}$, is $\{p\}$, so the cylindric set algebra with diagonal generated by R contains all the sets $X_i \times Y_i$ for $i < \omega$. In [8], the third author shows exactly which families of rectangles can be subsets of a cylindric set algebra (without diagonal) generated by a single set.

Now we need some lemmas. Let L_d be the line $y = x + d$ in $\omega \times \omega$. The upper triangular set $\Delta = \{(i, j) \mid i < j\}$ generates a cylindric algebra in $\omega \times \omega$ which contains all singletons by Lemma 5. As before, for $J \subseteq \omega$ and $d \in \omega$, $J + d = \{j + d \mid j \in J\}$.

LEMMA 12. *Let $\mathscr{A}$ be a finitely generated cylindric set algebra of dimension 2 on $U \times V$. Then there is a finite partition of $U \times V$, $\mathscr{H} = \{H_i \mid i < n\}$, so that every element L of $\mathscr{A}$ can be expressed as $L = \bigcup_{i<n} (B_i \cap H_i)$ where each B_i is a finite disjoint union of rectangles.*

Proof. Let $\{A_i \mid i < m\}$ be a set of generators for $\mathscr{A}$. Take $\mathscr{H}$ to be the collection of sets of the form $X_1 \cap X_2 \cap \cdots \cap X_m$ where each $X_j = A_i$ or A_i^c for some i. Let $\mathscr{B}$ be the Boolean algebra of all finite unions of rectangles in $\mathscr{A}$. The Boolean algebra generated by $\mathscr{B} \cup \mathscr{H}$ contains $\{A_i\}$ and is closed under cylindrification, thus it coincides with $\mathscr{A}$. The statement in the lemma follows easily.

LEMMA 13. *Let $\mathscr{C}$ be the cylindric set algebra on $\omega \times \omega$ generated by the lines L_d with $d \in \omega$ and Δ. There is no finitely generated cylindric set algebra $\mathscr{A}$ of dimension 2 on $\omega \times \omega$ such that $\mathscr{C}$ is a subalgebra of $\mathscr{A}$.*

Proof. Suppose to the contrary that such an $\mathscr{A}$ exists. Then by the previous lemma there is a finite partition of $\omega \times \omega$, $\mathscr{H} = \{H_i \mid i < n\}$, so that each L_d can be expressed as $L_d = \bigcup_{i<n} (B_i \cap H_i)$ where each B_i is a finite disjoint union of rectangles. We claim that we may assume that the B_i satisfy

(1) $B_i \subseteq c_y(H_i \cap L_d)$,

(2) B_i is a disjoint union of rectangles of the form $I \times (I + d)$.

To prove (1) note that since L_d is a function on $\omega \times \omega$, $c_y(H_i \cap L_d) \cap L_d = H_i \cap L_d$. Thus if we consider

$$L = \bigcup_{i<n} (B_i \cap c_y(H_i \cap L_d) \cap H_i) \subseteq L_d,$$

we have

$$L \cap L_d = \bigcup_{i<n} (B_i \cap c_y(H_i \cap L_d) \cap H_i \cap L_d)$$

$$= \bigcup_{i<n} (B_i \cap H_i \cap L_d) = L_d,$$

so $L = L_d$. To prove (2) note that if $B_i = R_1 \cup \cdots \cup R_k$ is a disjoint union of rectangles and $H_i \cap L_d \subseteq B_i$, then $H_i \cap L_d \subseteq R'_1 \cup \cdots \cup R'_k$ where $R'_j = c_y(R_j \cap L_d) \cap c_x(R_j \cap L_d)$. Notice that if $c_y(R_j \cap L_d) = I \times \omega$, then $c_x(R_j \cap L_d) = \omega \times (I + d)$, so $R'_j = I \times (I + d)$.

Now by using the distributive law, L_d can be expressed as a disjoint union of terms of the form $I \times (I + d) \cap H_i$. Call this expression *the given canonical decomposition of* L_d. The following facts about the given canonical decomposition of L_d follow from (1) and (2):

(3) The I's form a partition of ω,

(4) Each term $I \times (I + d) \cap H_i$ satisfies $L_d \cap (I + d)) = (I \times (I + d)) \cap H_i$.

Since the terms are disjoint, since $c_y(I \times (I + d)) = I \times \omega$ and since $c_y(L_d) = \omega \times \omega$, (3) is clear. To prove (4), we use $L_d \cap c_y(H_i \cap L_d) = L_d \cap H_i$ again and (1):

$$L_d \cap I \times (I + d) \subseteq L_d \cap c_y(H_i \cap L_d) = L_d \cap H_i \subseteq H_i,$$

so $L_d \cap (I \times (I + d)) \subseteq (I \times (I + d)) \cap H_i$. The other inclusion is clear.

We define J_i, $I_i \subseteq \omega$, $d_i \in \omega$ and $r(i) < n$ for $i \leq n$ by recursion. First set $d_0 = 0$. By (3) and remarks immediately preceeding Lemma 1, we can find a term $(I_0 \times I_0) \cap H_{r(0)}$ in the given canonical decomposition of L_0 so that I_0 has positive upper density. Set $J_0 = I_0$. Second, find $e_1 > 0$ and $J'_1 \subseteq J_0$ of positive upper density so that $J'_1 + e_1 \subseteq J_0$ (by Lemma 1). Set $d_1 = e_1 + d_0$. Find a term

$(L_1 \times (I_1 + d_1)) \cap H_{r(1)}$ in the given canonical decomposition of L_{d_1} so that $J_1 = J'_1 \cap I_1$ has positive upper density. Notice that not only is $J_1 \subseteq J_0$, but also $J_1 + d_1 = J_1 + e_1 + d_0 \subseteq J_0 + d_0$. At the stage $i+1$, find $e_{i+1} > 0$ and $J'_{i+1} \subseteq J_i$ of positive upper density so that $J'_{i+1} + e_{i+1} \subseteq J_i$. Set $d_{i+1} = d_i + e_{i+1}$. Find a term $(I_{i+1} \times (I_{i+1} + d_{i+1})) \cap H_{r(i+1)}$ in the given canonical decomposition of $L_{d_{i+1}}$ so that $J_{i+1} = I_{i+1} \cap J'_{i+1}$ has positive upper density. Notice that not only is $J_{i+1} \subseteq J_i$, but also $J_{i+1} + d_{i+1} = J_{i+1} + e_{i+1} + d_i \subseteq J_i + d_i$. Suppose $r(i) = r(j) = s$ with $j < i$. Then $J_i \times (J_i + d_i) \cap H_s \subseteq I_i \times (I_i + d_i) \cap H_s = L_{d_i} \cap (I_i \times (I_i + d_i))$ by (4) and $J_i = J'_i \cap I_i$. Let $p \in J_i$. Then $(p, p + d_i) \in L_{d_i}$ and $J_i \times (J_i + d_i) \subseteq I_i \times (I_i + d_i)$, so $(p, p + d_i) \in H_r$. In addition, $(J_i \times (J_i + d_i)) \cap H_s \subseteq (J_j \times (j_j + d_j)) \cap H_s \subseteq (I_j \times (I_j + d_j)) \cap H_s = L_{d_j} \cap (I_j \times (I_j + d_j))$. Thus $(p, p + d_i)$ is in both L_{d_i} and L_{d_j}, a contradiction. Thus we have defined a one-to-one function r from $n+1$ into n. This contradiction completes the proof of this lemma.

THEOREM 14. The cylindric set algebra $\mathscr{C}$ is not isomorphic to a subalgebra of any finitely generated set algebra of dimension 2.

Proof. The proof proceeds by contradiction, so assume that $\mathscr{D}$ on $U \times V$ is finitely generated and that $h : \mathscr{C} \to \mathscr{D}$ is an isomorphism of $\mathscr{C}$ into a subalgebra of $\mathscr{D}$.

Since h preserves cylindrifications and since a set R is a rectangle if and only if it is the intersection of its cylindrifications, $R = c_y R \cap c_x R$, the isomorphism h must take each rectangle of $\mathscr{C}$ to a rectangle of $\mathscr{D}$. Singletons are rectangles, and since $\Delta \in \mathscr{C}$, all singletons are in $\mathscr{C}$ (by Lemma 5). For each $i < \omega$, let $U_i \times V_i$ be the rectangle to which h takes $\{(i, i)\}$, that is, $h(\{i, i\}) = U_i \times V_i \neq \varnothing$. Now we show that if $i \neq j$, then $U_i \cap U_j = \varnothing$. It is enough to show that $(U_i \times V) \cap (U_j \times V) = \varnothing$. Now $U_i \times V = c_y(U_i \times V_i) = c_y(h\{i, i\}) = h(c_y(\{i, i\})) = h(\{i\} \times \omega)$. Similarly, $U_j \times V = h(\{j\} \times \omega)$. Since $(\{i\} \times \omega) \cap (\{j\} \times \omega) = \varnothing$, it follows that $(U_i \times V) \cap (U_j \times V) = \varnothing$ and $U_i \cap U_j = \varnothing$. Similarly, if $i \neq j$, then $V_i \cap V_j = \varnothing$. For each $i < \omega$, choose $u_i \in U_i$ and $v_i \in V_i$. Let $\bar{U} = \{u_i \mid i < \omega\}$ and $\bar{V} = \{v_i \mid i < \omega\}$. Then $\bar{U}$ and $\bar{V}$ are both infinite.

Let $\mathscr{E}$ be the algebra obtained from $\mathscr{D}$ by relativizing to $\bar{U} \times \bar{V}$. Then $\mathscr{E}$ is finitely generated by the relativizations of the generators of $\mathscr{D}$. Define $g : \mathscr{C} \to \mathscr{E}$ by $g(L) = h(L) \cap (\bar{U} \times \bar{V})$. We shall prove that g is an isomorphism of $\mathscr{C}$ onto a subalgebra of $\mathscr{E}$. It is not hard to check that g is a cylindric algebra homomorphism. To prove that g is one-to-one, let L and M be different sets in $\mathscr{C}$ and choose a point (i, j) that is in one but not the other, say $(i, j) \in L \setminus M$. Then $\{(i, j)\} \subseteq L$ and $\{(i, j)\} \cap M = \varnothing$. Now $h(\{(i, j)\}) = U_i \times V_j$ since h preserves cylindrifications and $\{(i, j)\}$ can be expressed in terms of cylindrifications of $\{(i, i)\}$ and $\{(j, j)\}$. Thus

$U_i \times V_j \subseteq h(L)$ and $(U_i \times V_j) \cap h(M) = \varnothing$. So $(u_i, v_j) \in g(L)$ and $(u_i, v_j) \notin g(M)$. Therefore, g is an isomorphism into $\mathscr{E}$.

Now define a function f from $\mathscr{E}$ into subsets of $\omega \times \omega$ by $f(L) = \{(i, j) \mid (u_i, v_j) \in L\}$. Then f essentially changes the names of the points in the underlying set $\bar{U} \times \bar{V}$ of $\mathscr{E}$. So f carries $\mathscr{E}$ isomorphically onto an algebra $\mathscr{A}$ on $\omega \times \omega$. Composing g and f gives an isomorphism η of $\mathscr{C}$ into $\mathscr{A}$. We prove that η is the identity on $\mathscr{C}$ as follows. If $(i, j) \in L$, then $\{(i, j)\} \subseteq L$, so $h(\{(i, j)\}) = U_i \times V_j \subseteq h(L)$. Thus $\{(u_i, v_j)\} \in (U_i \times V_j) \cap (\bar{U} \times \bar{V}) \subseteq h(L) \cap (\bar{U} \times \bar{V}) = g(L)$. So $\{(i, j)\} = f(\{(u_i, v_j)\}) \subseteq f(g(L)) = \eta(L)$. Thus $(i, j) \in \eta(L)$. Similarly, if $(i, j) \in L^c$, then $(i, j) \in \eta(L^c) = (\eta(L))^c$, so $L = \eta(L)$.

We have shown, therefore, that $\mathscr{C}$ is a subalgebra of $\mathscr{A}$. Now $\mathscr{A}$ is isomorphic to $\mathscr{E}$, so $\mathscr{A}$ is finitely generated. Thus $\mathscr{C}$ is a subalgebra of a finitely generated algebra $\mathscr{A}$ on $\omega \times \omega$, contradicting Lemma 13.

THEOREM 15. *Every finitely generated subalgebra of $\mathscr{C}$ is contained in a 2-generated algebra on $\omega \times \omega$.*

Proof. $\mathscr{C}$ is generated by $\{\Delta\} \cup \{L_d \mid d \in \omega\}$. Let $\mathscr{D}$ be a finitely generated subalgebra of $\mathscr{C}$. Each of the generators of $\mathscr{D}$ can be expressed in terms of a finite subset of $\{\Delta\} \cup \{L_d \mid d \in \omega\}$. Choose k so large that each generator of $\mathscr{D}$ can be expressed in terms of a subset of $\{\Delta\} \cup \{L_d \mid d < k\}$ and let $\mathscr{E}$ be the algebra on $\omega \times \omega$ generated by $\{\Delta\} \cup \{L_d \mid d < k\}$. Then $\mathscr{D}$ is a subalgebra of $\mathscr{E}$. So to prove that $\mathscr{D}$ is a subalgebra of a 2-generated algebra, it is enough to show that $\mathscr{E}$ is.

For each $i \in \omega$, let $[i]$ be the residue class of i modulo k. Let $R = (\cup\{L_d \mid d < k\}) \cup (\cup\{[i] \times \{i\} \mid i < k\})$. Let $\mathscr{F}$ be the cylindric set algebra on $\omega \times \omega$ generated by Δ and R. Since $\mathscr{D}$ is a subalgebra of $\mathscr{E}$, to prove the lemma it is enough to show that $\mathscr{E}$ is a subalgebra of $\mathscr{F}$. To prove this, we show that the generators of $\mathscr{E}$ are members of $\mathscr{F}$. Since Δ is a generator of $\mathscr{F}$, it is enough to show that each L_d is a member of $\mathscr{F}$ for $d < k$. First we show by a series of claims that $\cup\{L_d : d < k\}$ and the sets $[i] \times [j]$, where $i < k$ and $j < k$, are all members of $\mathscr{F}$.

CLAIM 1. For every $i < k$, the sets $[i] \times \{i\}$ and $[i] \times \omega$ are members of $\mathscr{F}$.

Proof. Since $\Delta \in \mathscr{F}$, by Lemma 5, every finite subset of $\omega \times \omega$ is a member of $\mathscr{F}$. Since $\{(i, i)\} = ([i] \times \{i\}) \cap (k \times \{i\})$ is finite, it is a member of $\mathscr{F}$. Now $[i] \times \{i\} = (([i] \times \{i\}) \cap (k \times \{i\})) \cup (([i] \times \{i\}) \cap ((\omega \setminus k) \times \{i\}))$. So to prove that $[i] \times \{i\}$ is a member of $\mathscr{F}$, it is enough to show that $([i] \times \{i\}) \cap ((\omega \setminus k) \times \{i\})$ is a member of $\mathscr{F}$. But $([i] \times \{i\}) \cap ((\omega \setminus k) \times \{i\}) = R \cap ((\omega \setminus k) \times \{i\})$ as the reader can easily check. So $([i] \times \{i\}) \cap ((\omega \setminus k) \times \{i\})$ and $[i] \times \{i\}$ are both members of $\mathscr{F}$. Since $[i] \times \omega = c_y([i] \times \{i\})$, it follows that $[i] \times \omega$ is also a member of $\mathscr{F}$.

CLAIM 2. The set $\cup\{L_d \mid d<k\}$ is a member of $\mathcal{F}$.

Proof. Since Δ is a member of $\mathcal{F}$, the sets $\{(0,0)\}, \{(1,1)\}, \ldots, \{(k-1),(k-1))\}$ are in $\mathcal{F}$; so the cylindrifications parallel to the x-axis of these sets are members of $\mathcal{F}$. That is, $\omega\times\{0\}, \omega\times\{1\}, \ldots, \omega\times\{k-1\}$ are members of $\mathcal{F}$. Thus the union, $\omega\times k$, of these cylindrifications is a member of $\mathcal{F}$, as is its complement, $\omega\times(\omega\setminus k)$.

Since

$$\cup\{L_d \mid d<k\}=((\cup\{L_d \mid d<k\})\cap(\omega\times k))\cup((\cup\{L_d \mid d<k\}\cap(\omega\times(\omega\setminus k))),$$

it is enough to show that $(\cup\{L_d \mid d<k\})\cap(\omega\setminus k)$ and $(\cup\{L_d \mid d<k)\cap(\omega\times(\omega\setminus k))$ are members of $\mathcal{F}$.

We show that $S=(\cup\{L_d \mid d<k\})\cap(\omega\times k)$ is a member of $\mathcal{F}$ by showing that it is finite. Suppose $(i,j)\in S$. Then $j<k$, and for some d with $d<k$, $j=i+d$. So for some d with $d<k$, both $j=i+d$ and $i=j-d<j<k$. That is, both i and j are less than k. This shows that S is finite and a member of $\mathcal{F}$.

Since

$$\cup\{[i]\times\{i\} \mid i<k\}\subseteq\cup\{\omega\times\{i\} \mid i<k\}=\omega\times k,$$

it follows that

$$(\cup\{[i]\times\{i\} \mid i<k\})\cap(\omega\times(\omega\setminus k))=\varnothing.$$

Using this fact, the definition of R and the distributive law, it is easy to see that $\cup\{L_d \mid d<k\}\cap(\omega\times(\omega\setminus k))=R\cap(\omega\times(\omega\setminus k))$, a member of $\mathcal{F}$.

CLAIM 3. The line L_0 is a member of $\mathcal{F}$.

Proof. Since Δ is a member of $\mathcal{F}$, $\{(i,j) \mid i\geq j\}=\Delta^c$ is a member of $\mathcal{F}$. If d and i are in ω and $d>0$, then the line $L_d=\{(i,i+d) \mid i<\omega\}$ has empty intersection with Δ^c. Thus by the distributive law, the intersection of $\cup\{L_d \mid d<k\}$ and Δ^c is $L_0\cap\Delta^c=L_0$. By claim 2, the set $\cup\{L_d \mid d<k\}$ is a member of $\mathcal{F}$, so L_0 is a member of $\mathcal{F}$.

CLAIM 4. For each i and j less than k, the set $[i]\times[j]$ is a member of $\mathcal{F}$.

Proof. Suppose i and j are each less than k. By Claim 1, the sets $[i]\times\omega$ and $[j]\times\omega$ are members of $\mathcal{F}$. Since $[i]\times[j]=([i]\times\omega)\cap(\omega\times[j])$, it is enough to show that $\omega\times[j]$ is a member of $\mathcal{F}$. By Claim 3, the line L_0 is in $\mathcal{F}$. Note that $c_x(L_0\cap([j]\times\omega))$ is the set of all pairs (n,m) for which $(m,m)\in[j]\times\omega$. Since $(m,m)\in[j]\times\omega$ if and only if $m\in[j]$, we have $c_x(L_0\cap([j]\times\omega))=\omega\times[j]$. Thus $\omega\times[j]$ is a member of $\mathcal{F}$. This establishes Claim 4.

Finally, we show that for all $d<k$, the set L_d is a member of $\mathcal{F}$. The sets of the form $[i]\times[j]$ for i and j less than k form a partition of $\omega\times\omega$, so for $d<k$, the line

L_d can be expressed as the union of the non-empty intersections of L_d with elements of this partition. For d, i and j less than k, the set $([i]\times[j])\cap L_d$ is non-empty if and only if $i+d=j$. Thus for $e<k$,

$$\begin{aligned}
L_e &= \{([i]\times[i+e])\cap L_e \mid i<k\} \\
&= \bigcup\{([i]\times[i+e])\cap L_d \mid i<k \text{ and } d<k\} \\
&= (\bigcup\{[i]\times[i+e] \mid i<k\})\cap(\bigcup\{L_d \mid d<k\}).
\end{aligned}$$

It follows by Claims 2 and 4 that L_e is a member of $\mathscr{F}$.

Remark 16. In [8], the third author shows that the cylindric subalgebra generated by Δ and L_0 in $\mathscr{C}$ is not isomorphic to a subalgebra of a 1-generated cylindric set algebra of dimension 2.

§3. Abstract projective algebras, defined by Everett and Ulam in [5], have been shown by Chinn and Tarski [3] to be definitionally equivalent to (diagonal-free) cylindric algebras of dimension 2 with a distinguished element p (which we shall call a *base point*) which is an atom and satisfies $c_xp\cap c_yp=p$. Thus many theorems concerning cylindric algebras which can be found in [7] apply to projective algebras. For example, by [7; pp. 252–3], both the projective algebra generated by the finite subsets of $\omega\times\omega$ and the algebra $\mathscr{B}_n$ of all subsets of the square of a finite set n are generated by a single element. (The latter fact was overlooked in [1] where the authors proved that $\mathscr{B}_n$ was generated by two elements.) Every cylindric algebra of sets of finite dimension is simple since the ideal of all sets congruent to the empty set must be closed under cylindrifications and hence contains the universe [7; p. 170 and p. 281]. It follows that there are no free (in the sense that no relations exist between the elements except those that are universally true) cylindric algebras of sets of finite dimension.

In a forthcoming series of papers [8], the third author proves the following results.

(A) There are exactly 7 non-isomorphic 1-generated cylindric algebras of sets of dimension 2 (without diagonal).
(B) There are $2^{\aleph_0}$ non-isomorphic 1-generated cylindric algebras of sets of dimension n with $2\le n<\omega$ (with diagonal).
(C) There are $2^{\aleph_0}$ non-isomorphic 1-generated projective algebras.
(D) There are $2^{\aleph_0}$ non-isomorphic 1-generated cylindric algebras of sets of dimension n (without diagonal) with $2<n<\omega$.

We shall now show that all the questions in [8] on projective algebras have been settled. The italics denote a direct quote from [8; pp. 12–13]. The answers follow the questions.

Given a countable class of sets in the plane, does there exist a finite number of sets which generate a projective algebra containing all sets of this countable class? Another statement might make this assertion for a countable class of sets given in E^m *with the generating sets required to be in some* E^n *with* $n<m$.

The answer is "no" by Theorem 14.

Does there exist a universal countable projective algebra, i.e., a countable projective algebra such that every countable projective algebra is isomorphic to some subalgebra of it?

The answer is "no" for otherwise there would only be countably many finitely generated projective algebras contradicting (C).

Is it true that, for every positive integer k, there exists a projective algebra generated by k sets in the plane and which is free in the sense that no relations exist between the generated sets except those that are true in every projective algebra? Can every projective algebra be obtained by a homomorphism of a free projective algebra?

The answer is "no" by the discussion at the beginning of this section.

How many non-isomorphic projective algebras exist with k generators?

The answer is $2^{\aleph_0}$ by (C).

§4. S. Comer has shown [4]:

If $2\le n<\omega$ *and* $\mathscr{A}$ *is a cylindric algebra of sets of dimension n with a base U such that* $|U|\le n$, *then* $\mathscr{A}$ *is generated by a single element.*

L. Henken has shown [6]:

For $2\le n$, $m<\omega$ *there is a cylindric algebra of sets* $\mathscr{A}$ *of dimension n with base U such that* $|U|=nm$ *and such that* $\mathscr{A}$ *cannot be generated by fewer than* $\log_2 m$ *elements.*

G. Bergman has shown [2]:

There is a monotone invariant, rank, on cylindric algebras of sets of dimension 2 and if $\operatorname{rank}(\mathscr{A})\ge r$, *then* $\mathscr{A}$ *requires at least* $\log_2 r$ *generators. If* $\mathscr{A}$ *is finitely generated, then* $\mathscr{A}$ *is contained in a cylindric algebra generated by at most* $\log_2 r+1$ *generators.*

J. D. Monk suggests the question:

Let $2\le n$, $m<\omega$. let $f(n,m)$ *be the largest* $k<\omega$ *such that there is a cylindric algebra* $\mathscr{A}$ *of sets of dimension n with base m such that* $\mathscr{A}$ *cannot be generated by* $<k$ *elements. Find f.*

Acknowledgement

The authors would like to thank the referee for considerable assistance in revising this paper. In particular, the short proof of Theorem 2 is entirely due to the referee and replaces the infinitely long original version. The authors would also like to thank J. D. Monk for his help, suggestions and references.

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