ON THE COMBINATORIAL PROBLEMS WHICH I WOULD MOST LIKE TO SEE SOLVED

by

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I was asked to write a paper about the major unsolved problems in combinatorial mathematics. After some thought it seemed better to modify the title to a less pretentious one. Combinatorial mathematics has grown enormously and a genuine survey would have to include not only topics where I have no real competence but also topics about which I never seriously thought, e.g. algorithmic combinatorics, coding theory and matroid theory. There is no doubt that the proof of the conjecture that several simply stated problems have no good algorithm is fundamental and may have important consequences for many other branches of mathematics, but unfortunately I have no real feeling for these questions and I feel I should leave the subject to those who are more competent.

I just heard that Khachiyan [59], has a polynomial algorithm for linear programming. (See also [50].) This is considered a sensational result and during my last stay in the U.S. many of my friends were greatly impressed by it.

I. Problems on Set Systems

First of all I will discuss some problems on set systems. I state only my three favourite problems, but before starting I refer to the survey paper [31].

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1. First our old problem with Rado on Δ-systems. Denote by $f_k(n)$ the smallest integer so that if $|A_i| = n$, $1 \leqq i \leqq f_k(n)$ then there are k A_i's which have pairwise the same intersection. Conjecture

$$f_k(n) < c_k^n, \tag{1}$$

where c_k depends only on k. I offer 1000 dollars for a proof or disproof of (1) for the case $k = 3$, which, I expect, contains the whole difficulty. The best upper bound $f_3(n) < (1+\varepsilon)^n n!$ is due to J. Spencer [77]. Lower bounds have been found by Abbott, Hansen and others. Abbott proved that $f_3(3) = 21$. (1) has fascinated me greatly – I really do not see why this question is so difficult. (See Erdős–Rado [34], Erdős–Milner–Rado [33], and for further problems and results Erdős–Szemerédi [41].)

2. Let $|A_k| = n$, $1 \leqq k \leqq n$, and $|A_i \cap A_j| \leqq 1$ for every $1 \leqq i < j \leqq n$. Prove that one can color the elements of $\bigcup_{k=1}^{n} A_k$ by n colors so that each A_k, $1 \leqq k \leqq n$ should contain elements of all the colors. Faber, Lovász and I made this harmless looking conjecture at a party in Boulder Colorado in September 1972. Its difficulty was realised only slowly. I now offer 500 dollars for a proof or disproof. (Not long ago I only offered 50; the increase is not due to inflation but to the fact that I now think the problem is very difficult. Perhaps I am wrong.)

Perhaps the following graph theoretic formulation is even more interesting Let $K_i(n)$, $i = 1, \ldots, n$, be n complete graphs of n vertices. Assume that every two of the $K_i(n)$'s have at most one vertex in common. Prove that the graph

$$\bigcup_{i=1}^{n} K_i(n)$$

has also chromatic number n.

There are many ways one can state more general problems e.g. let $\mathcal{G}_1, \ldots, \mathcal{G}_m$, be m graphs each of chromatic number n. Assume that no two $\mathcal{G}$'s have an edge in common. What is the smallest m for which $\bigcup_{i=1}^{m} \mathcal{G}_i$ has chromatic number greater than n? Perhaps one can further demand that any two $\mathcal{G}$'s have at most one vertex in common. I am vague because I am not sure which of these questions (if any) will lead to fruitful developments.

3. Problem of Lovász and myself, [32]. Let $f(n)$ be the smallest integer with the following property: There is a family A_k, $1 \leqq k \leqq f(n)$ satisfying $|A_k| = n$, $k = 1, \ldots, f(n)$, $|A_{k_1} \cap A_{k_2}| \geqq 1$ for every $1 \leqq k_1 < k_2 \leqq f(n)$, and for every $|S| = n-1$ there is an A_k with $A_k \cap S = \emptyset$. In other words our family can not be represented by fewer than n elements. We proved $f(n) < n^{3/2+\varepsilon}$ An improvement of our method very likely will give $f(n) < cn \log n$. I offer 500 dollars for a proof or disproof of $f(n) \leqq Cn$. In fact we can not even prove or disprove $f(n) < 3n$.

4. Chvátal [17] has the following nice conjecture. Let $\mathcal{F}$ be a family of sets such that if $A \in \mathcal{F}$ and $A' \subseteq A$ then $A' \in \mathcal{F}$. Then there is a maximal inter-

secting subfamily $\mathcal{F}_1$ of $\mathcal{F}$ (i.e. every two sets of $\mathcal{F}_1$ have nonempty intersection and $\mathcal{F}_1$ is maximal under this condition) such that all sets of $\mathcal{F}_1$ have an element in common. It is surprising that this attractive conjecture is probably very difficult.

5. Let $|S| = n$, $A + i \subset S$, $|A_i| = 3$, $1 \leqq i \leqq n + l$. V. T. Sós and I observed that then there are always two A's which have precisely one element in common and that the result fails to hold for n triples if $n \equiv 0 \pmod 4$. We then conjectured that if $A_i \subset S$, $|A_i| = k$, $l \leqq i \leqq \binom{n-2}{k-2} + 1$ then there are again two A's which have precisely one element in common. This conjecture was proved by Katona [57] for $k = 4$, and by P. Frankl [45] for all k. Then I asked the following question. Determine the smallest integer $T(n,r)$ with the following property: If $|S| = n$, $A_i \subset S$, $1 \leqq i \leqq T(n,r)$, then there are two indices i_1 and i_2 for which $|A_{i_1} \cap A_{i_2}| = r$.

Trivially $T(n,0) = 2^{n-1} + 1$ and P. Frankl [46] proved

$$T(n,1) = 2 + \begin{cases} \sum\limits_{i=\left[\frac{n+2}{2}\right]}^{n} \binom{n}{i} & n \text{ even} \\ \sum\limits_{i=\left[\frac{n+1}{2}\right]}^{n} 2n\binom{n-1}{i} & n \text{ odd} \end{cases}$$

For $r > 1$, $T(n,r)$ is not known. I conjectured that for every ε there is an η so that for $\varepsilon n < r < \left(\frac{1}{2} - \varepsilon\right) n$,

$$T(n,r) < (2-\eta)^n. \tag{2}$$

(2) seems to be an interesting and difficult conjecture. It also has some geometric application. Let $\mathcal{G}^{(n)}$ be a graph whose vertices are the points of the n-dimensional space, and two vertices are joined if their distance is 1. Denote by L_n the chromatic number of $\mathcal{G}^{(n)}$. As far as I know the problem of investigating L_n is due to Hadwiger [53] and Nelson. Larman and Rogers [62] proved $(3 + o(1))^n > L_n > cn^2$ and P. Frankl [47] proved that $L_n > n^k$ for every k if $n > n_0(k)$. (2) would imply that L_n tends to infinity exponentially. Presumably $L_n^{1/n}$ tends to a $c > 1$. P. Frankl and R. M. Wilson just proved $L_n > (1+\alpha)^n$, but they did not prove (2).

By a well known theorem of de Bruijn and myself [14] there is always a finite subgraph of $\mathcal{G}^{(n)}$ with chromatic number L_n. $\mathcal{G}^{(2)}$ received some attention and it was conjectured that $L_2 = 4$, but it is now generally believed $L_2 \geqq 5$. $L_2 \leqq 7$ is well known.

I asked the following question. Let S be a subset of the plane. Join two points of S if their distance is 1 and assume that the resulting graph $\mathcal{G}^{(2)}$ has girth k. Denote by $L_2^{(k)}$ the maximum value of the chromatic number of $\mathcal{G}^{(2)}$. Is there a k for which $L_2^{(k)} = 3$? Wormald [84] recently proved that if such a k exists it must be at least 5. Wormald's graph is constructed with the help of a computer.

More generally, let $L_2(r)$ be defined as follows. For given $0 < \alpha_1 < \ldots < \alpha_r$ join two points of the plane by an edge if their distance is one of $\alpha_1, \ldots, \alpha_r$. Take the maximum of the chromatic number of this graph for all $0 < \alpha_1 < \ldots < \alpha_r$ and denote it by $L_2(r)$. Does $L_2(r)$ increase polynomially or exponentially? I expect that $L_2(r) < r^c$ for some absolute constant c.

I offer 250 dollars for a proof or disproof of (2).

II. Problems in Combinatorial Number Theory

As I stated in a previous paper this subject is perhaps closest to my heart or rather to my brain. Here I state some of my favourite conjectures.

1. Let $r_k(n) = l$ be the smallest integer for which if $1 \leqq a_1 < \ldots < a_l \leqq n$ is a sequence of integers then the a's contain an arithmetic progression of k terms. Turán and I [42] conjectured nearly 50 years ago that for every $k \geqq 3 r_k(n) = o(n)$. I offered 1000 dollars for a proof or disproof of this conjecture. In 1972 Szemerédi [79] proved it. Later Fürstenberg [48] gave a new proof using ergodic theory and in a recent paper Fürstenberg and Katznelson [49] proved the n-dimensional version of Szemerédi's theorem.

Very little is known about the order of magnitude of $r_k(n)$. We have

$$n \exp\left(-\left(c_1 (\log n)^{1/2}\right)\right) < r_3(n) < \frac{c_2 n}{\log\log n}. \tag{1}$$

The upper bound is due to K. F. Roth [73] and the lower bound to F. Behrend [3]. It would be very desirable to improve (1) and if possible to obtain an asymptotic formula for $r_3(n)$ and more generally for $r_k(n)$. This problem is probably enormously difficult and I offer 10 000 dollars for such an asymptotic formula. In particular is it true that

$$r_k(n) = o\left(\frac{n}{(\log n)^c}\right) \tag{2}$$

for every k and c? It is an old problem in number theory whether for every k there are k primes in an arithmetic progression? At the moment the longest known arithmetic progression whose terms are all primes has 17 terms and is due to Weintraub. In this connection I conjecture that if $\sum_{r=1}^{\infty} \frac{1}{a_r} = \infty$ then for every k there are k a_r's in an arithmetic progression. Since Euler proved that the sum of the reciprocals of the primes diverges, our conjecture would settle the conjecture on primes. (2) of course also would settle it. I offer 3000 dollars for the proof or disproof of the conjecture.

2. One of my oldest conjectures going back to the early 1930's and clearly influenced by van der Waerden's theorem [82] states as follows:

Let $f(n) = \pm 1$ (i.e. we divide the integers into two classes: if n is in the first class then $f(n) = +1$, and $f(n) = -1$ if n is in the second class). Is it true that to every c there is a d and an m so that

$$\left|\sum_{k=1}^{m} f(kd)\right| > c? \tag{3}$$

Another question: Is it true that

$$\max_{md<x}\left|\sum_{k=1}^{m} f(kd)\right| > c\log x \tag{4}$$

where the maximum in (4) is to be taken for every m and d satisfying $md < x$. It is easy to see that (4) if true is best possible.

3. Let $1 \leqq a_1 \ldots < a_k \leqq n$ and the sums $\sum_{j=1}^{k} \varepsilon_j a_j$, $\varepsilon_j = 0$ or 1 be all distinct. Then

$$\max k = \frac{\log x}{\log 2} + O(1)?$$

4. Here are two old conjectures of Turán and myself [43] (a) Let $1 \leqq a_1 < \ldots < a_k \leqq n$, the sums $a_i + a_j$ be all distinct. Prove that $\max k = n^{1/2} + O(1)$.

(b) Let $1 \leqq a_1 < \ldots < a_k < \ldots$ be a sequence of integers. Denote by $f(n)$ the number of solutions of $n = a_i + a_j$. Assume that for $n > n_0$, $f(n) > 0$. Then

$$\limsup_{n\to\infty} f(n) = \infty. \tag{5}$$

A slightly stronger conjecture states that $a_k < ck^2$ $k = 1, 2, \ldots$ also implies (5).

Sidon asked: Let $1 \leqq a_1 < \ldots$ be an infinite sequence of integers. Assume that the sums $a_i + a_j$ are all distinct (i.e. $f(n) \leqq 1$): How slowly can a_k increase? One can easily show the existence of such a sequence for which $a_k < ck^3$. Recently Ajtai, Komlós and Szemerédi [1] proved that there is such a sequence for which $a_k = o(k^3)$. Rényi and I proved that to every $\varepsilon > 0$ there is a K_ε and a sequence $1 \leqq a_1 < \ldots$ for which $a_k < k^{2+\varepsilon}$ for every k and $f(n) \leqq K_\varepsilon$ for all n. Presumably there is a sequence for which $a_k < ck^{2+\varepsilon}$ and $f(n) \leqq 1$ for all but finitely many integers n. Perhaps $a_k < ck^{2+\varepsilon}$ can be strengthened to $a_k < ck^2(\log k)^\alpha$ (for sufficiently large $\alpha > 0$).

Is it true that if $a_k < ck^3$ holds for all k then the sums $a_i + a_j + a_i$ can not all be different?

Let $1 \leqq a_1 < \ldots < a_k$ be a sequence of integers for which the sums $a_i + a_j$ are all distinct. Is there a perfect difference set mod $(p^2 + p + l)$ for some prime p which contains all the a's?

I offer 500 dollars for the proof or disproof of each of these conjectures.

5. Finally a word about van der Waerden's theorem. Van der Waerden [82] proved that there is a smallest integer $f(n)$ so that if we divide the integers not exceeding $f(n)$ into two classes at least one of them contains an arithmetic progression of n terms. Van der Waerden's proof gives no usable upper bound for $f(n)$-it increases as fast as Ackermann's well known function, which increases faster than every primitive recursive function.

What is the true order of magnitude of $f(n)$? I offer 100 dollars for a proof (or disproof) of $f(n)^{1/n} \to \infty$. Until recently nearly everybody was sure that

$f(n)$ increases much slower than Ackermann's function. I first heard doubt expressed by Solovay which I more or less dismissed as a regrettable aberration of an otherwise great mind. After the surprising results of Paris and Harrington [68] Solovay's opinion seems much more reasonable, and certainly should be investigated as much and as soon as possible.

Is it true that $f(n+1)/f(n) \to \infty$? Can one at least prove that $f(n+1) - f(n) \to \infty$?

III. Problems on Extremal Graph Theory

Let $\mathcal{G}^{(r)}(k;l)$ be an r-uniform hypergraph of k vertices and l hyperedges. $f(n;\mathcal{G}^{(r)}(k;l))$ is the smallest integer such that every r-uniform hypergraph of n vertices and $f(n;\mathcal{G}^{(r)}(k;l))$ edges contains a $\mathcal{G}^{(r)}(k;l)$ as a subgraph.

1. The study of extremal graph theory was started by Turán [81] who determined $f(n;K^{(2)}(k))$ for every k ($K^{(2)}(k)$ is the complete graph of k vertices). He asked for the determination of $f(n;K^{(r)}(k))$ for all r and $k > r$ ($K^{(r)}(k)$ is the complete r-graph of k vertices and $\binom{k}{r}$ hyperedges). Turán made some plausible conjectures for $r = 3$, $k = 4$ and $r = 3$, $k = 5$. I offer 500 dollars for the determination of

$$\lim_{n\to\infty} f(n;K^r(k))/\binom{n}{k} = C_{r;k},$$

for even a single $k > r > 2$. $c_{2,k} = \frac{1}{2}\left(1 - \frac{1}{k-1}\right)$ was proved by Turán. I offer 1000 dollars for clearing up the whole set of problems.

Very recently a comprehensive book of Bollobás [5] appeared on extremal graph theory.

2. Both Simonovits and I published many papers and problems (some jointly) on this subject. Here I state first of all some of our favourite joint problems. (See [37], [38], [39], [40], [75], [76].)

Let $\mathcal{G}$ be bipartite. Is it true that for some $\alpha = \alpha(\mathcal{G})$, $(0 \leqq \alpha < 1)$

$$\lim f(n;\mathcal{G})/n^{1+\alpha} = c_{\mathcal{G}}, \quad 0 < c_{\mathcal{G}} < \infty. \tag{1}$$

I offer 500 dollars for a proof or disproof of this conjecture. We know that it does not hold for hypergraphs. (See Ruzsa–Szemerédi [74].)

Is the $\alpha = \alpha(\mathcal{G})$ in (1) always rational? Is it true that for every rational α, $1 \leqq \alpha < 2$ there is a $\mathcal{G}$ satisfying (1)?

Is it true that

$$f(n;\mathcal{G}) > cn^{8/5}, \tag{2}$$

where in (2) $\mathcal{G}$ is the graph determined by the edges of a cube? Is it true that

$$f(n;K(r,r)) > cn^{2-1/r}? \tag{3}$$

$f(n;\mathcal{G}) < c_1 n^{8/5}$ is a theorem of Simonovits and myself $f(n;K(r,r)) < c_2 n^{2-1/r}$ is an old theorem of Kővári, T. Sós, Turán [61] and myself. (See also [10], [36].)

3. Now I state a few miscellaneous extremal problems. Sauer and I investigated the following problem: Denote by $F(n;r)$ the smallest integer for which every $\mathcal{G}(n;F(n;r))$ contains a regular subgraph of valency r. Trivially $F(n;1) = 1$, $F(n;2) = n$. The trouble is that we really know nothing about $F(n;3)$. We could not disprove $F(n;3) < cn$ and our only upper bound for $F(n;3)$ is $cn^{8/5}$. We conjectured that

$$F(n,r) = O(n^{1+\varepsilon})$$

for every r and $\varepsilon > 0$.

Berge conjectured that every regular graph of valency 4 contains a subgraph of valency 3. As far as I know it is not known whether there is an r for which every regular graph of valency r contains a regular graph of valency 3.

W. Brown, Vera T. Sós and I [12] conjectured that

$$f(n;\mathcal{G}^{(3)}(6,3)) = o(n^2). \tag{4}$$

Ruzsa and Szemerédi [72] proved (4). In fact, they showed

$$n^{2-\varepsilon} < cnr_3(n) < f(n;\mathcal{G}^{(3)}(6,3)) = o(n^2) \tag{5}$$

where $r_3(n)$ is the function defined in (2) of II. The discovery of this connection was a great and unexpected surprise. (5) shows that (1) is not true for hypergraphs.

Probably

$$f(n;\mathcal{G}^{(3)}(k,k-3)) = o(n^2)$$

holds for every k. (See [11].)

IV. Some Problems in Graph Theory

1. First I mention two classical problems. Berge calls a graph perfect if for all its induced subgraphs H the chromatic number of H equals its clique number, i.e. is as small as possible. Berge [4] formulated two conjectures:

1. A graph is perfect if and only if its complement is perfect.
2. A graph is perfect if and only if it does not contain an induced subgraph which is either a cycle $C_{2k+1}, 2k+1 \geqq 5$ or the complement of a such a cycle.

Conjecture (a) has been proved a few years ago by Lovász [63], conjecture (b) is one of the principal open problems of graph theory.

Perfect graphs have a large literature. I do not deal with this problem any more since I have nothing original to contribute. I just refer to a very recent paper of V. Chvátal, R. L. Graham, A. F. Perold and Susan H. Whitesides [18] which has many relevant references.

The second conjecture is the reconstruction conjecture of Ulam and P. Kelly, which has an immense literature. The conjecture states that two graphs $\mathcal{G}_1(n)$ and $\mathcal{G}_2(n)$ are isomorphic if the set of their induced subgraphs of size $n-1$ are isomorphic. I never worked on this fascinating problem and thus I just refer to a recent survey paper of J. A. Bondy–R. L. Hemminger [8]. For a related problem see J. A. Bondy [6].

2. The most famous conjecture of graph theory or perhaps of the whole Mathematics, the four colour conjecture, became recently the theorem of Appel and Haken [2]. There were two conjectures which generalized the four color conjecture. Hadwiger [54] conjectured that every r-chromatic graph can be contracted to a $K(r)$. This conjecture is still open for $r \geqq 5$, it has been proved for $r \leqq 4$. The conjecture of Hajós [55] stated that if $\mathcal{G}$ has chromatic number r then $\mathcal{G}$ contains a subdivision of $K(r)$ which I called $K_{\text{top}}(r)$ (i.e. a topologically complete graph or r vertices). This conjecture was also proved for $r \leqq 4$, but was recently disproved for $r \geqq 7$ by Catlin [16]. In this journal Fajtlowicz and I [22] disprove it in a very strong form. Let $\mathcal{G}(n)$ be a labelled graph of n vertices $\chi(\mathcal{G}(n))$ its chromatic number and $t(\mathcal{G}(n))$ be the size of its largest topologically complete subgraph. The conjecture of Hajós states that $\chi(\mathcal{G}(n)) \leqq t(\mathcal{G}(n))$. Put

$$S(n) = \max_{\mathcal{G}(n)} \chi(\mathcal{G}(n)) l t(\mathcal{G}(n)).$$

Fajtlowicz and I prove that

$$S(n) > c_1 n^{1/2} / \log n. \tag{1}$$

In fact we prove that (1) holds for almost all of the graphs $\mathcal{G}(n)$.

Very likely (1) is best possible i.e. $S(n) < c_2 n^{1/2}/\log n$, but this conjecture remains open for the time being.

G. Dirac [20] proved that every $\mathcal{G}(n; 2n-2)$ contains a $K_{\text{top}}(4)$ and observed that $2n-2$ is best possible. He conjectured that every $\mathcal{G}(n; 3n-5)$ contains a $K_{\text{top}}(5)$ – if true this is clearly best possible. (Pelikán [69] has shown that every 5-chromatic graph contains a topological $[K5 - \{\text{anedge}\}]$.)

Hajnal, Mader and I conjectured that every $\mathcal{G}(n; cr^2 n)$ contains a $K_{\text{top}}(r)$. Mader [65] proved the weaker $K_{\text{top}}(r) \subset \mathcal{G}(n; 2^{\binom{r}{2}} n)$. (See Erdős–Hajnal [26].)

3. Goodman, Pósa and I [23] proved that every $\mathcal{G}(n)$ is the union of at most $\left[\frac{n^2}{4}\right]$ edge disjoint cliques. These cliques can be chosen to be edges and triangles.

Sauer and I conjectured that every r-graph $\mathcal{G}^{(r)}(n)$ is the union of at most $f(n; K^{(r)}(r+1)) - 1$ cliques where no two of the cliques have a $K^{(r)}(r)$ in common and the cliques are either $K^{(r)}(r)$'s or $K^{(r)}(r+1)$'s. We hoped that this conjecture can be proven without knowing the value of $f(n; K^{(r)}(r+1))$.

Gallai and I conjectured that the edges of every $\mathcal{G}(n)$ can be covered by $\leqq Cn$ edge disjoint circuits or edges of our $\mathcal{G}(n)$. We easily showed that the result holds with $cn \log n$ replacing Cn.

Gallai further conjectured that the edges of every $\mathcal{G}(n)$ can be covered by at most $\left[\frac{n+1}{2}\right]$ paths and Hajós conjectured that every Eulerian graph of n vertices is the union of $\left[\frac{n}{2}\right]$ circuits.

Lovász conjectured that every vertex transitive connected graph has a Hamilton line.

V. Problems of Ramsey Theory

There is a very nice survey paper on the so called generalized Ramsey numbers by S. Burr [15] and a forthcoming book by Graham, Rothschild and Spencer [52]. Not to make this paper too long I only state somewhat arbitrarily 3 or 4 problems with which I spent lots of time. $r(\mathcal{G}_1, \dots, \mathcal{G}_r)$ is the smallest integer n so that if one colors the edges of $K(n)$ by r colors then for some i, $1 \leqq i \leqq r$ the ith color contains $\mathcal{G}_i$ as a subgraph. As far as I know, the problem in this generality was first formulated by Harary.

Prove that

$$\lim_{n\to\infty} r(K(n), K(n))^{1/n} \tag{1}$$

exists and determine the value of the limit. I offer 100 dollars for the first problem and 500 for the second. The proof of the existence of the limit in (1) will perhaps be easy, the determination of its value will probably be difficult (it is between $2^{1/2}$ and 4).

Let $C(n)$ denote a circuit of n vertices. Prove that

$$r(K(n), C(4)) < n^{2-\varepsilon}, \tag{2}$$

Bondy and I [7] conjectured

$$r(C_n, C_n, C_n) \leqq 4n - 3. \tag{3}$$

It is easy to see that if (3) is true then for odd n it is best possible. Vera Rosta [72] and independently R. Faudree and Schelp [44] determined $r(C_n, C_m)$ for every n and m.

A problem of Faudree, Rousseau, Schelp and myself: Let $\hat{r}(P_n)$ be the smallest integer for which there is a graph $\mathcal{G}$ of $\hat{r}(P_n)$ edges so that if we color the edges of $\mathcal{G}$ with two colors, there is always a monochromatic path P_n of length n. Is it true that

$$\hat{r}(P_n)/n \to \infty, \quad \hat{r}(P_n)/n^2 \to 0? \tag{4}$$

Both of these questions seemed very interesting to us but we had no success at all with (4). It would be useful to have an asymptotic formula or at least a good inequality for $\hat{r}(P_n)$, but the first step is clearly to settle (4). I offer 100 dollars for a proof or disproof of (4).

Just one more problem of Hajnal, Rado and myself:

Is it true that

$$\log\log r(K^{(3)}(n), K^{(3)}(n)) > cn? \tag{5}$$

In other words, does $r(K^{(3)}(n), K^{(3)}(n))$ tend to infinity like a double exponential. This question is fundamental and I offer 500 dollars for a proof or disproof of (5).

For further problems I have to refer to the very extensive literature. Before closing I just want to point out that perhaps it might be worthwhile to study in analogy of the Ramsey numbers the van der Waerden numbers. $f(u, v)$ is the smallest integer for which if we divide the integers not exceeding $f(u, v)$ into two classes either Class 1 contains an arithmetic progression of u terms or Class II contains an arithmetic progression of length v. As far as I know nothing is known about the growth of $f(u, v)$ for $u \neq v$. It immediately follows from (1) of II., that

$$f(3, v) < \exp\exp v, \tag{6}$$

but probably (6) is very far from being best possible. I have no non-trivial lower bound for $f(3, v)$, and would not be surprised if $f(3, v) < \exp v^{\alpha}$ would hold for some $\alpha < 1$.

VI. Some problems on designs and elementary geometry

I hope the reader will forgive me some personal reminiscences on these problems-due to my advanced age I can not be sure if I will have many opportunities to tell these stories. I heard from the book of Netto, Kombinatorik from my father — reading it in 1930. I read about Steiner triples — immediately the question occurred to me: Let $|S| = n$, $n > n_0(k, r)$, $k > r$. Is it true that one can always find a family of $\binom{n}{k}\binom{k}{r}^{-1}$ subsets of size k of S so that all r-tuples of S are contained in precisely one of our k-tuples, unless there is a trivial congruential reason that such a system can not exist. I realised that this is a fundamental problem but did not know that it was already formulated by Kirkman nearly a century earlier, more than 10 years before Steiner — and in fact Kirkman settled the case $r = 2$, $k = 3$ — thus Steiner triples should really be called Kirkman triples.

I was in Israel in 1955 when Hanani told me that he settled the case $r = 3$, $k = 4$ — thus settling a problem which was open for more than a century — I urged him to publish this as fast as possible — his paper [56] was a starting point of many further investigations. Later Hanani settled the cases $r = 2$, $k = 4$ and $r = 2$, $k = 5$. R. M. Wilson [83] a few years ago proved that for $r = 2$ and any k there is an $n_0(k)$ so that if $n > n_0(k)$ there is always a system of $\binom{n}{2}\binom{k}{2}^{-1}$ k-tuples so that every pair is contained in one and only one k-tuple unless a trivial congruential reason prevents the existence of such a system.

Unfortunately (or perhaps fortunately — a subject needs challenging unsolved and not quite hopeless problems to keep alive) very little is known of the cases $r > 2$ (except Hanani's case $r = 3$, $k = 4$). In fact, as far as I know no such design is known for $r \geqq 6$. Perhaps Wilson's theorem remains true for every k and r if $n > n_0(k, r)$. This is certainly one of the fundamental unsolved problems of our subject. Many problems would remain even if Wilson's theorem can be extended for every k and r. One of the most famous ones states as follows: Let $n = k^2 + k + 1$, $|S| = k^2 + k + 1$. Is there a system of $k^2 + k + 1$ $(k + 1)$-tuples of S so that every pair is contained in exactly one of them. This is the famous problem on the existence of finite geometries. Such a geometry always exists if $k = p^\alpha$ and perhaps for no other $k^2 + k + 1$.

The classical theorem of Bruck–Ryser [13] excluded infinitely many values of $k^2 + k + 1$; $k = 10$ is the first unknown case — I expect that no such system exists and hope to see a solution before I leave. This question is an outstanding challenge for the ingenuity of mathematicians and computer scientists.

In about 1931 I asked myself: Determine or estimate the maximum of all $n \times n$ determinants all whose entries are 0 or 1. I soon realised that this connects up with the problem of orthogonal matrices all whose entries are ± 1. I mistakenly thought that they exist only if $n = 2^k$. Kalmár soon pointed out to me that I am wrong and the problem in fact is due to Sylvester and Hadamard. It is generally believed that such matrices always exist if $n \equiv 0 \pmod 4$. Much work has been done on this fascinating conjecture which has a very large literature.

Another challenging and interesting problem is the maximum number of pairwise orthogonal Latin squares of order n. Chowla, Straus and I were present at the Boulder meeting on number theory in 1959 when Bose, Parker and Shrikhande [9] disproved the old conjecture of Euler: Denote by $f(n)$ the maximum number of pairwise orthogonal Latin squares of order n. Euler conjectured that $f(4n+2) = 1$. This was proved for $n = 1$ but Bose, Parker and Shrikhande disproved it for every $n > 1$. Using their ideas we proved that $f(n) \to \infty$ and using Brun's method we proved that, for $n > n_0$, $f(n) > n^{l/91}$. Our result has been improved a great deal. The current record is due to R. M. Wilson who proved $f(n) > n^{1/17}$. The "final Truth" is perhaps $f(n) > cn^{1/2}$.

Kaplansky and I proved: Let $k = o((\log n)^{3/2-\varepsilon})$. Then the number of $k \times n$ Latin rectangles equals

$$(1 + o(1))\, e^{-\binom{k}{2}} n!^k,$$

(see Erdős–Kaplansky [30]). Our result has been extended first for $k < n^{1/3-\varepsilon}$ and then until $n^{1/2-\varepsilon}$. (If $k > n^{1/3-\varepsilon}$, the asymptotic formula becomes much more complicated). An asymptotic formula for the $n \times n$ Latin squares is nowhere in sight. For literature about Latin squares and other related subjects see the comprehensive book of Dénes and Keedwell [19] — this book contains an immense material and a very extensive list of references.

Let me now state a few problems on block designs. Since I am not an expert, I can not be entirely sure that the problems are new and fruitful.

1. Is there an absolute constant C so that every finite plane has a blocking set which meets every line but each in at most C points? Or more generally: Is it true that to every c_1 there is a c_2 so that if $|S| = n$ and $A_1, \dots, A_m$ is a pairwise balanced block design satisfying $|A_i| > c_1 n^{1/2}$ then there is a $Q \subset S$ for which

$$1 \leqq |Q \cap A_i| \leqq c_2, \quad (i = 1, \dots, m).$$

2. Let $\{A_i\}$, $1 \leqq i \leqq m$ be a pairwise balanced block design. Put $|A_i| = X_i$, $X_1 \geqq \dots \geqq X_m$. One could ask: Give necessary and sufficient conditions for the $\{X_i\}$, that there should be a pairwise balanced block design $\{A_i\}$ satisfying $|A_i| = X_i$? Trivially we must have $\sum_{i=1}^{m} \binom{X_i}{2} = \binom{n}{2}$. Perhaps there will not be a reasonable necessary and sufficient condition.

On the other hand the following problem should not be hopeless: consider the families $\{X_1, \dots, X_m\}$ of sequences for which there is a pairwise balanced block design with $|A_i| = X_i$, $1 \leqq i \leqq m$. Denote by $F(n)$ the number of such sequences. Is it true that

$$\exp(c_1 n^{1/2} \log n) < F(n) < \exp(c_2 n^{1/2} \log n)? \tag{1}$$

The upper bound of (1) is easy to prove, but I had no success with the lower bound.

I expect that the following problem is much more difficult: Let there be given n points in the plane, join any two of them by a line. This gives a pairwise balanced block design. Let $X_1 \geqq X_2 \geqq \dots \geqq X_m$ be the number of points on the lines and denote by $f(n)$ the number of possible choices for the $\{X_i\}$. I am convinced that here

$$\exp(c_3 n^{1/2}) < f(n) < \exp(c_4 n^{1/2}). \tag{2}$$

The lower bound in (2) is easy, but the upper will probably be difficult. If (2) is correct one should try to prove that

$$\lim_{n \to \infty} \log f(n)/n^{1/2} = c,$$

and try to determine the value of c.

Prove that there is a pairwise balanced block design, for which, for every t the number of indices i for which $|A_i| = t$ is less than $cn^{1/2}$ where c is an absolute constant. This will probably not be very difficult to prove but so far I was not succesful. A theorem of de Bruijn and myself states that for every pairwise balanced block design $A_1, \dots, A_m : m \geqq n$. This easily implies that there is always a t such that the number of indices i with $|A_i| = t$ is greater than $cn^{1/2}$. Thus our conjecture if true is best except for the value of the constant c. Now I state a few problems in combinatorial geometry. Since I published a survey paper ([21]) on this subject, here I mention only a few of my favourite problems, but first I ask the indulgence of the reader for giving a few historical reminiscences.

In 1933 I was reading the beautiful book of Hilbert and Cohn–Vossen Anschauliche Geometrie (the English title was "Geometry and the Imagination"). One of the chapters is on configurations and while looking at it suddenly the following question occurred to me: Let there be given n points in the plane not all on a line. Is it true that there is always a line which goes through precisely two of our points? This seemed to me to be true but I could not prove it. I told my conjecture to T. Gallai who soon found his well known proof. I observed that Gallai's result implies that if we join any two of these points we get at least n distinct lines.

Three years later at the international congress in Oslo Karamata asked me about this theorem. He read it without proof in an old book on mechanics and could not prove it. I of course told him Gallai's proof. Sometime in the early 1940's L. M. Kelly found that Sylvester posed the same problem in 1893 in the Educational Times. No solution was received. As far as one can tell the first proof is due to T. Gallai and the conjecture was first stated by Sylvester (the first published proof is due to Melchior in 1940). For further results and historical remarks see the papers of Motzkin [88] and Grünbaum [87].

Let there be given n points in the plane, at most $n - k$ of them on a line. Is it true that these points determine at least ckn distinct lines where c is independent of k and n? For $k < cn^{1/2}$ a stronger result is proved by L. M. Kelly and W. Moser [58]:

Let there be given n points in the plane. Is it true that they determine at least $cn/(\log n)^{1/2}$ distinct distances? If true this is best possible. Is it true that the same distance can occur at most $n^{1+c/\log\log n}$ times? If true this is also best possible.

Conjecture of G. Szekeres [78]: Let there be given $2^{k-2} + 1$ points in the plane, no three on a line. Is it true that there are always k of them which are the vertices of a convex k-gon? If true this is best possible.

Is there an n_k so that for any set of n_k points in the plane no three on a line there are always k of them which determine a convex k-gon which contains no other of the points in its interior? $n_4 = 5$ is easy and Harborth recently proved $n_5 = 10$. It is not at all certain if n_6 exists.

The following interesting and probably difficult problem is due to U. S. R. Murty [66]: Let there be given n points in the plane, when can one give positive weights to the points so that the sum of the weights of the points on every line is the same? Murty conjectures that there are only four possibilities, three of them trivial; all points on a line; no three on a line; $n-1$ of them on a line and finally a triangle the angle bisectors and the incentre, or a projective equivalent.

VII. Miscellaneous Problems

I restricted myself so far entirely to finite problems. Now I state a few problems on infinitary combinatorics, for many further such problems see my papers with Hajnal (e.g. [24]) on solved and unsolved problems in set theory.

Let α be an ordinal number which has no predecessor. Let $\mathcal{G}$ be a graph whose vertices form a well ordered set of type α. Hajnal, Milner and I [27] conjectured that if $\mathcal{G}$ has no infinite path then it contains an independent set of type α. We proved this for all $\alpha < \omega_1^{\omega+2}$. Our proof breaks down at $\alpha = \omega_1^{\omega+2}$, but we expect that the result remains true for $\alpha = \omega_1^{\omega+2}$ and for all other α which have no immediate predecessor.

The following very attractive conjecture is due to Walter Taylor. Let be an arbitrary graph of chromatic number $\aleph_1$. Then for every cardinal number m there is a graph $\mathcal{G}_m$ of chromatic number m so that every finite subgraph of $\mathcal{G}_m$ is contained in $\mathcal{G}$. More generally, characterise families $\mathcal{F}_\alpha$ of finite graphs such that there is a graph $\mathcal{G}_\alpha$ of chromatic number $\aleph_\alpha$ all whose finite subgraphs are contained in $\mathcal{F}_\alpha$. In our triple paper [28] with Hajnal and Shelah several further problems are stated and some are proved, e.g. we prove that if $\mathcal{G}$ has chromatic number $\aleph_1$ then there is an integer n so that $\mathcal{G}$ contains a circuit C_m for every $m \geqq n$. The simplest of the many questions which we cannot answer, states: Is it true that if $\mathcal{G}$ has chromatic number $\aleph_1$ then it has an edge e for which $\mathcal{G}$ contains a C_m containing e for every $m > m_0$.

Hajnal and I proved [25] that if $\mathcal{G}$ has chromatic number $\aleph_1$ then it must contain every finite bipartite graph, but it does not have to contain any C_{2k+1} for $k \leqq n$. In fact for any non-bipartite graph $\mathcal{G}_1$ and any cardinal number m there is a graph of chromatic number m which does not contain $\mathcal{G}_1$.

If we only know that $\mathcal{G}$ has chromatic number $\aleph_0$ then $\mathcal{G}$ can have arbitrarily large girth, thus it does not have to contain any fixed graph which is not a tree. On the other hand Hajnal and I conjectured that if the lengths of all cycles are $n_1 < n_2 < \dots$ in the graph $\mathcal{G}$ of infinite chromatic number then $\sum \frac{1}{n_i} = \infty$ and perhaps the n's have positive upper density. This conjecture was recently proved by Gyárfás, Komlós and Szemerédi.

A finite form of our conjecture states: let $\mathcal{G}$ be a graph of v vertices and kv edges. Let $u_1 < u_2 < \dots$ be the lengths of all cycles of $\mathcal{G}$. Then

$$\sum 1/u_i > c \log k. \tag{1}$$

Perhaps (1) can be strengthened as follows: Let $n \geqq 2k$ then for every $\mathcal{G}(n; k(n-k))$

$$\sum \frac{1}{u_i} \geqq \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right). \tag{2}$$

In other words: $\sum \frac{1}{u_i}$ is minimum for the complete bipartite graph of k white and $n-k$ black vertices. I just formulated this conjecture and I hope the reader will forgive me if there is an easy counterexample. Gyárfás conjectured that if $\mathcal{G}$ has infinite chromatic number and no triangle (or more generally no $K(n)$) then $\mathcal{G}$ contains every tree as an induced subgraph. I find this conjecture very attractive and only regret that I did not think of it myself. Gyárfás also formulated his conjecture in a finite form. For some partial results see the forthcoming paper of Gyárfás, Szemerédi and Tuza [80].

Let $m > n$ be infinite cardinals. Galvin asked the following very pretty question: Let $\mathcal{G}$ have chromatic number m. Is it true that $\mathcal{G}$ always has a subgraph $\mathcal{G}'$ of chromatic number n? Galvin. proved that if $2^{\aleph_0} > \aleph_1$ the answer is negative if we insist that the subgraph $\mathcal{G}'$ should be induced. Hajnal and I asked: Let $\mathcal{G}$ be a graph of chromatic number m. Does $\mathcal{G}$ have a subgraph $\mathcal{G}'$ of chromatic number m the smallest odd circuit of which has length $\geqq 2k+1$? Rödl [71] proved this if $m = \aleph_0$ and $k = 2$. We also asked: Let $m = \aleph_0$, does $\mathcal{G}$ have a subgraph $\mathcal{G}'$ of chromatic number $\aleph_0$ and girth $\geqq k$? Clearly this can be formulated in a finite form, Rödl settled it for $k = 4$ but probably his bounds are far from being best possible.

Let $\mathcal{G}$ be a graph whose vertices are the integers. Consider sup $\sum \frac{1}{\log n_i}$ where the set of vertices $n_1 < n_2 < \dots$ either are independent or form a complete graph. I conjecture sup $\sum \frac{1}{\log n_i} = \infty$. Ramsey's theorem is just too weak to give this and I could not settle the conjecture. Clearly it also has a finite form, one would have to estimate how fast max $\sum\limits_{n_i<X} \frac{1}{\log n_i}$ tends to infinity.

Murty and Plesnik [67] conjectured that if $\mathcal{G}(n)$ has diameter two and if the omission of any edge increases the diameter of $\mathcal{G}(n)$ then $\mathcal{G}(n)$ has at most $\left[\frac{n^2}{4}\right]$ edges. I several times tried to prove this surprising conjecture but without success. The complete bipartite graph of $\left[\frac{n}{2}\right]$, white and $\left[\frac{n+1}{2}\right]$ black vertices shows that the conjecture if true is best possible.

A famous conjecture of van der Waerden states: The permanent of an $n \times n$ doubly stochastic determinant is $\geqq \frac{n!}{n^n}$, equality holds only if all the entries are $\frac{1}{n}$. If I remember right this conjecture was first published in the problem section of the Jahresbericht der Deutschen Math. Vereinigung in about 1924. The question was unnoticed for a long time but now has a very large literature. Very recently T. Bang [85] and Shmuel Friedland [86] proved that the permanent is greater than e^{-n}, which is the best inequality so far.

Another attractive conjecture of Gyárfás states that if T_k, $k = 2, 3, \dots, n$ is any set of $n-1$ trees, and $v(T_k) = k$, then $K(n)$ is the edge disjoint union of the T_k's.

An old problem of mine states as follows: Let $\mathcal{G}$ be a graph and x, y be two vertices of $\mathcal{G}$. Is it true that there are always a separating set S and a family $\mathcal{P}$ of vertex disjoint paths joining x and y so that for every $z_\alpha \in S$ there is a path P_α of $\mathcal{P}$ passing through z_α and no other point of S? If S can be chosen as finite (i.e. if x and y can be separated by a finite set) then this is the well known theorem of Menger. Nothing is known even if $|S| = \aleph_0$.

In a forthcoming paper [29] of Hajnal, Szemerédi and myself we state the following conjecture: Is it true that for $k > k_0$ there is a function $f(n;k)$ tending to infinity as n tends to infinity so that if $\mathcal{G}$ is a critical k-chromatic graph of n vertices then the graph can not be made two-chromatic by the omission of $f(n;k)$ edges? This result no doubt holds already for $k = 4$ (odd circuits show that it is false for $k = 3$) but we made no progress at all with it.

A result of Gallai [51] shows that $f(n;4) < cn^{1/2}$ and an extension of Gallai's construction by Lovász [64] gives $f(n;k) < cn^{1-1/(k-2)}$. We would certainly expect $f(n;4) > c\log n$ but perhaps the Gallai–Lovász examples are essentially best possible.

Another problem of Hajnal, Szemerédi and myself states: Let $\mathcal{G}$ have chromatic number $\geqq \aleph_1$. Prove that for every c there is a finite m so that $\mathcal{G}$ has a subgraph of m vertices which can not be made bipartite by the omission of cm edges. On the other hand we believe that for every $\varepsilon > 0$ there is a $\mathcal{G}$ of chromatic number $\aleph_1$ every subgraph of m vertices of which can be made bipartite by the omission of fewer than $m^{l+\varepsilon}$ edges.

VIII. Two Problems on Random Graphs and Hypergraphs

Let $\mathcal{G}(n;k)$ be a random graph of n vertices and k edges. (By random graph we mean: consider all possible labelled — or unlabelled — graphs of n vertices and k edges. We try to find theorems which hold for almost all of these graphs.)

Rényi and I investigated the dependence of the size of the largest component of $\mathcal{G}(n;k)$ on k and we found an unexpected singularity at $k = \frac{n}{2}$. The discovery of this singularity is perhaps our most interesting result. We always planned (but Rényi's untimely death intervened) to investigate what happens to the second largest component? I expect that it almost surely will never be large, perhaps not much larger than log n and certainly $o(n^{\varepsilon})$, but nothing definite is known. (Added in proof:

These questions were cleared up by Komlós and Szemerédi.)

Rényi and I [35] proved that almost all graphs $\mathcal{G}(2n;[(1+\varepsilon)n\log n])$ have a perfect matching and that this result is best possible. We conjectured the same for the graph being Hamiltonian, our conjecture was settled by Pósa [70], Komlós and Szemerédi [60].

During my visit to Jerusalem in March 1979 A. Shamir surprised me with the following beautiful question. Consider $\mathcal{G}^{(3)}(3n;l_n)$ the random 3-uniform hypergraph of $3n$ vertices and l_n hyperedges. How large must l_n be that almost all of these hypergraphs should contain n vertex-disjoint hyperedges? Many of the problems on random hypergraphs can be settled by the same methods as used for ordinary graphs and usually one can guess the answer almost immediately. Here we have no idea of the answer. I rather felt foolish for not having thought of this interesting and natural question.

References

[1] M. Ajtai, J. Komlós and E. Szemerédi, On infinite Sidon sequences, *European J. Comb.,* to appear.

[2] K. Appel and W. Haken, Every planar map is four colourable, *Illinois J. of Math.* **21** (1977) 429–490.

[3] F. A. Behrend, On sets of integers which contain no three terms in arithmetic progression, *Proc. Nat. Acad. Sci. USA* **32** (1946) 331–332.

[4] C. Berge, *Graphs and Hypergraphs,* North Holland Publ. Comp. Amsterdam–London 1973.

[5] B. Bollobás, *Extremal Graph Theory,* London Math. Soc. Mon. No. 11. 1978.

[6] J. A. Bondy, Reflections on the legitimate deck problem, *Lecture Notes in Math.,* **686**, Springer 1977, 1–12

[7] J. A. Bondy and P. Erdős, Ramsey numbers for cycles in graphs, *J. Comb. Theory* **14** (1973) 46–54.

[8] J. A. Bondy and R. L. Hemminger, Graph reconstruction, *J. of Graph Theory* **1** (1977), 227–268.

[9] R. C. Bose, E. T. Parker and S. Shrikhande, Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture, *Can. J. Math.* **12**, 189–203.

[10] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966) 281–285.

[11] W. G. Brown, P. Erdős and V. T. Sós, Some extremal problems on r-graphs, *New Directions in the Theory of Graphs* (ed. F. Harary), Academic Press 1973, 53–64.

[12] W. G. Brown, P. Erdős and V. T. Sós, On the existence of triangulated spheres in 3-graphs and related questions, *Periodica Math. Hung.* **3** (1973) 221–228.

[13] R. H. Bruck and H. J. Ryser, The nonexistence of certain projective planes, *Canad. J. Math.* **1** (1949) 88–93.

[14] N. G. de Bruijn and P. Erdős, A colour problem for infinite graphs and a problem in theory of relations, *Nederl. Akad. Wetensch. Proc. Ser. A* **54** (1951) 371–373.

[15] S. A. Burr, Generalized Ramsey theory for graphs — a survey, *Graphs and Combinatorics, Lecture Notes in Math.* **406**, Springer 1974, 52–75.

[16] P. A. Catlin, Subgraphs of graphs I, *Discrete Math.* **10** (1974) 225–233.

[17] V. Chvátal, Intersection families of edges in hypergraphs having the hereditary property, *Hypergraph Seminar, Lecture Notes in Math.* **411**, Springer 1972, 61–66.

[18] V. Chvátal, R. L. Graham, H. A. Perold and S. H. Whitesides, Combinatorial designs related to the strong perfect graph conjecture, *Discrete Math.* **26** (1979) 83–92.

[19] J. Dénes and A. D. Keedwell, *Latin Squares and their applications,* Academic Press, New York 1974.

[20] G. Dirac, In abstrakten Graphen vorhandene 4-Graphen and ihre Unterteilungen, *Math. Nachrichten* **22** (1960) 61–85.

[21] P. Erdős, Combinatorial problems in geometry and number theory, to appear. See also P. Erdős, On some problems of elementary and combinatorial geometry, *Annali di Mat. ser.* 4, **104** (1975) 99–108.

[22] P. Erdős and S. Fajtlowicz, On the conjecture of Hajós, *to appear in Combinatorica.*

[23] P. Erdős, A. W. Goodman and L. Pósa, The representation of a graph by set intersection, *Canad. J. Math.* **18** (1966) 106–112.

[24] P. Erdós and A. Hajnal, Unsolved and solved problems in set theory, *Proc. Symp. Pure Math.* **XXIII**. Univ. Calif. Berkeley, A.M.S. 1971, 17–48.

[25] P. Erdős and A. Hajnal, On chromatic number of graphs and set systems, *Acta Math. Acad. Sci. Hung.* **17** (1966) 61–99.

[26] P. Erdős and A. Hajnal, On complete topological subgraphs of certain graphs, Annales Univ. Sci. Budapest **7** (1969) 193–199.

[27] P. Erdős, A. Hajnal and E. C. Milner, A problem on well ordered sets, *Acta Math. Acad. Sci. Hung.* **20** (1969) 323–329.

[28] sc P. Erdős, A. Hajnal and S. Shelah, On some general properties of chromatic numbers, *Coll. Math. Soc. J. Bolyai* **8**, *Topics in Topology,* North-Holland 1972, 243–255.

[29] P. Erdős, A. Hajnal and E. Szemerédi, On almost bipartite large chromatic graphs, *to appear in the volume dedicated to the 60th birthday of A. Kotzig.*

[30] P. Erdős and I. Kaplansky, The asymptotic number of Latin rectangles, *Amer. J. Math.* **69** (1946) 230–236.

[31] P. Erdős and D. Kleitman, Extremal problems among subsets of a set, *Discrete Math.* **8/3** (1974) 281–294. See also G. O. H. Katona, Extremal problems for hypergraphs, *Combinatorics* (M. Hall and J. H. van Lint, eds.) Math. Centre Amsterdam 1974, 215–244.

[32] P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, *Coll. Math. J. Bolyai* **10** *Infinite and Finite Sets,* 1973, 609–617.

[33] P. Erdős, E. C. Milner and R. Rado, Intersection theorems for systems of sets III. *Austral. J. Math.* **18** (1974) 22–40.

[34] P. Erdős and R. Rado, Intersection theorems for systems of sets, *J. London Math. Soc.* **35** (1960) 85–90.

[35] P. Erdős and A. Rényi, On the existence of a factor of degree one of a connected random graph, *Acta Math. Acad. Sci. Hung.* **17** (1966) 359–379.

[36] P. Erdős, A. Rényi and V. T. Sós, On a problem of graph theory, *Studia Sci. Math. Hung.* **1** (1966) 215–235.

[37] P. Erdős and M. Simonovits, Some extremal problems in graph theory, *Comb. Theory and Appl.* (P. Erdős et al. eds.) *Coll. Math. Soc. J. Bolyai* **4** (1969) 377–390.

[38] P. Erdős and M. Simonovits, An extremal graph problem, *Acta Math. Acad. Sci. Hung,* **22/3–4** (1971) 275–282.

[39] P. Erdős and M. Simonovits, On a valence problem in extremal graph theory, *Discrete Math,* **5** (1973) 323–334.

[40] P. Erdős and M. Simonovits, On the chromatic number of geometric graphs, to appear in *Ars Combinatoria.*

[41] P. Erdős and E. Szemerédi, Combinatorial properties of systems of sets, *J. of Comb. Theory A* **24** (1978) 308–313.

[42] P. Erdős and P. Turán, On some sequences of integers, *J. of London Math. Soc.* **11** (1936) 341–363.

[43] P. Erdős and P. Turán, On a problem of Sidon in additive number theory and related problems, *J. London Math. Soc.* **16** (1941) 212–216.

[44] R. J. Faudree and R. H. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Math.* **8** (1974) 313–329.

[45] P. Frankl, On families of finite sets no two of which intersect in a singleton, *Bull. Austral. Math. Soc.* **17** (1977) 125–134.

[46] P. Frankl, An intersection problem for finite sets, *Acta Math. Acad. Sci. Hung.* **30** (1977) 371–373.

[47] P. Frankl, Extremal problems and coverings of the space,. *European J. of Comb.* **1** (1980), *to appear.*

[48] H. Fürstenberg, Ergodic behaviour of diagonal measures and a theorem of Szemerédi on arithmetic progressions I–III, *J. Analyse Math.* **31** (1977) 204–256.

[49] H. Fürstenberg and Y. Katznelson, An ergodic Szemerédi theorem for commuting transformations, *J. Analyse Math.* **34** (1978) 275–291.

[50] P. Gács and L. Lovász, Khachian's Algorithm for linear programming, *Math. Progr. Studies,* to appear

[51] T. Gallai, On covering of graphs, *Theory of Graphs, Proc. Coll. Tihany, Hungary* 1968, (P. Erdős and G. Katona, eds.) 231–236.

[52] R. L. Graham, B. Rothschild and J. Spencer, *Ramsey Theory, John Wiley*, New York 1980.

[53] H. Hadwiger, Ungelöste Probleme No. 40, *Elemente der Math.* **16** (1961) 103–104.

[54] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierte Naturforsch. Ges. Zürich* **88** (1943) 133–142.

[55] G. Hajós, Über eine Konstruktion nicht n-färbbarer Graphen, *Wiss. Zeitschr. Martin Luther Univ. Halle–Wittenberg A* **10** (1961) 116–117.

[56] H. Hanani, The existence and construction of balanced incomplete block designs, *Ann. Math. Stat.* **32** (1961) 361–386.

[57] G. O. H. Katona, *unpublished.*

[58] L. M. Kelly and W. Moser, On the number of ordinary lines determined by n points, *Canad. J. Math.* **10** (1958) 210–219.

[59] L. G. Khachiyan, A polynomial algorithm in linear programming (in Russian) *Doklady Akademii Nauk SSSR,* **244**, (1979), 1093–1096.

[60] J. Komlós and E. Szemerédi, Limit distribution for the existence of Hamilton cycles in a random graph, *Discrete Math., to appear.*

[61] T. Kővári, V. T. Sós and P. Turán, On a problem of Zarankiewicz, *Coll. Math.* **3** (1954) 50–57.

[62] D. G. Larman and C. A. Rogers, The realization of distances within sets in Euclidean space, *Mathematica* **19** (1972) 1–24.

[63] L. Lovász, Normal hypergraphs and the perfect graph conjecture, *Discrete Math.* **2** (1972) 253–267.

[64] L. Lovász,

[65] W. Mader, Homomorphieeigenschaften and mittlere Kantendichte von Graphen, *Math. Annalen* **174** (1967) 265–268.

[66] U. S. R. Murty, How many magic configurations are there, *Amer. Math. Monthly* **1978/9**, 1001–1002.

[67] U. S. R. Murty, *unpublished.* See L. Caccetta and R. Häggkvist, On diameter critical graphs, *Discrete Math.* **28** (1979) 223–229.

[68] J. Paris and L. Harrington, A Mathematical Incompleteness in Peano Arithmetic, *Handbook of Math. Logic*, North Holland, Amsterdam 1977, 1133–1142.

[69] J. Pelikán, Valency conditions for the existence of certain subgraphs, *Theory of Graphs, Proc. Coll. Tihany, Hungary* 1968, 251–258.

[70] L. Pósa, Hamiltonian cycles in random graphs, *Discrete Math.* **14** (1976) 359–364.

[71] V. Rödl, *to appear.*

[72] Vera Rosta, On a Ramsey type problem of J. A. Bondy and P. Erdős I–II, *J. Comb. Theory B* **15** (1973) 94–105, 105–120.

[73] K. F. Roth, On certain sets of integers II, *J. of London Math. Soc.* **29** (1954) 20–26.

[74] I. Z. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles, *Coll. Math. Soc. J. Bolyai* **18** *Combinatorics* (A. Hajnal and V. T. Sós, eds.) North-Holland 1978, 939–946.

[75] M. Simonovits, Paul Turán's influence on graph theory, *J. Graph Theory* **2** (1977). 102–116.

[76] M. Simonovits, A method for solving extremal problems in graph theory, *Theory of Graphs, Proc. Coll. Tihany, Hungary* 1966, 279–319.

[77] J. Spencer, Intersection theorems for systems of sets, *Canad. Math. Bull.* **20** (1977) 249–254.

[78] G. Szekeres, On an extremum problem in the plane, *Amer. J. Math.* **63** (1941) 208–211.

[79] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arithm.* **27** (1975) 199–245.

[80] E. Szemerédi, A. Gyárfás and Zs. Tuza, Induced subtrees in graphs of large chromatic number, *Discrete Math.* **30** (1980) 235–244.

[81] P. Turán, On an extremal problem in graph theory (in Hungarian) *Mat. Fiz. Lapok* **48** (1941) 436–152 and On theory of graphs, *Coll. Math.* **3** (1954) 19–30.

[82] B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw. Arch. Wisk.* **15** (1928) 212–216.

[83] R. M. Wilson, An existence theory for pairwise balanced designs I–II, *J. Comb. Theory A* **13** (1972) 220–273.

[84] N. Wormald, A 4-chromatic graph with a special plane drawing, *J. Austral. Math. Soc. Ser. A.* **28** (1979) 1–8.

[85] T. Bang, On matrix functioner som med et numerisk lille deficit viser v. d. Waerdens permanenthypotese, *Proc. Scandinavian Congress,* Turku 1976

[86] S. Friedland, A lower bound for the permanent of a doubly stochastic matrix, *Ann. Math.* **110** (1979), 167–176

[87] B. Grünbaum, *Arrangements and Spreads,* A. M. S., Providence, R. I. 1972

[88] T. S. Motzkin, The lines and planes connecting the points of a finite set, *Trans. A. M. S.* **70** (1951), 451–464.