Reprint from ISNM 60
Functional Analysis and Approximation

ON THE LEBESGUE FUNCTION OF INTERPOLATION

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Solving an old problem of P.Erdős, we prove the best possible in order estimation for the Lebesgue function of Lagrange interpolation.

1. Introduction

Let $Z=\{x_{kn}\}$, $n=1,2,\ldots;$ $1\le k\le n$, be a triangular matrix where

$$(1.1)\qquad -1\le x_{nn}<x_{n-1,n}<\ldots<x_{1n}\le 1 \qquad (n=1,2,\ldots)$$

are n arbitrary points in $[-1,1]$ (shortly $x_k=x_{kn}$).

Putting

$$(1.2)\qquad \omega(x)=\omega_n(Z,x)=\prod_{k=1}^{n}(x-x_k) \qquad (n=1,2,\ldots),$$

$$(1.3)\qquad \ell_k(x)=\ell_{kn}(Z,x)=\frac{\omega(x)}{\omega'(x_k)(x-x_k)} \qquad (k=1,2,\ldots,n)$$

are the corresponding fundamental polynomials of the Lagrange interpolation. It is well known that the so called Lebesgue function and Lebesgue constant

$$\lambda_n(x)=\lambda_n(Z,x)=\sum_{k=1}^{n}|\ell_k(x)|\ ,\qquad \lambda_n=\lambda_n(Z)=\max_{-1\le x\le 1}\lambda_n(x)$$

play a decisive role in the convergence and divergence properties of Lagrange interpolation.

G.Faber [1] proved that

$$\lambda_n > \frac{1}{12} \ln n$$

for arbitrary matrix Z. Later S.Bernstein [1] obtained that for any system of nodes (1.1)

$$\overline{\lim_{n\to\infty}} \lambda_n(x_o) = \infty \tag{1.4}$$

for a certain $x_o \in (-1,1)$.

In 1961, P.Erdős [5] improved an earlier result of P.Erdős and P.Turán [6] proving

$$\lambda_n > \frac{2}{\pi} \ln n - c \qquad (n \geq n_o)$$

for all system (1.1) again. (Here and later $c, c_1, c_2, \ldots,$ will denote positive absolute constants.)

Finally we quote the result of P.Erdős [4] which says as follows.

THEOREM 1.1. <u>Let</u> ε <u>and</u> A <u>be any given positive numbers. Then, considering arbitrary matrix</u> Z, <u>the measure of the set in</u> x $(-\infty<x<\infty)$ <u>for which</u>

$$\lambda_n(x) \leq A \quad \underline{\text{if}} \quad n \geq n_o(A,\varepsilon), \tag{1.5}$$

<u>is less than</u> ε.

2. Results

Here we prove the following improvement of Theorem 1.1.

THEOREM 2.1. <u>Let</u> $\varepsilon > 0$ <u>be any given number. Then for arbitrary matrix</u> Z <u>there exist sets</u> H_n <u>with</u> $|H_n| \leq \varepsilon$ <u>and</u> $\eta(\varepsilon) > 0$

<u>such that</u>

(2.1) $\lambda_n(x)>\eta(\varepsilon)\ln n$ <u>whenever</u> $x\in[-1,1]\setminus H_n$ <u>and</u> $n\geq n_o(\varepsilon)$.

The case of Chebyshev nodes shows that the order of (2.1) is best possible.

By this theorems it is easy to obtain the following

COROLLARY 2.2. <u>Let</u> $\varepsilon>0$ <u>and</u> $\eta(\varepsilon)>0$ <u>be as above</u>. <u>If</u> $S_n\subseteq[-1,1]$ <u>are arbitrary measurable sets then for any matrix</u> Z

(2.2) $\int_{S_n}\lambda_n(x)dx>(|S_n|-\varepsilon)\eta(\varepsilon)\ln n$ <u>whenever</u> $n\geq n_o(\varepsilon)$.

The case $S_n\equiv S=[a,b]$ was treated by P.Erdős and J.Szabados [7].

<u>2.1.</u> The relation (2.1) is obviously valid if $|x|\geq 1+\varepsilon$ because of $x^{n-1}\equiv\sum_{k=1}^{n}x_k^{n-1}\ell_k(x)$ which means $|x|^{n-1}\leq\sum_{k=1}^{n}|\ell_k(x)|$. So we have (2.1) on the whole real line apart from a set of measure $\leq 3\varepsilon$ $(n\geq n_o(\varepsilon))$.

<u>2.2.</u> Nearly 50 years ago S.Bernstein [1] conjectured that

$$\min_Z \lambda_n(Z)$$

is assumed if all the $n+1$ maxima in $(-1,1)$ of $\lambda_n(x)$ are the same. P.Erdős conjectured that the smallest of these $n+1$ maxima is largest again if all these $n+1$ maxima are the same. Erdős further conjectured that if the z_i are on the unit circle then the corresponding extremal problems are solved if the z_i are the n-th roots of unity.

All these conjectures were recently proved in a series of remarkable papers by T.A.Kilgore [10], C.de Boor and A.Pinkus [2] and L.Bratman [3].

3. Proof

3.1. In what follows, sometimes omitting the superfluous notations, let $x_{on}\equiv 1$, $x_{n+1,n}\equiv -1$ and

(3.1) $$J_{kn}=[x_{k+1,n},x_{kn}] \qquad (k=0,1,\dots,n;\ n=1,2,\dots)\ .$$

Let us define the index-sets K_{1n} and K_{2n}, further the sets D_{1n} and D_{2n} by

(3.2) $$|J_{kn}| \begin{cases} \le n^{-1/6} \overset{\text{def}}{=} \delta_n & \text{iff} \quad k\in K_{1n}\ , \\ > \delta_n & \text{iff} \quad k\in K_{2n}\ , \end{cases}$$

$$D_{1n}=\bigcup_{k\in K_{1n}} J_{kn}\ , \qquad D_{2n}=[-1,1]\setminus D_{1n}\ .$$

If $|J_k|\le\delta_n$ (which means $k\in K_{1n}$ and $J_k\subset D_{1n}$) we say that the interval is short; the others are the long ones.

3.2. In our common paper [8] we proved

LEMMA 3.1. Let $|J_{kn}|>\delta_n$ (k is fixed, $0\le k\le n$). Then for any fixed $0<\bar{q}<1/4$ we can define the index $t=t(k,n)$ and the set $h_{kn}\subset J_{kn}$ so that $|h_{kn}|\le 4\bar{q}|J_{kn}|$, moreover

(3.3) $$|\ell_{tn}(x)|\ge 3^{n\delta_n^5} \quad \text{if} \quad x\in J_{kn}\setminus h_{kn} \quad \text{and} \quad n\ge n_1(\bar{q})\ .$$

(See [8], Lemma 4.4. In [8] $\delta_n=1/\ln n$ but this does not make any difference in the proof.)

Now, if $\bar{q}=\varepsilon/32$, for the long intervals we obtain (2.1) (see (3.3)) if $x\in D_{2n}\setminus H_{1n}$.

Here $H_{1n} \overset{\text{def}}{=} \bigcup_{k\in K_{2n}} h_{kn}$, which means $|H_{1n}| \le 4\bar{q} \sum_k |J_k| \le \varepsilon/4$ $(n \ge n_2(\varepsilon))$.

3.3. To settle the short intervals we introduce the following notations

$$J_k(q) = J_{kn}(q) = [x_{k+1} + q|J_k|,\ x_k - q|J_k|] \qquad (0 \le k \le n)$$

where $0 \le q \le 1/2$. Let $z_k = z_{kn}(q)$ be defined by

(3.4) $\quad |\omega_n(z_k)| = \min_{x\in J_k(q)} |\omega_n(x)|, \quad k=0,1,\dots,n,$

finally let

$$|J_i, J_k| = \max(|x_{i+1} - x_k|, |x_{k+1} - x_i|) \qquad (0 \le i,\ k \le n) .$$

In [8], Lemma 4.2 we proved

LEMMA 3.2. If $1 \le k,\ r < n$ then for arbitrary $0 < q \le 1/2$

(3.5) $$|\ell_k(x)| + |\ell_{k+1}(x)| \ge q^2 \frac{|\omega_n(z_r)|}{|\omega_n(z_k)|} \frac{|J_k|}{|J_r, J_k|} \quad \text{if} \quad x \in J_r(q) .$$

3.4. Later we shall also use the

LEMMA 3.3. Let $I_k = [a_k, b_k]$, $1 \le k \le t$, $t \ge 2$, be any t intervals in $[-1,1]$ with $|I_k \cap I_j| = 0$ $(k \ne j)$, $|I_k| \le \rho$ $(1 \le k \le t)$, $\sum_{k=1}^{t} |I_k| = \mu$. Supposing that for certain integer $R \ge 2$ we have $\mu \ge 2^R \rho$, there exists the index s, $1 \le s \le t$, such that

(3.6) $$S = \sum_{k=1}^{t} \frac{|I_k|}{|I_s, I_k|} \ge \frac{R}{8} \mu .$$

I_s will be called accumulation interval of $\{I_k\}_{k=1}^{t}$.

(Here and later mutatis mutandis we apply the notations of 3.3. for arbitrary intervals.)

Note that we do not require $b_k \le a_{k+1}$.

The lemma and its proof correspond to [8], 4.1.3. Indeed, dropping the interval I_j containing the middle point of $[-1,1]$ and bisecting the same interval $[-1,1]$, we have (say) in $[0,1]$ a set of measure $\geq(\mu-|I_j|)/2 \geq (\mu-\rho)/2$ consisting of certain I_k. Doing the same, after the ℓ-th bisection we obtain that interval of length $2^{1-\ell}$ which contains certain I_k's of aggregate measure $>2^{-\ell}\mu-\rho\geq 2^{-\ell-1}\mu\geq\rho$ for $1\leq\ell\leq p \overset{\text{def}}{=} R-1$.

Consider these intervals $L_1^*,L_2^*,\dots,L_p^*$ (Fig. 1).

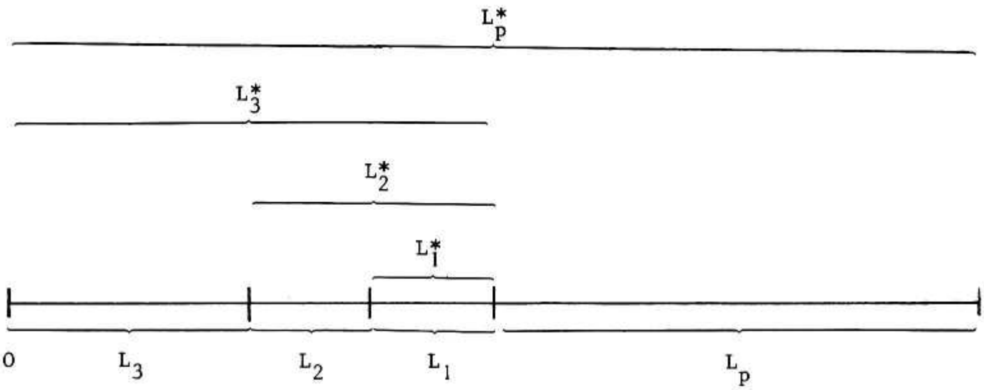

Figure 1.

Obviously $|L_\ell^*|=2^{\ell-p}$. Further each L_ℓ^* contains at least $2^{\ell-1}$ intervals I_k because

$$(3.7)\qquad \sum_{\substack{k\\ I_k\subset L_\ell^*}} |I_k| \geq 2^{\ell-p-2}\mu \qquad (1\leq\ell\leq p)\ .$$

Let $L_1=L_1^*$, further $L_\ell=L_\ell^*\setminus L_{\ell-1}^*$ $(2\leq\ell\leq p)$ (see Figure 1). If s is an index, for which $I_s\subseteq L_1$, we can write

$$(3.8)\qquad s \geq \sum_{\ell=1}^{p} \sum_{\substack{k\\ I_k\subset L_\ell}} \frac{|I_k|}{|I_s,I_k|} \overset{\text{def}}{=} B\ .$$

To estimate B, let

$$(3.9)\qquad \sum_{\substack{k\\ I_k\subset L_\ell}} |I_k| \overset{\text{def}}{=} \alpha_\ell\mu \qquad (1\leq\ell\leq p)\ .$$

By (3.7) and construction we can write

$$(3.10)\qquad \mu \sum_{\ell=1}^{i} \alpha_\ell \ge 2^{i-p-2}\mu \qquad (1\le i\le p),$$

$$(3.11)\qquad |I_s, I_i| \le 2^{\ell-p} \quad \text{if} \quad I_i \subset L_\ell \qquad (1\le \ell\le p).$$

It is worth to remark that

$$(3.12)\qquad \alpha_\ell \le 2^{\ell-2}\alpha_1 \qquad (2\le \ell\le p).$$

(Indeed, by construction $\alpha_2\le\alpha_1$, $\alpha_\ell \le \sum_{i=1}^{\ell-1} \alpha_i \le 2\sum_{i=1}^{\ell-2} \alpha_i$, $3\le\ell\le p$, from where we get (3.12).)

Now by (3.11), (3.9), (3.10), finally by the Abel transformation we obtain as follows

$$B\ge\mu 2^p \sum_{\ell=1}^{p} 2^{-\ell}\alpha_\ell = \mu 2^p\Big[\sum_{\ell=1}^{p-1} 2^{-\ell-1}\Big(\sum_{i=1}^{\ell}\alpha_i\Big) + 2^{-p}\sum_{i=1}^{p}\alpha_i\Big] \ge$$

$$\ge \mu 2^p\Big(\sum_{\ell=1}^{p-1} 2^{\ell-p-2-\ell-1} + 2^{-p-2}\Big) = [2^{-3}(p-1)+2^{-2}]\mu = \frac{p+1}{8}\mu,$$

which was to be proven.

3.5. Suppose $x\in J_{kn}(q)\subset D_{1n}$ $(1\le k\le n-1)$; whenever $\lambda_n(x)\le \le\eta(\varepsilon)\ln n$ (η will be determined later), the point x, the intervals J_{kn} and $J_{kn}(q)$, finally the index k will be called e x c e p t i o n a l . Let $q=\varepsilon/12$.

W e s h a l l p r o v e

$$(3.13)\qquad \sum_k{}' |J_{kn}| \overset{\text{def}}{=} \mu_n \le \frac{\varepsilon}{6} \qquad (n\ge n_0 = n_0(\varepsilon)).$$

Here and later the dash indicates that the summation is extended only over the exceptional indices k. To prove (3.13) it is enough to consider those indices $\{n_i\}_{i=1}^{\infty} \overset{\text{def}}{=} N$ for which $\mu_{n_i} \geq \varepsilon/10$.

We can apply Lemma 3.3 for the exceptional J_{kn}'s with $\mu=\mu_n$, $\rho=\delta_n$ and $R=[\log^2 n^{1/7}]+1$ if $n\in N$ and $n\geq n_o(\varepsilon)$ (shortly $n\in N_1$).

Denote by $M_1=M_{1n}$ the accumulation interval. Dropping M_1, we apply Lemma 3.3. again for the remaining exceptional intervals with $\mu=\mu_n-|M_1|>\mu_n/2$ and the above ρ and R, supposing $\mu_n\geq\rho 2^{R+1}$ whenever $n\in N_1$. We denote the accumulation interval by M_2. At the i-th step $(2\leq i\leq\psi_n)$ we drop $M_1, M_2, \ldots M_{i-1}$ and apply Lemma 3.3. for the remaining exceptional intervals with $\mu=\mu_n - \sum_{j=1}^{i-1} |M_i|$ using the same ρ and R.

Here ψ_n is the first index for which

$$(3.14)\quad \sum_{i=1}^{\psi_n-1} |M_i|\leq\frac{\mu_n}{2} \quad\text{but}\quad \sum_{i=1}^{\psi_n} |M_i|>\frac{\mu_n}{2}, \qquad n\in N_1 .$$

If we denote by M_{ψ_n+1}, M_{ψ_n+2} ... M_{φ_n} the remaining (i.e. not accumulation) exceptional intervals (by $|M_i|\leq\delta_n$, $(\varepsilon/20)n^{1/6}<\psi_n<\varphi_n)$, by (3.6) we can write

$$(3.15)\qquad \sum_{k=r}^{\varphi_n} \frac{|M_k|}{|M_r,M_k|} \geq \frac{\mu_n \ln n}{112} \qquad \text{if} \quad 1\leq r\leq\psi_n \quad (n\in N_1) .$$

<u>3.6.</u> To go further in proving (3.13) let $\eta=c_1\varepsilon^3/6$, $u_{in}\in M_{in}(q)$ $(1\leq i\leq\varphi_n,\ n\in N_1)$ be exceptional points, where c_1 will be determined later.

If for a fixed $n\in N_1$ there exists t, $1\leq t\leq\varphi_n$, such that

$$(3.16)\qquad \lambda_n(u_{tn}) \geq c_1\varepsilon^2\mu_n \ln n ,$$

by $\eta \ln n\geq\lambda_n(u_{tn})$ we obtain (3.13) for this n. We shall prove (3.16) for arbitrary $n\in N_1$. Indeed, let us suppose that for a certain $m\in N_1$

$$\lambda_m(u_{rm})<c_1\varepsilon^2\mu_m \ln m \quad \text{where} \quad u_{rm}\in M_{rm}(q), \quad 1\le r\le \varphi_m. \tag{3.17}$$

By (3.17) we obtain

$$\sum_{r=1}^{\varphi_m} |M_{rm}|\lambda_m(u_{rm})<c_1\varepsilon^2\mu_m^2 \ln m \quad \text{where} \quad m\in N_1 . \tag{3.18}$$

On the other hand, by (3.5), for a r b i t r a r y $n\in N_1$

$$|M_r| \sum_{k=1}^{n} |\ell_k(u_r)| \ge \frac{1}{2} |M_r| \sum_k{}' [|\ell_k(u_r)|+|\ell_{k+1}(u_r)|] \ge$$

$$\ge \frac{q^2}{2} \sum_{k=1}^{\varphi_n} \left|\frac{\omega(\bar z_r)}{\omega(\bar z_k)}\right| \frac{|M_r||M_k|}{|M_r,M_k|}, \quad (1\le r\le \varphi_n),$$

so, by (3.14) and (3.15) we have

$$\sum_{r=1}^{\varphi_n} |M_r|\lambda_n(u_r)=\sum_{r=1}^{\varphi_n} |M_r| \sum_{k=1}^{n} |\ell_k(u_r)| \ge \frac{q^2}{2} \sum_{r=1}^{\varphi_n} \sum_{k=1}^{\varphi_n} \left|\frac{\omega(\bar z_r)}{\omega(\bar z_k)}\right| \frac{|M_r||M_k|}{|M_r,M_k|} \ge$$

$$\ge \frac{1}{2}\frac{q^2}{2} \sum_{r=1}^{\varphi_n} \sum_{k=r}^{\varphi_n} \left[\frac{\omega(\bar z_r)}{\omega(\bar z_k)} + \frac{\omega(\bar z_k)}{\omega(\bar z_r)}\right] \frac{|M_r||M_k|}{|M_r,M_k|} \ge$$

$$\ge \frac{q^2}{4} \sum_{r=1}^{\phi_n} |M_r| \sum_{k=r}^{\varphi_n} \frac{|M_k|}{|M_r,M_k|} > \frac{q^2}{4}\frac{\mu_n}{2}\frac{\mu_n \ln n}{112} = c_1\varepsilon^2\mu_n^2 \ln n$$

if $c_1=8.144.112$. This contradicts to (3.18), i.e. (3.16) is valid for arbitrary $n\in N_1$, which proves (3.13).

<u>3.7.</u> By definition, if the short J_{kn} is not exceptional, then for any $x\in J_{kn}(q)$ (2.1) valid, supposing that $k\ne 0,n$. If J_{0n} is short it should belong to H_n. The same should be done with

J_{nn}. Moreover, the sets $J_{kn}\setminus J_{kn}(q)$ of aggregate measure c_2 should belong to H_n, too. Obviously $c_2 \le 2q\sum_{k=0}^{n} |J_{kn}| = 4q = \varepsilon/3$. So using these, 3.2 and (3.13), we obtain

$$|H_n| \le |H_{1n}| + \mu_n + 2\delta_n + c_2 \le \varepsilon/4 + \varepsilon/6 + \varepsilon/4 + \varepsilon/3 = \varepsilon ,$$

which completes the proof.

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The authors are indebted to G.Halász for his valuable remarks and suggestions.

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