OFFPRINTS FROM THE THEORY OF APPLICATIONS OF GRAPHS
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Existence of Complementary Graphs with Specified Independence Numbers

P. ERDÖS
S. SCHUSTER*

ABSTRACT

G. Chartrand and S. Schuster established inequalities of the Nordhaus-Gaddum type for the independence and edge-independence numbers. This paper considers the existence of complementary graphs that realize the independence and edge-independence numbers in the ranges permitted by the Chartrand-Schuster results.

1. Introduction.

A set of vertices (edges) is *independent* if no two vertices (edges) in the set are adjacent. The *independence-number* $\beta(G)$ of a graph G is the maximum number of elements in an independent set of vertices of G; the *edge-independence number* $\beta_1(G)$ is the maximum number of elements in an independent set of edges of G. As is usual, $V(G)$ will denote the set of vertices of G, $E(G)$ the set of edges of G, $\overline{G}$ the complement of G, K_n the complete graph of order n, and P_n the path of order n.

Chartrand and Schuster [1] established inequalities of the

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Nordhaus-Gaddum type for the parameters β and β_1 by determining best possible upper and lower bounds for $\beta(G) + \beta(\overline{G})$, $\beta(G) \cdot \beta(\overline{G})$, $\beta_1(G) + \beta_1(\overline{G})$, and $\beta_1(G) \cdot \beta_1(\overline{G})$. Here, we consider the question of the existence of complementary graphs that realize the independence and edge-independence numbers in the ranges permitted by the Chartrand-Schuster results.

2. Edge-Independence.

The Nordhaus-Gaddum inequalities for β_1 established in [1] are

$$[p/2] \leq \beta_1(G) + \beta_1(\overline{G}) \leq 2[p/2] \quad \text{and} \quad 0 \leq \beta_1(G) \cdot \beta_1(\overline{G}) \leq [p/2]^2, \qquad (1)$$

which hold for every graph G of order $p \geq 3$.

We begin by showing that $\beta_1(G)$ and $\beta_1(\overline{G})$ cannot both be "small", simultaneously.

Theorem 1. If G is any graph of order p, then

$$\max\{\beta_1(G), \beta_1(\overline{G})\} \geq \left[\frac{p+1}{3}\right].$$

Proof. We confine attention to the case in which $p \equiv 0 \pmod 3$, and proceed by induction.

The theorem obviously holds for $p = 3$. We assume that it holds for all graphs of order $3k$, and consider G to be an arbitrary graph of order $3k + 3$. Clearly, the theorem is true if $G = K_{3k+3}$ or $G = \overline{K}_{3k+3}$, so we exclude these cases from further consideration. Having done this, we assert that G contains three vertices v_1, v_2 and v_3 such that the induced subgraph $\langle v_1, v_2, v_3 \rangle$ is either P_3 or $P_1 \cup P_2$. Deleting v_1, v_2 and v_3 from G, we obtain $G' = G - \{v_1, v_2, v_3\}$, which is a graph to which the induction hypothesis applies; i.e., $\max\{\beta_1(G'), \beta_1(\overline{G'})\} \geq k/3$. But the deletion of v_1, v_2 and v_3 removed at least one independent edge from both G and $\overline{G}$. Hence, $\beta_1(G) \geq k/3 + 1$ or

$\beta_1(\overline{G}) \geq k/3 + 1$, completing the induction.

The very same type of argument proves the theorem for the cases in which $p = 1, 2 \pmod 3$. ∎

We now ask: If $\beta_1(G) = m$, what values can $\beta_1(\overline{G})$ assume? A moment's reflection shows that $\beta_1(\overline{G})$ is not restricted on the upper end, except by the upper bounds in (1) and the obvious condition that $\beta_1(\overline{G}) \leq [p/2]$. Indeed, if the graph H of order p whose edge set consists of m independent edges, then $\beta_1(\overline{H}) = [p/2]$. Therefore, the question to ask is: How small can $\beta_1(\overline{G})$ be if $\beta_1(G) = m$?

We begin investigating this question, again for the case $p \equiv 0 \pmod 3$. Since the roles of G and $\overline{G}$ can be interchanged, we assume, without loss of generality, that it is $\beta_1(G)$ that is at least $p/3$; more precisely, we assume that $\beta_1(G) = p/3 + r$, where $0 \leq r \leq [p/2] - p/3$, and we shall seek graphs G with this edge-independence number but with $\beta_1(\overline{G})$ as small as possible. Let $v_{2i-1}v_{2i}$, $i = 1, 2, \ldots, p/3 + r$, be independent edges. Call $A = \{v_1, v_2, \ldots, v_{2(p/3+r)}\}$ and B the set of remaining $p/3 - 2r$ vertices of G. If $\langle A \rangle$, the subgraph induced by the vertices of A, is $K_{2(p/3+r)}$, then $\beta_1(\overline{G}) \leq p/3 - 2r$, for the largest number of independent edges of $\overline{G}$ would be attained by having each vertex of B joined (in $\overline{G}$) to a vertex of A, forming $p/3 - 2r$ independent edges. If $\langle A \rangle = K_{2(p/3+r)}$ and, in addition, all the vertices of A are adjacent to a single vertex of B (in which case $K_{2(p/3+r)+1} \subset G$), then

$$\beta_1(\overline{G}) \leq p/3 - 2r - 1. \tag{2}$$

Our claim is that inequality in (2) is never possible; that is, there is no graph G of order $p \geq 3$ for which $\beta_1(\overline{G}) < p/3 - 2r - 1$. For suppose there exists a graph G such that $\beta_1(G) = p/3 + r$ and $\beta_1(\overline{G}) = p/3 - 2r - 2$. As before, let $v_{2i-1}v_{2i}$, $i = 1, 2, \ldots, 2(p/3+r)$, be independent edges, $A = \{v_1, v_2, \ldots, v_{2(p/3+r)}\}$ and B the set of remaining vertices of G. Consider the subgraph $\overline{H}$ of $\overline{G}$ whose vertices are all the vertices of G and whose edges are those that join, in $\overline{G}$, vertices of A to vertices of B. That is, $V(\overline{H}) = V(G) = V(\overline{G})$ and $E(\overline{H}) = \{v_jv_k \mid v_j \in A, v_k \in B, v_jv_k \in E(\overline{G})\}$. The graph $\overline{H}$ is bipartite, so we may invoke König's Theorem, which states that the number of edges in a maximum matching equals the number of vertices in a minimum covering. The number of edges in a maximum matching in $V(\overline{H})$ is at most $\beta_1(\overline{G}) = p/3 - 2r - 2$, while a minimum cover of $E(\overline{H})$ can be taken from B, which contains $p/3 - 2r$ vertices. This implies that two vertices of B are not part of the cover, which means that these two vertices are adjacent in G to every vertex in A. It follows that G has $p/3 + r + 1$ independent edges, contradicting the fact that $\beta(G) = p/3 + r$. Thus, our claim is established. ∎

The minimum value for $\beta_1(\overline{G})$ is attained by the graph $G = K_{2(p/3+r)+1} \cup \overline{K}_{p/3-2r-1}$. It is easy to see that judicious removal of edges from this graph without reducing $\beta_1(G) = p/3 + r$ will result in increasing $\beta_1(\overline{G})$ until it reaches $[p/2]$.

Completely analogous analyses produce the companion results in the cases in which $p \equiv 1, 2 \pmod 3$. One needs only to note that base number for $\beta_1(G)$ - that number being drawn from Theorem 1 - is different in the two remaining cases. For $p \equiv 1 \pmod 3$, take $\beta_1(G) = [p/3] + r$; and for $p \equiv 2 \pmod 3$, take $\beta_1(G) = \{p/3\} + r$.

We summarize the results for the three cases in the following.

Theorem 2. For all integers p, r, n with $p \geq 3$,

$0 \le r \le \left[\frac{p}{2}\right] - \left[\frac{p+1}{3}\right]$ and $\left[\frac{p+1}{3}\right] - 2r - 1 \le n \le \left[\frac{p}{2}\right]$, there exists a graph G such that $\beta_1(G) = \left\lceil\frac{p+1}{3}\right\rceil + r$ and $\beta_1(\overline{G}) = n$. Conversely, if G is any graph of order $p \ge 3$ and $\begin{Bmatrix}\beta_1(G)\\ \beta_1(\overline{G})\end{Bmatrix} = \left[\frac{p+1}{3}\right] + r$, with $0 \le r \le \left[\frac{p}{2}\right] - \left[\frac{p+1}{3}\right]$, then

$$\left[\frac{p+1}{3}\right] - 2r - 1 \le \begin{Bmatrix}\beta_1(\overline{G})\\ \beta_1(G)\end{Bmatrix} \le \left[\frac{p}{2}\right].$$

Note. Our Theorem 1 contradicts, and therefore indicates an error in, the last sentence of Theorem 1 of [1]. Our Theorem 2 provides a complete correction of that error.

3. Vertex Independence.

The question of existence of complementary graphs with specified independence numbers is answered by some simple observations concerning the results in [1]. We therefore begin by reviewing the pertinent parts of that paper.

For $m, n \ge 2$, the *Ramsey number* $r(m,n)$ is the least integer p such that for every graph G of order p, either G contains the subgraph K_m or $\overline{G}$ contains K_n. For each positive integer p, let

$$R_p = \{(m,n) \mid r(m,n) > p\},$$

and let σ_p and μ_p denote, respectively, the minima of $(m-1) + (n-1)$ and $(m-1)\cdot(n-1)$ for (m,n) in R_p. The following Nordhaus-Gaddum type inequalities were proved in [1]. For every graph G of order p,

$$\sigma_p \le \beta(G) + \beta(\overline{G}) \le p+1 \quad\text{and}\quad \mu_p \le \beta(G)\cdot\beta(\overline{G}) \le \left[\frac{p+1}{2}\right]\left\{\frac{p+1}{2}\right\}, \tag{3}$$

where all four bounds are best possible.

Thus, we look at the table of Ramsey numbers for the solution to our problem. Within the table, we define the cone with vertex (m,n) as the set

$$C(m,n) = \{(m',n') \mid m' \ge m \text{ and } n' \ge n\}$$

and observe that if (m,n) is in R_p, then $C(m,n)$ is a subset of R_p. We say that (m,n) is a *cornerpoint* of R_p if (m,n) lies in R_p but neither $(m-1,n)$ nor $(m,n-1)$ is in R_p. Then R_p is the union of those cones each of whose vertices is a cornerpoint of R_p. Then σ_p and μ_p, the minima of the respective functions $\sigma = (m-1) + (n-1)$ and $\mu = (m-1)\cdot(n-1)$ for (m,n) in R_p, are attained at cornerpoints of R_p.

TABLE OF RAMSEY NUMBERS

m \ n	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10
3	3	6	9	14	18	23			
4	4	9	18						
5	5	14							
6	6	18							
7	7	23							
8	8								
9	9								
10	10								

We now turn attention to our problem of existence. Let (m,n) be any cornerpoint of R_p. Since $r(m,n) > p$, there exists a graph G of order p with $\beta(G) < n$ and $\beta(\overline{G}) < m$. Neither $(m,n-1)$ nor $(m-1,n)$ lies in R_p, so $\beta(G) \not< n-1$ and $\beta(\overline{G}) \not< m-1$. Therefore, $\beta(G) = n-1$ and $\beta(\overline{G}) = m-1$. Now, by judiciously adding edges to G without reducing $\beta(G)$, we can increase $\beta(\overline{G})$ until reaching the upper bound for $\beta(G) + \beta(\overline{G})$, namely $p+1$. Thus, if $s(i,j)$ is the square in row i and column j of the table of Ramsey numbers, then there is a graph G of order p having $\beta(G) = j$ and $\beta(\overline{G}) = i$ if $s(i,j)$ abuts the region R_p. Conversely, suppose $s(i,j)$ neither belongs to the region defined by R_p nor abuts R_p, then $(i+1,j+1)$ does not lie in R_p. This implies that $r(i+1,j+1) \leq p$, which means that if G is any graph of order p, then $\beta(G) \geq j+1$ or $\beta(\overline{G}) \geq i+1$; i.e., there exists no graph G of order p having $\beta(G) = j$ and $\beta(\overline{G}) = i$.

We summarize our discussion in the following theorem.

Theorem 3. There exists a graph G of order p having $\beta(G) = j$ and $\beta(\overline{G}) = i$ if and only if $(i+1,j+1)$ lies in R_p, but (i,j) does not.

REFERENCE

1. G. Chartrand and S. Schuster, On the independence number of complementary graphs, *Transactions of the New York Academy of Sciences*, Series II, 36(1974), 247-251.

Hungarian Academy of Sciences and Carleton College