SOME COMBINATIONAL PROBLEMS IN GEOMETRY

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In this short survey I mainly discuss some recent problems which occupied me and my colleagues and collaborators for the last few years. I will not give proofs but will either give references to the original papers or to the other survey papers. I hope I will be able to convince the reader that the subject is "alive and well" with many interesting, challenging and not hopeless problems. Development in geometry of the kind which Euclid surely would recognize as geometry was perhaps not very significant - one very striking theorem was proved about 75 years ago by Morley which states that if one trisects the angles α, β, γ of a triangle the pairs of trisectors of the angles meet in an equilateral triangle. Surely a striking and unexpected result of great beauty. In fact because the trisection can not be carried out by ruler and compass it is not quite sure if Euclid would recognize this as legitimate (if and when I meet him [soon?] I plan to ask him).

On the other hand, many new results have been found on geometric inequalities - I won't deal with them in this paper and state only one of them, the so called Erdös-Mordell inequality (which was one of my first conjectures - I conjectured it in 1932 - two years later Mordell found the first proof). Let A, B, C be any triangle, O a point in its interior, OX is perpendicular to AB, OY to BC and OZ to AC. Then,

$$\overline{OA} + \overline{OB} + \overline{OC} \geq 2(\overline{OX} + \overline{OY} + \overline{OZ})$$

equality only if ABC is equilateral and O is the center.

I first of all give references to some earlier papers and books which deal with similar or related questions:

P. Erdös, "On Some Problems of Elementary and Combinational Geometry," *Annali di Mat. Ser IV, V 103* (1975), p. 99-108. We will refer to this paper as I.

Very interesting problems and results are in the monograph of B. Grünbaum, "Arrangements and Spreads," *Conference Board on Math. Sciences*, Amer. Math. Soc., No.10. This monograph has a very extensive and useful bibliography.

The book of Hadwiger, Debrunner and Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart and Winston, New York, 1964, contains much interesting information about geometric and combinational results. It can be used as a textbook to learn the subject but contains few unsolved problems.

Assoc. Sympos. Pure Math., Vol.7 (Convexity), 1963, contains many papers on related problems - in particular the beautiful papers of Danzer, Grünbaum and Klee are relevant to our subject matter.

Very interesting geometric questions of a related but somewhat different kind are in the books of L. Fejes-Toth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer-Verlag, Berlin 1953, and *Regular Figures*, Pergamon Press Macmillan Co., New York, 1964.

G. Purdy and I plan to write a book on some of the questions and their extensions which we considered in our joint papers - if we live - the book should appear sometime in the next decade.

I apologize to the reader and to the authors for the many references which I omitted here, these omissions are partly due to limitations of time and space and partly to ignorance.

1. Let $f_k(n)$ be the largest integer so that there are n distinct points $x_1, \ldots, x_n$ in k-dimensional Euclidean space E_k so that there are $f_k(n)$ pairs x_i, x_j with $d(x_i, x_j) = 1$. ($d(x_i, x_j)$ denotes the distance between x_i and x_j). $g_k(n)$ is the largest integer so that for every such choice of n points in E_k there are at least $g_k(n)$ distinct numbers among the $d(x_i, x_j)$. These problems are extensively studied in I, here I just state the outstanding open problems (we restrict ourselves to $k = 2$).

$$n^{1+c/\log\log n} < f_2(n) = o(n^{3/2}) \tag{1}$$

Whereas $f_2(n) < C n^{3/2}$ is very simple the seemingly slight improvement (due to E. Szemerédi is very difficult). I conjecture that the lower bound in (1) gives the right order of magnitude and I offer 250 dollars for a proof or disproof. I give 100 dollars for a proof of $f_2(n) < n^{1+\epsilon}$.

Perhaps the following stronger conjecture holds: There always is an x_i so that there are at most $o(n^{\epsilon})$ ($n^{c/\log\log n}$?) x_j's equidistant from it.

$$c_1 n^{2/3} < g_2(n) < c_2 \frac{n}{(\log n)^{1/2}} \tag{2}$$

The lower bound in (2) is due to L. Moser. I believe that the upper bound gives the correct order of magnitude and I offer 250 dollars for a proof or disproof and 100 dollars for $g_2(n) > n^{1-\epsilon}$. (This would of course be implied by $f_2(n) < n^{1+\epsilon}$).

I conjectured and Altman proved that if the x_i are the vertices of a convex n-gon then $g_2(n) = \left[\frac{n}{2}\right]$. I also conjectured that there is an x_i so that there are at least $\left[\frac{n}{2}\right]$ distinct distances amongst the $d(x_i, x_j)$, $j=1,\ldots n$, $j \neq i$. This problem is still open. I also conjectured that there is an x_i which does not have three other x's equidistant from it. This was disproved by Danzer (unpublished), but perhaps there is always an x_i which does not have four other vertices equidistant from it.

Szemerédi made the pretty conjecture that Altman's $g_2(n) = \left[\frac{n}{2}\right]$ remains true if we only assume that no three of the x_i are on a line, but he only proved $g_2(n) \geq \left[\frac{n}{3}\right]$, in this case.

L. Moser and I conjectured that if the x_i are the vertices of a convex polygon then $f_2(n) < Cn$. It is annoying that no progress was made with this elementary conjecture. We have a simple example which shows $f_2(5n+1) \geq 3n$ and as far as I know this is all that is known.

G. Purdy and I observed that if no three of the x_i are on a line then $f_2(n) > cn\log n$ is possible, this follows easily by a method of Kárteszi. We have no idea for the exact order of magnitude of $f_2(n)$ in this case.

Most of the results stated in this Chapter are referenced in I.

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P. Erdos and G. Purdy, "Some Extremal Problems in Geometry," *IV Assoc. Seventh Southeastern Conference on Combinations Graph Theory and Computing*, 1976, p.307-322. (Congress Numerantium XVII). (For related problems see our paper III and V, same conference, 1975 and 1977, p. 291-308 and p. 569-578).

2. An old problem of E. Klein (Mrs. Szekeres) states: Let $H(n)$ be the smallest integer so that every set of $H(n)$ points in the plane, no three on a line, contains the vertices of a convex n-gon (it is not at all obvious that $H(n)$ exists for every n). She proved $H(4) = 5$ and Szekeres conjectured $H(n) = 2^{n-2} + 1$. Makai and Turan proved $H(5) = 9$ and Szekeres and I proved

$$2^{n-2} + 1 \leq H(n) \leq \binom{2n-4}{n-2}.$$

All this is in I (See Introduction). Recently I found the following interesting modification of this problem: Let M_n be the smallest integer so that every set of M_n points no three on a line always contains the vertices of a convex n-gon which contains no x_i in its interior. Trivially $M_4 = 5$, and Ehrenfeucht proved that M_5 exists, Harborth proved $M_5 = 10$. It is not at all clear that M_n exists and at present it is possible that even M_6 does not exist (in other words, for every m there are m points in the plane no three on a line so that every convex hexagon determined by these points contains at least one other point in its interior).

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3. Let there be given n points in the plane not all on a line. Is it true that there always is a line which goes through precisely two of the points? Such a line

is called an ordinary line. This beautiful result was conjectured by Sylvester in 1893. I rediscovered the conjecture in 1933 and a few days later T. Gallai found a proof. The first proof was published by Melchior who rediscovered it quite independently in 1940. Extensive references and the history of this problem can be found in II and I and in a paper of Motzkin. Here I only state a few recent results and problems. Denote by $t_k(n)$ the largest integer for which there is a set of n points in the plane for which there are $t_k(n)$ lines containing exactly k of the points. $t_3(n)$ Has been studied for more than 150 years. The sharpest results on $t_3(n)$ are due to Burr, Grünbaum and Sloane. They conjecture that

$$t_3(n) = 1 + [n(n-3)/6] \quad \text{for} \quad n \neq 7, 11, 16, 19. \tag{1}$$

They prove ( ] × [is the least integer ≥ x)

$$1 + \left[\frac{n(n-3)}{6}\right] \leq t_2(n) \leq \left[\left(\binom{n}{2} - \right] \frac{3n}{7} \left[\right)/3\right] .$$

Croft and I prove that for every $k \geq 3$

$$t_k(n) > c_k n^2 \tag{2}$$

c_k is an absolute constant. The simple proof of (2) is given in II. The best possible value of c_k in (2) is not known. Denote by $t_k'(n)$ the largest integer for which there is a set of n points in the plane no $k+1$ of them on a line for which there are $t_k'(n)$ lines containing exactly k of the points. I conjectured that for $k > 3$, $t_k'(n) = o(n^2)$ and could not even prove $t_k'(n)/n \to \infty$. Karteszi proved $t_k'(n) > c_k\, n \log n$ and Grünbaum showed that $t_k'(n) > cn^{1+1/k-2}$. Further problem: Assume $k = [cn^{1/2}]$. Determine or estimate $t_k'(n)$. It is true that

$$t_k'(n) > \alpha n^{1/2}/c$$

where α is independent of n and c?

Let $x_1, \ldots, x_n$ be n points in E_2. Join every two of them. Prove (or disprove) that one gets at least ckn distinct lines where c is an absolute constant independent of n and k. This (and more) was proved by Kelly and Moser if $k < c_1 n^{1/2}$.

Let $x_1, \ldots, x_n$ be n points in the plane not all on a line and let $L_1, \ldots, L_m$ be the set of lines determined by these points. Graham conjectured that if S is a subset of $\{x_1, \ldots, x_n\}$ so that every line L_i intersects S, then for at least one i, $L_i \subset S$. This conjecture was recently proved by Rabin and Motzkin.

I then asked the following question: Does there exist for every k a finite set S of points in the plane so that if one colors the points of S by two colors in an arbitrary way, there always should be a line which contains at least k points and all whose points are of the same color. Graham and Selfridge gave an affirmative answer for $k = 3$, but the cases $k > 3$ seem to be open.

Finally, I want to call attention to a nearly forgotten problem of Serre: Let

A_n be the projective n space over the complex numbers. A finite subset is a Sylvester-Gallai configuration if every line through two of its points also goes through a third. Characterize all planar Gallai-Sylvester configurations. Is there a non-planar Gallai-Sylvester configuration?

For generalization of the Gallai-Sylvester theorem to matroids, see, e.g. the book of D.J.A. Welsh, *Matroid Theory*, p. 286-297, Academic Press, 1976.

For a generalization of different nature, see, e.g., M. Edelstein, "Generalizations of the Sylvester Problem," *Math. Magazine, 43* (1970), p. 250-254, and M. Edelstein, F. Herzog, and L.M. Kelly, "A Further Theorem of the Sylvester Type," *Proc. Amer. Math. Soc., 14* (1963), p. 359-363.

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4. In this last Chapter I state a few miscellaneous problems. Recently "we" (Graham, Montgomery, Rothschild, Spencer, Straus and I) published several papers on a subject which we called Euclidean Ramsey theorems. A subset S of E_m is called Ramsey if for every k there is an m_k so that if we decompose E_{m_k} into k subsets, $E_{m_k} = \bigcup_{i=1}^{k} S_i$ at least one S_i has a subset congruent to S. We prove that every brick (i.e., rectangular parallelepiped) is Ramsey and that every S which is Ramsey is inscribed in a sphere. The most striking open problems are: Is the regular pentagon Ramsey? Is there an obtuse angled triangle which is Ramsey? Are in fact all obtuse angled triangles Ramsey?

Let $S_1 \cup S_2$ be the plane. Is it true that if T is any triangle (with the possible exception of equilateral triangles of one fixed height) then either S_1 or S_2 contains the vertices of a triangle congruent to T? Many special cases of this startling conjecture have been proved by us and Schader but so far the general case eluded us. There surely will be interesting generalizations for higher dimensions but these have not yet been investigated.

Let S be a set of points in the plane no two points of S are at distance one. We conjectured that the complement of S contains the vertices of a unit square. This conjecture was proved by R. Juhász. She in fact showed that if X_1, X_2, X_3, X_4, are any set of four points then the complement of S contains a congruent copy. It is not known at present if this remains true for 5 points; she showed that there is a k so that it fails for k points.

Clearly many more problems can be stated here, and in fact many have been stated in our papers. I hope more people will work on this subject in the future and our results will soon become obsolete.

The following problem is due to Hadwiger and Nelson: Join two points of r-dimensional space if their distance is one. Denote by α_r the chromatic number of this graph. Is it true that $\alpha_2 = 4$? It is known that $4 \leq \alpha_2 \leq 7$. I am sure that $\alpha_2 > 4$ but cannot prove it. By a well known theorem of the Bruijn and myself if $\alpha_2 > 4$ then there is a finite set of points $x_1, \ldots, x_n$ in the plane so that the graph whose edges are (x_i, x_j), $d(x_i, x_j) = 1$ has chromatic number greater than four. The determination of such a graph may not be easy since perhaps n must be very large.

α_r for large r was first studied by Lavman and Rogers. The sharpest known result is due to P. Frankl, $\alpha_r > r^c$ for every c if $r > r_0(c)$. It seems certain that there is a fixed $\varepsilon > 0$ so that $\alpha_r > (1+\varepsilon)^r$. ($\alpha_r < 3^r$ is proved by Lavman and Rogers.) This conjecture would easily follow from the following purely combinatorial conjecture (which perhaps is very hard). Let $|S| = n$, $A_i \subset S$, $1 \leq i \leq u_n$ be a family of subsets of S satisfying for every $1 \leq i_1 < i_2 \leq u$, $|A_{i_1} \cap A_{i_2}| \neq [\frac{n}{4}]$. Then there is an $\varepsilon > 0$ independent of n for which

$$\max u_n < (2 - \varepsilon)^n \tag{1}$$

(1) no doubt remains true if the assumption $|A_{i_1} \cap A_{i_2}| \neq [\frac{n}{4}]$ is replaced by: There is a t, $\eta n < t < (\frac{1}{2} - \eta)n$ so that $|A_{i_1} \cap A_{i_2}| \neq t$ for every $1 \leq i_1 < i_2 \leq u_n$, only here ε will depend on η. At present no proof seems to be in sight.

Let $x_1, \ldots, x_n$ be n distinct points in the plane. Denote by $C(x_1, \ldots, x_n)$ the number of distinct circles of radius one which go through at least three of the x_i. Put

$$F(n) = \max C(x_1, \ldots, x_n) \tag{2}$$

where the maximum in (2) is taken for all possible choices of distinct points $x_1, \ldots, x_n$. I conjectured more than two years ago that

$$F(n)/n^2 \to 0\ ,\ F(n)/n \to \infty \tag{3}$$

It seems that (3) is trivial but I could not prove it and I have no idea about the true order of magnitude of $F(n)$, probably $F(n) < n^{1+\varepsilon}$ for every $\varepsilon > 0$, if $n > n_0(\varepsilon)$.

Let $x_1, \ldots, x_n$ be n points in E_r satisfying $d(x_i, x_j) \geq 1$. Determine or estimate

$$D_r(n) = \min \max_{1 \leq i < j \leq n} d(x_i, x_j)$$

where the minimum is taken over all choices of $x_1, \ldots, x_n$ in E_r satisfying $d(x_i, x_j) \geq 1$. The exact value of $D_r(n)$ is known only for very few values of r and n. A classical result of Thue states

$$\lim_{n=\infty} D_2(n)/n^{1/2} = \left(\frac{2\,3^{1/2}}{\pi}\right)^{1/2}$$

The value of $\lim_{n=\infty} D_3(n)/n^{1/3}$ is not known and is an outstanding open problem in the geometry of numbers.

Let $x_1,\ldots,x_n$ be n points in the plane. Denote by $L_1,\ldots,L_m$ the set of lines determined by these points. Denote by u_i the number of points on L_i. $u_1 \geq u_2 \geq \ldots \geq u_m$. Clearly

$$\sum_{i=1}^{m} \binom{u_i}{2} = \binom{n}{2} \tag{4}$$

Let $\{u_i\}$ be a set of integers satisfying (4). It would be of interest to obtain nontrivial conditions on the u_i which would assure that there is a set of points in the plane for which there are u_i points on L_i. Perhaps there is no simple necessary and sufficient condition. Denote by $f(n)$ the number of distinct sequences $u_1 \geq \ldots \geq u_m$ (m is also a variable) for which there is a set of points $x_1,\ldots,x_n$ with u_i points on L_i. It is easy to see that

$$\exp[c_1 n^{1/2}] < f(n) < \exp[c_2 n^{1/2}] \tag{5}$$

I expect that the lower bound gives the correct order of magnitude in (5), but I had not the slightest success in proving this.

One can formulate this problem in a more combinatorial way. Let $|S| = n$, $A_i \subset S$, $1 \leq i \leq m$ are subsets of S ($|A_i| \geq 2$). Assume that every pair x,y of elements of S are contained in exactly one A_i. Put $|A_i| = u_i$, $u_1 \geq u_2 \geq \ldots \geq u_m$. Clearly (4) holds here too. Denote by $F(n)$ the number of possible choices for the u's. It is not hard to prove that (5) holds for $F(n)$ too, but here I expect that the upper bound gives the correct order of magnitude, but again I had no success. ($F(n) > f(n)$ easily follows since by Gallai-Sylvester $u_m = 2$ in the geometric case.)

A well known theorem of de Bruijn and myself states that (unless $|A_1| = n$) we must have $m \geq n$. This easily implies that there are $c_1 n^{1/2}$ A_i's of the same size. I believe that this is best possible, in other words: There is a system of subsets $A_i \subset S$ $m > 1$, every pair of elements of S is contained in exactly one A_i and there are at most $c_2 n^{1/2}$ values of i for which the A_i are of the same size. Perhaps it is not hard to construct such a design and my lack of success was due to lack of experience with construction of block designs.

Assume $u_1 \leq Cn^{1/2}$. Purdy and I recently obtained fairly accurate asymptotic formulas in the general combinatorial case for

$$\max \sum \binom{u_i}{3}$$

in terms of u_1. On the other hand, we had no success in the geometric case (i.e., when the x_i are points in the plane and the L_i are lines). We conjectured that if $u_1 < c_1 n^{1/2}$ then

$$\sum \binom{u_i}{3} < c_2 n^{3/2}$$

where $c_2 = c_2(c_1)$.

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