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RAMSEY-MINIMAL GRAPHS FOR THE PAIR STAR, CONNECTED GRAPH

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Abstract

In this paper it is proved that $(G, K_{1,k})$ is Ramsey-infinite for any non-trivial two-connected graph G and any star with $k \geqq 2$ edges. Also it is shown that $(H, K_{1,2})$ is Ramsey-infinite if H is a bridgeless connected graph.

Introduction

Let F, G and H be graphs (without loops or multiple edges). We write $F \to (G, H)$ if whenever each edge of F is colored either red or blue, then either the red subgraph of F, denoted $(F)_R$, contains a copy of G or the blue subgraph of F, denoted $(F)_B$, contains a copy of H. The graph F is (G, H)*-minimal* if $F \to (G, H)$ but $F' \not\to (G, H)$ for any proper subgraphs F' of F. In particular if F, G and H have no isolated vertices, F' can be replaced by $F-e$ for any edge e of F. The class of all (G, H)-minimal graphs will be denoted by $\mathcal{R}(G, H)$. The pair (G, H) will be called *Ramsey-finite* or *Ramsey-infinite* depending upon whether $\mathcal{R}(G, H)$ is finite or infinite.

In [4] it was shown that if M is a disjoint union of edges (a matching), then $\mathcal{R}(M, H)$ is finite for any graph H. It was also conjectured that if G is any graph such that $\mathcal{R}(G, H)$ is finite for each graph H, then G must be a matching. There is considerable support for this conjecture. Nešetřil and Rödl proved in [7] that if G and H are both 3-connected, then $\mathcal{R}(G, H)$ is infinite. Also the results in [2], [3] and [8] imply that the conjecture is true for graphs which are forests. In this paper we will add to this evidence. We will show that $\mathcal{R}(K_{1,k}, H)$ is infinite if $k \geqq 2$ ($K_{1,k}$ is a star with k edges) and H is a 2-connected graph. We will also show that $\mathcal{R}(K_{1,2}, H)$ is infinite if H is any connected graph none of whose blocks are edges.

Notation

Before proving the main two results, some additional notation and terminology will be introduced. Notation not specifically mentioned will generally follow that of [1] and [7]. For a graph G, $V(G)$ and $E(G)$ will denote the vertex set and edge set respectively, and $H \leqq G$ will denote that H is a subgraph of G. The degree of a vertex v of G will be written $d_G(v)$.

The word "coloring" will always refer to coloring each edge of some graph red or blue. A coloring of a graph F with neither a red G or a blue H will be called a

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(G, H)-*good* coloring, or if G and H are obvious, simply a good coloring. A useful concept introduced in [5] was a (G, H, γ)-*determiner*, which is a graph that has a (G, H)-good coloring but under all such good colorings the edge γ must be colored red. Normally one might call such a graph a "red" determiner, but since we will only work with red determiners the word red will be dropped. In a (G, H, γ)-determiner, the edge γ will be referred to as the *determined edge*.

Let $\{G_\alpha : \alpha \in A\}$ be a non-empty finite family of graphs. Sometimes in a coloring instead of requiring a red G it is only necessary to have one red G_α from some family, that is the "or" of the family $\{G_\alpha : \alpha \in A\}$. Thus we write $F \to (\bigvee_{\alpha \in A} G_\alpha, H)$ if when F is colored there is either a red G_α for some α in A or a blue H. Likewise the "and" $(\bigwedge_{\alpha \in A} G_\alpha)$ and the "disjoint union" $(\bigcup_{\alpha \in A} G_\alpha)$ can be considered. These would require a red G_α for each α in A and vertex disjoint red G_α for each α in A respectively. Concepts such as minimal, Ramsey-finite, good coloring and determiner are defined in the same way if G is replaced by the "or", "and" or "union" of a family of graphs. One can also generalize H in the same fashion.

In many of the constructions in this paper a graph F will be enlarged to a graph L by "attaching" a graph G onto F at vertex v and identifying the vertex v of F with the vertex u of G. Anytime this "attaching" is done it will be asssumed in L that F and G are disjoint except for the identified vertices. Also if copies of G are "attached" to several or all vertices of F it will be assumed that these distinct copies are pairwise disjoint.

For any real number x, $[x]$ will denote the greatest integer less than or equal to x and $\{x\}$ will denote the least integer greater than or equal to x.

Star with 2-connected graph

We first state the main result of this section.

THEOREM 1. *Let* $\{G_\alpha : \alpha \in A\}$ *be a non-empty finite family of* 2*-connected graphs. If* $k \geqq 2$, *then* $\mathcal{R}(\bigvee_{\alpha \in A} G_\alpha, K_{1,k})$ *is infinite.*

Theorem 1 as stated will be needed to prove the main result of the next section. As far as this section is concerned the special case in which A has precisely one element is of interest.

COROLLARY 2. *If G is a* 2*-connected graph and* $k \geqq 2$, *then the pair* $(G, K_{1,k})$ *is Ramsey-infinite.*

The main idea used in the proof of Theorem 1 is that of a determiner. Its power is illustrated in the following theorem, which is a generalization of a result in [3]. It is clear from the proof of the result that it can be stated in this more general form.

THEOREM 3. *Let* $\{G_\alpha : \alpha \in A\}$ *be a non-empty finite family of* 2*-connected graphs and let T be a tree with at least* 3 *vertices. If there is a* $(\bigvee_{\alpha \in A} G_\alpha, T, \gamma)$*-determiner with* γ *a free edge, then* $(\bigvee_{\alpha \in A} G_\alpha, T)$ *is Ramsey-infinite.*

PROOF of Theorem 1. To shorten the notation, $\bigvee_{\alpha\in A} G_\alpha$ will be denoted by $\mathcal{G}$, and since we will only be interested in the degree of vertices in the blue graph, $d_{(F)_B}$ will be written d_F throughout this proof. In view of Theorem 3 it is sufficient to show that there is an appropriate $(\mathcal{G}, K_{1,k}, \gamma)$-determiner. Clearly, if a graph F can be good colored but in any such good coloring a fixed vertex v has $d_F(v)=k-1$, then a $(\mathcal{G}, K_{1,k}, \gamma)$-determiner is formed by attaching an edge γ to v. We will assume that this does not occur and show that this leads to a contradiction.

Let F be a graph in $\mathcal{R}(\mathcal{G}, K_{1,k})$ and $e=ab$ an edge of F with endvertices a and b. Consider the graph $F'=F-e$. The minimality of F implies that F' can be good colored. Also either $d_{F'}(a)$ or $d_{F'}(b)=k-1$ since F has no good coloring. If $d_{F'}(a)=k-1$ (or equivalently $d_{F'}(b)=k-1$) for each good coloring of F' then this gives a contradiction. We thus have $d_{F'}(a)<k-1$ for some good colorings of F' and $d_{F'}(b)<k-1$ for others. For all good colorings of F', let s be the minimum of $d_{F'}(a)+d_{F'}(b)-(k-1)$. With no loss of generality we can assume that there is a good coloring of F' with $d_{F'}(a)=s$ and $d_{F'}(b)=k-1$. Let $F_i\,(i\geqq 1)$ be a countably infinite family of disjoint copies of F' with corresponding vertices a_i and b_i.

The remainder of the proof will be broken into three cases depending upon s. Also the third case will be split into two subcases.

Case I. $s=0$.

Select an integer n strictly greater than the number of vertices in any $G_\alpha, \alpha\in A$. Construct a graph L from the graphs $\{F_i\colon 1\leqq i\leqq n)$ by identifying a_{i+1} and b_i $(1\leqq i\leqq n-1)$ and finally by identifying a_1 and b_n. The graph L can be good colored since there is a good coloring of F' with $d_{F'}(a)=0$ and $d_{F'}(b)=k-1$. There is no $G_\alpha\subseteq (L)_R$ for any $\alpha\in A$ since each G_α is 2-connected and n is large. Also in any good coloring of $L, d_L(a_i)=k-1$ since $d_{F'}(a)+d_{F'}(b)\geqq k-1$ in any good coloring of F'. This gives a contradiction so we can assume $s>0$.

Case II. $0<s<k/2$.

Let $t=[(k-1)/s]$. Let L be the graph formed by attaching each F_i to F_1 by identifying a_i with a_1 for $2\leqq i\leqq t$. There is a good coloring of L since each F_i can be good colored with $d_{F_i}(a_i)=s$. Since $t\geqq 2, d_{F_i}(a_i)<k-1$ for each i in any good coloring of L. Thus $d_{F_i}(b_i)=k-1$ in any good coloring of L, again a contradiction.

Case III. $s\geqq k/2$.

Let L be the graph formed by attaching F_2 to F_1 by identifying a_1 and a_2. Since $s\geqq k/2, L\to(\mathcal{G}, K_{1,k})$. Let L' be a subgraph of L such that $L'\in\mathcal{R}(\mathcal{G}, K_{1,k})$. Set $F_i'=L'\cap F_i\,(i=1,2)$; then each of F_1' and F_2' contains at least one edge. In any good colorings of F_1' and $F_2', d_{F_1'}(a_1)+d_{F_2'}(a_2)\geqq k$ since there is no good coloring of L'. The good colorings of F_1' and F_2' are independent of each other. Therefore we can assume with no loss of generality that $d_{F_2'}(a_2)\geqq k/2$ in any good coloring of F_2'.

Let L'' be the graph $L'-a_1c$, where c is a vertex of F_1', and let $F_1''=F_1'-a_1c$. Thus L'' can be good colored and hence there is a good coloring of F_1'' with $d_{F_1''}(a_1)<k/2$. Let s_1 be the minimum of $d_{F_1''}(a_1)$ over all good colorings of F_1''. There is a good coloring of L'' with $d_{L''}(a_1)<k-1$, for otherwise there would be a determiner. This implies that F_2' has a good coloring with $d_{F_2'}(a_2)<k-1-s_1$. Therefore in any good coloring of $F_1'', d_{F_1''}(c)=k-1$ when $d_{F_1''}(a_1)=s_1$, the minimum.

Consider all graphs H' which can be good colored and have non-adjacent vertices u and v such that in any good coloring in which $d_{H'}(v)$ is a minimum, $d_{H'}(u)=k-1$. The graph F_1'' with vertices a_1 and c is an example of such a graph. If r is the minimum of $d_{H'}(v)$ over all possible colorings of all possible such graphs, let H with vertices u and v be a graph in which this minimum r is attained. Clearly $0 \leqq r \leqq s_1 < k/2$.

Let $\{H_i : i \geqq 1\}$ be disjoint copies of H with corresponding vertices u_i and v_i. Using the graphs $\{H_i : i \geqq 1\}$ we will show that there is a determiner, which will give a contradiction. The remaining argument will be broken into two cases depending on r.

Subcase i. $r>0$.

We will show that the case does not occur. Let $t=\{k/r\}$ and let N be the graph constructed by attaching H_i to H_1 by identifying v_1 and v_i for each $2 \leqq i \leqq t$. Since $rt \geqq k$, $N \to (\mathcal{G}, K_{1,k})$. In the same way that F_1'' with vertices c and a_1 was obtained from L, we get from N a graph H_1'' with vertices w and v_1. The graph H_1'' is a subgraph of H_1 and has the same properties as F_1''. In this case there is a good coloring of H_1'' with $d_{H_1''}(v_1) < k/t$. Since $k/t \leqq r$, this contradicts the minimality of r. We can thus assume $r=0$.

Subcase ii. $r=0$.

Construct a graph L from the graphs $\{H_i : 1 \leqq i \leqq n\}$ by identifying u_{i+1} and v_i for $1 \leqq i \leqq n-1$, and then identifying u_1 and v_n. Recall that n is strictly greater than the order of any G_α for $\alpha \in A$. This graph has a good coloring since H can be good colored with $d_H(u)+d_H(v) \leqq k-1$. If $d_H(u)+d_H(v)=k-1$ for every good coloring of H then for every good coloring of $L, d_L(u_i)=k-1$ for each i. Therefore we can assume that there is a good coloring of H with $d_H(u)+d_H(v)<k-1$. Let r' be the minimum of $d_H(v)$ over all colorings in which $d_H(u)+d_H(v)<k-1$. By the definition of $H, r' \geqq 1$.

Let G be a copy of a graph in $\{G_\alpha : \alpha \in A\}$ with a minimum number of vertices. Construct a graph L' from the graphs $\{H_i : 1 \leqq i \leqq n\}$ by identifying u_{i+1} and v_i for $1 \leqq i \leqq n-1$. To each vertex g of G except for a fixed vertex g_0 attach a copy L_g' of L' by identifying the copy of u_1 in L_g' with g. Finally identify all of the copies of v_n in each L_g' and call this vertex x. Denote this graph by L. Let $t'=[(k-1)/r']$. Consider t' copies $L_1, L_2, \ldots, L_{t'}$ of L with corresponding graphs G_i and vertices $x_i, (1 \leqq i \leqq t')$. Construct a graph by identifying all of the vertices x_i from each $L_i\,(1 \leqq i \leqq t)$. Denote this vertex by y. Attach H (actually a copy of H) to this graph by identifying y and v and call this graph N.

We will show that N has a good coloring and in any such good coloring $d_N(u)=k-1$. Consider the graph L and select a vertex g_1 in G which is adjacent to g_0. Color the edge g_0g_1 blue and the remaining edges of G red. For each $g \neq g_0, g_1$, good color each copy of H in L_g' such that $d_H(u)=k-1$ and $d_H(v)=0$. Good color each copy of H in L_{g_1}' such that $d_H(v)=r'$. This is a good coloring of L with $d_L(x)=r'$. If each copy L_i of L in N is colored as just described and H is good colored such that $d_H(v)=0$ and $d_H(u)=k-1$, then this gives a good coloring of N.

In any good coloring of N each copy of G will have a blue edge. Let g be a vertex of G incident to this blue edge. Thus the good coloring induced on L_g' must have $d_{L_g'}(v_n)=d_{L_g'}(y) \geqq r'$. This implies $d_N(y) \geqq t'r'+d_H(v)$. Hence $d_H(v)=0$ and $d_H(u)=k-1$. This contradiction completes the proof.

The following theorem was proved in [3] and applies to the results proved in this section.

THEOREM 4. *Let* $\{G_\alpha: \alpha \in A\}$ *and* $\{H_\beta: \beta \in B\}$ *be a non-empty finite family of connected graphs. If* (G_α, H_β) *is Ramsey-infinite for each* α *and* β, *then*

$$\mathscr{R}\Big(\bigwedge_{\alpha\in A} G_\alpha,\ \bigwedge_{\beta\in B} H_\beta\Big) \quad \textit{is infinite and}$$

$$\mathscr{R}\Big(\bigcup_{\alpha\in A} G_\alpha,\ \bigcup_{\alpha\in B} H_\beta\Big) \quad \textit{is infinite.}$$

Let $\mathscr{G}$ be a collection of 2-connected graphs and $\mathscr{H}$ a collection of non-trivial stars. Consider any expression E_1 (or E_2) using graphs from $\mathscr{G}$ (or $\mathscr{H}$) and $\vee$, $\wedge$ and $\cup$. Then Theorems 1 and 4 imply that the pair (E_1, E_2) is Ramsey-infinite.

THEOREM 5. *Let G be a graph all of whose components are 2-connected and let H be a star forest without isolated edges. Then* (G, H) *is Ramsey-infinite.*

$K_{1,2}$-connected graph

The main result to be proved in this section is the following:

THEOREM 6. *If G is a connected graph all of whose blocks are not edges, then* $\mathscr{R}(G, K_{1,2})$ *is infinite.*

Before we give the proof some additional notation and results must be introduced. The Ramsey number $r(G, K_{1,2})$ for any arbitrary G will be used frequently in the proof. This was calculated in [6]. In the following, $\beta_1(G)$ will denote the edge independence number of G.

THEOREM 7. *For any graph G with no isolated vertices,*

$$r(G, K_{1,2}) = \begin{cases} |V(G)| & \textit{if } \bar{G} \textit{ has a perfect matching,} \\ 2|V(G)| - 2\beta_1(\bar{G}) - 1 & \textit{otherwise.} \end{cases}$$

The location as well as the existence of monochromatic graphs in two-colored graphs will be important. This is the motivation for the following concept. Assume that $F \to (G, K_{1,2})$; assume in addition that for any coloring of F in which there is no blue $K_{1,2}$, it is true that for every $v \in V(G)$ and $w \in V(F)$ there is a red copy of G with w corresponding to v. If this is true for every v in G we will write $F \twoheadrightarrow (G, K_{1,2})$. It is easily seen that if $F - M$, where M is a perfect matching of F, is vertex-symmetric, then $F \to (G, K_{1,2})$ implies $F \twoheadrightarrow (G, K_{1,2})$. For example if n is even, then $K_n \to (G, K_{1,2})$ implies $K_n \twoheadrightarrow (G, K_{1,2})$. Also if $m \leqq n$ and $K_m \twoheadrightarrow (G, K_{1,2})$, then $K_n \twoheadrightarrow (G, K_{1,2})$. If $r(G, K_{1,2}) > |V(G)|$ then Theorem 7 implies that for any fixed maximal matching of $\bar{G}$ there are at least 2 vertices not on the matching. If $K_{r(G, K_{1,2})}$ is colored such that there is no blue $K_{1,2}$ then there are red copies of G with any vertex not on a maximal matching of $\bar{G}$ appearing at any vertex of the complete graph. The following useful result is a consequence of the above observations.

LEMMA 8. *If* $K_n \to (G, K_{1,2})$ *then* $K_{n+1} \twoheadrightarrow (G, K_{1,2})$. *If* $K_n \to (G, K_{1,2})$ *and* $n > |V(G)|$ *then* $(K_{n+1} - e) \twoheadrightarrow (G, K_{1,2})$.

If H is a subgraph of a graph G then by $G-H$ we will mean in what follows the graph obtained from G by deleting the edges of H and then deleting any isolated vertices.

The proof of Theorem 6 will be preceded by several lemmas.

One construction appears so frequently throughout the proofs of the lemmas that we describe it now and merely refer to it later in the proofs. Let G be a connected graph with no blocks which are edges and $\{H_1, H_2, ..., H_l\}$ be a subset of the blocks of G. By Theorem 1, $\mathcal{R}(\bigvee_{i=1}^{l} H_i, K_{1,2})$ is infinite. Let $\{F_i: i\geqq 1\}$ be an infinite family in $\mathcal{R}(\bigvee_{i=1}^{l} H_i, K_{1,2})$ and let D be a fixed graph with a specified vertex v. For each $i\geqq 1$ attach to each vertex u of F_i a copy of D by identifying u and v. Denote this graph by F_i'. In many cases one can show, for an appropriate choice of D, that $F_i'\to(G, K_{1,2})$ and moreover, that if $F_i''\leqq F_i'$ and $F_i''\to(G, K_{1,2})$ then $F_i\leqq F_i''$ for all $i\geqq 1$. In particular one can usually show that $F_i''-e\nrightarrow(G, K_{1,2})$ for any edge e of F_i. The usual conditions on D are that $D\twoheadrightarrow C$ for any component C of $G-H_j$ for $i\leqq j\leqq l$ and that there is a $(G, K_{1,2})$-good coloring of D such that no blue edge is incident to the specified vertex v. This would imply that $(G, K_{1,2})$ is Ramsey-infinite. If this is true we will say that $\mathcal{R}(G, K_{1,2})$ is infinite by a $(\bigvee_{i=1}^{l} H_i, D)$-construction.

Throughout the remainder of this section G will denote a connected graph with blocks $B_1, B_2, ..., B_l$, such that each B_i has at least 3 vertices.

As mentioned above, the graph D attached at each vertex of a graph F_i generally satisfies $D\twoheadrightarrow C$ for any component C of $G-B_j$ $(1\leqq j\leqq l)$. There is one special case where the attached graph does not "double arrow". This special case is covered by the conditions given in Lemma 9 (ii).

LEMMA 9. (i) *If* $r(G-B_i, K_{1,2})\leqq r(G, K_{1,2})-2$ *for each endblock* B_i *of* G, *then* $(G, K_{1,2})$ *is Ramsey-infinite.*

(ii) *Let* $r(G, K_{1,2})=|V(G)|$. *For any component* C *of* $G-B_j$ $(1\leqq j\leqq l)$ *let* $u=u(B_j, C)$ *be the vertex common to both* C *and* $G-B_j$. *If each good coloring of* $K_{r(G,K_{1,2})-1}$ *has a copy of* C *with vertex* u *at any of its vertices, then* $(G, K_{1,2})$ *is Ramsey-infinite.*

PROOF. Let $t=r(G, K_{1,2})$. If C is any component of $G-B_j$ for an arbitrary block B_j, then C is a subgraph of $G-B_i$ for some endblock B_i. Therefore $r(C, K_{1,2})\leqq t-2$ if $r(G-B_i, K_{1,2})\leqq t-2$ for each endblock B_i.

If $t=|V(G)|$ let $D=K_{t-1}$ and v be any vertex of D. If $t>|V(G)|$ then let $D=K_{t-1}-e$ for some edge e and let v be an endvertex of e. If (i) is satisfied then Lemma 8 implies $D\twoheadrightarrow(C, K_{1,2})$ for any component C of $G-B_j$ for any j. The corresponding hypothesis is built into (ii). Also, there clearly is a $(G, K_{1,2})$-good coloring of D with no blue edge incident to v. Therefore in both cases $(G, K_{1,2})$ is Ramsey-infinite by a $(\bigvee_{i=1}^{l} B_i, D)$-construction.

Failure of the conditions of Lemma 9 to hold places severe restrictions on the structure of G as the following lemma indicates.

LEMMA 10. *Assume the conditions of Lemma 9 are not satisfied.*

(i) *If* $r(G, K_{1,2})$ *is odd then* G *has a block* B *such that* $B \cong K_s$ *with* $s > |V(G)| - |V(B)|$.

(ii) *If* $r(G, K_{1,2})$ *is even then* G *has a block* B *such that* $B \cong K_s$ *with* $s \geqq |V(G)| - |V(B)|$.

PROOF of (i). Let B_1 be an endblock of G such that $r(G-B_1, K_{1,2}) \geqq r(G, K_{1,2})-1$. Let b be the number of vertices in B_1 not in $G-B_1$. Theorem 7 implies

$$2(|V(G)|-b)-2\beta_1(\overline{G-B_1})-1 \geqq 2|V(G)|-2\beta_1(\bar{G})-1-1.$$

This can only occur if $\beta_1(\bar{G}) \geqq \beta_1(\overline{G-B_1})+b$. Clearly $\beta_1(\bar{G}) \leqq \beta_1(\overline{G-B_1})+b$, so we have equality.

Let M be a maximal matching in $\bar{G}$ and M' be the matching of $G-B_1$ induced by M. Results in the previous paragraph imply that M' is a maximal matching in $\overline{G-B_1}$ and that there are b edges in M with one endvertex in B_1 and the other endvertex in $G-B_1$. Since M is not a perfect matching for $\bar{G}$, there is a vertex u of $\overline{G-B_1}$ which is not incident to M, hence not incident to M'. Let R be the vertices of $G-B_1$ which are not on any edge of M'. Thus R includes those b vertices of $\overline{G-B_1}$ which were matched via M with vertices of $\bar{B}_1$ together with the vertex u. Hence $|R| \geqq b+1 \geqq 3$. The maximality of M' implies that the vertices of R form a complete graph in $G-B_1$. Let B be the block of G containing R.

All of the edges of M' must have one endvertex in $V(\bar{B})$. Assume that this is not true. Then there is an edge xy in M' with $x, y \notin V(\bar{B})$. The edge xy can be replaced by two edges r_1x and r_2y for appropriate r_1 and r_2 in R, yielding a matching of $\overline{G-B_1}$ larger than M', a contradiction. Also, any vertex y not in $\bar{B}$ is incident to some edge of M', for otherwise M' can be enlarged by adding the edge yr for an appropriate $r \in R$.

Let R' be the vertices of $\bar{B}$ which are incident to some edge of M' with one endvertex not in $\bar{B}$. Thus $|R'|$ is the same as the number of vertices of $G-B_1$ not in B. Hence $s=|R \cup R'| > |V(G)|-|V(B)|$. Any two vertices of R' can be interchanged with an appropriate pair of vertices of R, still leaving a matching of $\overline{G-B_1}$ the same size as M'. Therefore, since the vertices of R form a complete graph, the vertices of $R \cup R'$ also form a complete graph in B. This completes the proof of (i).

PROOF of (ii). Let $2t=r(G, K_{1,2})=|V(G)|$. For any i, any component C of $G-B_i$ has at most $2t-2$ vertices and hence $r(C, K_{1,2}) \leqq 2t-1$. We first consider the case where Lemma 9 (ii) fails to hold for some endblock, which we will denote by B_1. Therefore $r(G-B_1, K_{1,2})=2t-1$; and there is a good coloring of K_{2t-1} which does not have a red copy of $G-B_1$ appropriately placed on a vertex v of K_{2t-1}. Let u be the vertex of $G-B_1$ which is also in B_1. If M is a fixed maximal matching in $\overline{G-B_1}$ and w is a vertex not on M, then $r(G-B_1, K_{1,2}) > |V(G-B_1)|$. Moreover the discussion preceding Lemma 8 implies that any good coloring of K_{2t-1} will contain a red copy of $G-B_1$ with the vertex w at any given vertex of the K_{2t-1}. Therefore, in this case any maximal matching of $\overline{G-B_1}$ must contain u. In addition the good coloring of K_{2t-1} which does not contain an appropriate red $\overline{G-B_1}$ is the one with a maximal blue matching avoiding the specified vertex v.

Let B, b, R and R' be defined just as in (i) of this lemma. In this case $|R\cup\{u\}|\geqq 3$ where in (i) we knew that $|R|\geqq 3$. Using the fact that u must be in every maximal matching of $\overline{G-B_1}$ as well as the fact that M' is a maximal matching of $\overline{G-B_1}$, one can mimic the proof of (i) of this lemma to obtain that $|R|\geqq b$, $s=|R\cup R'|\geqq \geqq |V(G)|-|V(B)|$, and that the vertices of $R\cup R'$ form a complete subgraph K_s of the block B.

The case where the conditions of Lemma 9 (ii) fail for some block B_j which is not an endblock is similar. In this case the proof parallels exactly the argument given for the endblock B_1, except that instead of working with $G-B_1$ and B_1 one must work with C and $G-C$ where C is a component of $G-B_j$ for which the conditions of Lemma 9 (ii) fail to hold. This completes the proof of Lemma 10.

A special graph with two blocks occurs which has to be dealt with differently. This graph is obtained by attaching a K_n onto a K_n by identifying a vertex from each graph. This gives a graph with $2n-1$ vertices which we will denote by $K_n\cdot K_n$.

LEMMA 11. *For* $s\geqq 3$, *the pair* $(K_s\cdot K_s, K_{1,2})$ *is Ramsey-infinite.*

PROOF. Consider the graph $F=K_{2s-1}-e$ for some edge e. Note that $F\nrightarrow(K_s\cdot K_s, K_{1,2})$ since F can be colored with no blue $K_{1,2}$ such that every vertex has red degree at most $2s-3$. In fact, this is the only good coloring of F. If v is a vertex of F not incident to e, then there must be a blue edge incident to v in any good coloring of F. Also, in any good coloring of F there is a red K_s containing the vertex v.

Let u_1 and v_1 be two vertices of a copy of K_s. For each vertex w of K_s other than u_1 and v_1 attach a copy of F by identifying w and v. This graph can be good colored but in any such coloring the edge u_1v_1 must be colored blue. Therefore if we attach an edge γ_1 to v_1 we have a $(K_s\cdot K_s, K_{1,2}, \gamma_1)$-determiner. Let D_1 be a subgraph of this graph which is a minimal $(K_s\cdot K_s, K_{1,2}, \gamma_1)$-determiner. Also note that there is a good coloring of D_1 such that there is no red K_s containing v_1.

Consider a copy of $K_s\cdot K_s$ and select an edge f which is not incident to the cut-vertex of $K_s\cdot K_s$. Let v_2 be an endvertex of f. To each vertex w of $K_s\cdot K_s$ not incident to f attach a copy of $D_1-\gamma_1$ by identifying v_1 and w. This graph can be good colored but in any such coloring the edge f must be colored blue. Thus if we attach an edge γ_2 onto v_2 we have a $(K_s\cdot K_s, K_{1,2}, \gamma_2)$-determiner. Let D_2 be a subgraph which is a minimal $(K_s\cdot K_s, K_{1,2}, \gamma_2)$-determiner. Clearly D_2 has more edges than D_1 since at least one copy of $D_1-\gamma_1$ must remain intact in D_2. Repetition of this gives an infinite sequence $\{D_i: i\geqq 1\}$ of $(K_s\cdot K_s, K_{1,2}, \gamma_i)$-determiners which are minimal.

If one attaches a copy of $D_i-\gamma_i$ to each vertex u of a $K_s\cdot K_s$ by identifying u and v_i, one obtains a graph F_i such that $F_i\rightarrow(K_s\cdot K_s, K_{1,2})$. Also, there is a subgraph F_i' of F_i such that $F_i\in\mathcal{R}(K_s\cdot K_s, K_{1,2})$ and F_i' has at least as many vertices as D_i. Thus $\mathcal{R}(K_s\cdot K_s, K_{1,2})$ is infinite.

PROOF of Theorem 6. If G is 2-connected then Theorem 1 implies that $(G, K_{1,2})$ is Ramsey-infinite. Thus G has at least two blocks. By Lemma 9 and 10 we can assume that G has a block $B\geqq K_s$ with $s>|V(G)|-|V(B)|$ if $r(G, K_{1,2})$ is odd and $s\geqq \geqq|V(G)|-|V(B)|$ if $r(G, K_{1,2})$ is even. Let $r=r(B, K_{1,2})$. Hence $r\geqq 2s-1$. If $r>|V(B)|$ let $D=K_{r-1}-e$ for some edge e of K_{r-1} and let v be a vertex of K_{r-1} incident to e. If $r=|V(B)|$ let $D=K_{r-1}$ and v be any vertex of K_{r-1}. If $r(C, K_{1,2})\leqq \leqq r-2$ for every component C of $G-B$, then just as before, $\mathcal{R}(G, K_{1,2})$ is infinite by a (B, D)-construction. We can thus assume that there is a component C of $G-B$ such that $r(C, K_{1,2}) \geqq r-1 \geqq 2s-2$. Since $|V(G)|-|V(B)| \leqq s$ in both cases, $G-B$ has precisely one such component C, which must be a block. In our previous notation $C=B_1$ and G has precisely two blocks B and B_1.

First consider the case when $r(G, K_{1,2})$ is odd. In this case B_1 has at most s vertices. Hence B_1 is isomorphic to K_s and $r(B_1, K_{1,2})=2s-1$. Thus $r=2s-1$ or $r=2s$. If $r=2s$ then $\mathscr{R}(G, K_{1,2})$ is infinite by a (B, K_{2s-1})-construction, so we assume that $r(B, K_{1,2})=2s-1$. Note that $B=G-B_1$ so $r(B, K_{1,2}) \geqq r(G, K_{1,2})-1$. Hence $r(G, K_{1,2})=2s-1$, also. This implies B has precisely s vertices and $G \cong K_s \cdot K_s$ for $s \geqq 3$. The proof of this case is complete by Lemma 11.

We now consider the case when $r(G, K_{1,2})$ is even. Let $2t=r(G, K_{1,2})$. Since $B=G-B_1$, $r(B, K_{1,2}) \geqq 2t-1$. Thus $r=r(B, K_{1,2})=2t-1$ since B does not have $2t$ vertices. Also $r(B_1, K_{1,2}) \geqq r-1 \geqq 2t-1$ and thus $r(B_1, K_{1,2})=2t-1$. With no loss of generality we can assume that $|V(B)| \geqq |V(B_1)|$. Thus B_1 has t vertices and $B_1 \cong K_t$. The block B has $t+1$ vertices and thus is isomorphic to a K_{t+1} with some edges incident to the cutvertex u of G deleted. Let $D=K_{2t-1}$. In this case $\mathscr{R}(G, K_{1,2})$ is infinite by a (B, G)-construction. This completes the proof of the theorem.

One question of interest that this paper leaves unanswered is the following. Is $\mathscr{R}(G, K_{1,2})$ Ramsey-infinite for any connected graph G which is not an edge? This would of course settle the conjecture mentioned in the introduction.

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