ESTIMATES FOR SUMS INVOLVING THE LARGEST PRIME FACTOR OF AN INTEGER AND CERTAIN RELATED ADDITIVE FUNCTIONS

by

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Abstract

Let $P(n)$ denote the largest prime factor of an integer $n \geqq 2$, and let

$$\beta(n) = \sum_{p|n} p, \quad B(n) = \sum_{p^a \| n} ap, \quad B_1(n) = \sum_{p^a \| n} p^a.$$

Asymptotic formulas for sums of quotients of these functions are derived. The estimates are made to depend on $\psi(x, y)$, the number of integers not exceeding x, all of whose prime factors do not exceed y.

1. Introduction

Let $P(n)$ denote the largest prime factor of an integer $n \geqq 2$, and let us define additive functions $\beta(n)$, $B(n)$ and $B_1(n)$ as

$$\beta(n) = \sum_{p|n} p, \quad B(n) = \sum_{p^a \| n} ap, \quad B_1(n) = \sum_{p^a \| n} p^a,$$

where $p^a \| n$ means that p^a divides n, but p^{a+1} does not. The importance of the above functions comes from the fact that they represent partitions of n into sums of primes or prime powers which divide n, and recently several results concerning these functions have appeared (see [1], [2], [4], [5] and [6]). Thus it was proved in [2], eq. (5.33), that one has

$$\tag{1.1} \begin{aligned} \sum_{2 \leqq n \leqq x} \beta(n)/P(n) &= x + O(x \log\log x/\log x), \\ \sum_{2 \leqq n \leqq x} B(n)/P(n) &= x + O(x \log\log x/\log x), \end{aligned}$$

and [4] contains a proof of

$$\tag{1.2} \sum_{2 \leqq n \leqq x} B(n)/\beta(n) = x + O\big(x \exp(-C(\log x \cdot \log\log x)^{1/2})\big), \ C > 0,$$

and the same asymptotic formula holds for sums of $\beta(n)/B(n)$. Sharp formulas for sums of reciprocals of $P(n)$, $\beta(n)$ and $B(n)$ are obtained in [6], where it was shown

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that

$$(1.3)\qquad \sum_{2\le n\le x} 1/P(n) = x\exp\left\{(-2\log x\cdot\log\log x)^{1/2}+O\left((\log x\cdot\log\log\log x)^{1/2}\right)\right\},$$

and the same formula holds for sums of $1/\beta(n)$ and $1/B(n)$.

Sums of quotients like those appearing in (1.1) or (1.2) provide us with information about the degree of compositeness of n, and it turns out (see [1]) that it is $P(n)$ which dominates the values of $\beta(n)$ and $B(n)$. Our Lemma 4 shows that the same is also true for $B_1(n)$. The main goal of our paper is to give estimates for the twelve distinct sums of the type $\sum_{2\le n\le x} f(n)/g(n)$ when $f\ne g$ and

$$f, g\in\left\{P(n), \beta(n), B(n), B_1(n)\right\}.$$

Estimates for some of these sums are already provided by (1.1) and (1.2) and some follow easily hencefrom, but a number of these estimates are non-trivial and will be given in theorems of this paper.

The notation used throughout the text is standard: p and q are always primes; m, n, r, s are natural numbers; $\psi(x, y)=\sum_{n\le x,\, P(n)\le y} 1$; $f=O(g)$ and $f\ll g$ both mean $|f(x)|<Cg(x)$ for some $C>0$ and $x\ge x_0$; $C, C_1, \ldots$ denote positive absolute constants, not necessarily the same ones; ε denotes a positive number which may be chosen arbitrarily small. The notation $\beta(n)=\sum_{p|n} p$ and $B(n)=\sum_{p^a\| n} ap$ stresses the analogy between the relation of "large" additive functions $\beta(n)$ and $B(n)$ and the relation between the well-known "small" additive functions $\omega(n)=\sum_{p|n} 1$ and $\Omega(n)=\sum_{p^a\| n} a$. Moreover, $\beta(n)$ may be regarded as the additive analogue of the multiplicative function $\alpha(n)=\prod_{p|n} p$, which represents the greatest square-free divisor of n.

2. Statement of results

THEOREM 1.

$$(2.1)\qquad \sum_{2\le n\le x} P(n)/B_1(n) = x+O(x\log\log x/\log x),$$

$$(2.2)\qquad \sum_{2\le n\le x} \beta(n)/B_1(n) = x+O(x\log\log x/\log x),$$

$$(2.3)\qquad \sum_{2\le n\le x} B(n)/B_1(n) = x+O(x\log\log x/\log x).$$

Let now γ denote Euler's constant, so that $e^\gamma=1.78107\ldots$, and let $D>1$ denote an absolute constant whose genesis will be precisely given in § 4. We have then

THEOREM 2.

$$(2.4)\qquad \sum_{2\le n\le x} B_1(n)/P(n) = e^\gamma x\log\log x+O(x).$$

THEOREM 3.

$$\sum_{2\leq n\leq x} B_1(n)/B(n) = Dx+O(x\log^{-1/3}x). \tag{2.5}$$

THEOREM 4.

$$\sum_{2\leq n\leq x} B_1(n)/\beta(n) = e^{\gamma}x\log\log x+O(x). \tag{2.6}$$

Therefore it still remains to estimate two of the twelve sums $\sum_{2\leq n\leq x} f(n)/g(n)$ that are mentioned in the Introduction. These are

$$\sum_{2\leq n\leq x} P(n)/\beta(n) = x+O(x\log\log x/\log x), \tag{2.7}$$

and

$$\sum_{2\leq n\leq x} P(n)/B(n) = x+O(x\log\log x/\log x). \tag{2.8}$$

To obtain (2.7) note that the sum in question is trivially $<x$ and from (1.1) and the Cauchy—Schwarz inequality we infer

$$x+O(1) = \sum_{2\leq n\leq x} 1 \leq \Big(\sum_{2\leq n\leq x} P(n)/\beta(n)\Big)^{1/2}\Big(\sum_{2\leq n\leq x} \beta(n)/P(n)\Big)^{1/2} \leq$$
$$\leq \Big(\sum_{2\leq n\leq x} P(n)/\beta(n)\Big)^{1/2}\big(x+O(x\log\log x/\log x)\big)^{1/2},$$

whence

$$\sum_{2\leq n\leq x} P(n)/\beta(n) \geq x+O(x\log\log x/\log x).$$

This gives (2.7), and (2.8) is proved analogously. The error term in (1.1) (and consequently in (2.7) and (2.8)) can be improved to $O(x/\log x)$, which will be shown at the end of § 4.

3. The necessary lemmas

We begin the preparation for proofs of our theorems by proving several lemmas of which some seem to be interesting in themselves. Our proofs will be made to depend on estimates for $\psi(x, y)$, the number of positive integers not exceeding x, all of whose prime factors do not exceed y. From the wealth of results concerning $\psi(x, y)$ we shall need several estimates whose proofs are to be found in DE BRUIJN [3] (to see that (3.2) holds for $\log y>\log^{5/8+\varepsilon}x$ one has to use the strongest form of the prime number theorem), and apart from Lemma 1 below our proofs are self-contained.

LEMMA 1. *Let* $y\leq x$ *and* $u=\log x/\log y$. *If* $3<u<4y^{1/2}/\log y$, *then there exist constants* $c_1, c_2>0$ *such that*

$$\psi(x, y) < c_1 x\log^2 y\cdot\exp\big(-u(\log u+\log\log u-c_2)\big). \tag{3.1}$$

If $\varrho(u)$ *is defined as* $\varrho(u)=1$ *for* $0\leq u\leq 1$, $u\varrho'(u)=-\varrho(u-1)$ *for* $u\geq 1$, *then for* $\log y>\log^{5/8+\varepsilon}x$ *we have*

$$\psi(x, y) = x\varrho(u)\big(1+O(\log\log x/\log y)\big). \tag{3.2}$$

If $s \geqq 0$, $x \geqq y^s$, *then*

$$\psi(x, y) \ll x/s!, \tag{3.3}$$

and for $2 \leqq y \leqq x$ *and some* $C > 0$

$$\psi(x, y) \ll x \exp(-C \log x/\log y). \tag{3.4}$$

LEMMA 2. *Let* $S(x)$ *denote the number of integers* $n \leqq x$ *such that* $P^2(n)|n$. *Then for some* $C > 0$

$$S(x) \ll x \exp\left(-C(\log x \cdot \log\log x)^{1/2}\right). \tag{3.5}$$

PROOF. We have

$$S(x) = \sum_{p^2 m \leqq x,\, P(m) \leqq p} 1 = \sum_{p^2 \leqq x} \psi(xp^{-2}, p) = S_1 + S_2, \tag{3.6}$$

where in S_1 we have $p > \exp\left((\log x \cdot \log\log x)^{1/2}\right) = z$, and in S_2 we have $p \leqq z$. We have

$$S_1 \ll \sum_{p > z} \sum_{m \leqq xp^{-2}} 1 \ll x \sum_{p > z} p^{-2} \ll x \exp\left(-(\log x \cdot \log\log x)^{1/2}\right). \tag{3.7}$$

For S_2 we use (3.1) to obtain with $u_p = (\log xp^{-2})/\log p$, $C_1 > 0$

$$\begin{aligned} S_2 = \sum_{p \leqq z} \psi(xp^{-2}, p) &\ll x \log^2 x \sum_{p \leqq z} p^{-2} \exp(-C_1 u_p \log u_p) \ll \\ &\ll x \sum_{p \leqq z} p^{-2} \exp\left(-C(\log x \cdot \log\log x)^{1/2}\right) \ll x \exp\left(-C(\log x \cdot \log\log x)^{1/2}\right), \end{aligned} \tag{3.8}$$

since $u_p \geqq (\log xz^{-2})/\log z \gg (\log x/\log\log x)^{1/2}$ for $p \leqq z$.

LEMMA 3. *Let* $T(x)$ *denote the number of integers* $n \leqq x$ *such that there exists* $q^a|n$, $q < P(n)$, q *prime, for which* $q^a > P(n)\log^{-A} x$, *where* $A > 0$ *is arbitrary but fixed. Then*

$$T(x) \ll (x \log\log x)/\log x. \tag{3.9}$$

PROOF. With $y = (\log x \cdot \log\log x)^{1/2}$ we have $\psi(x, \exp y) \ll x \exp(-Cy)$ by (3.1), so we have only to consider those $n \leqq x$ for which $P(n) > \exp y$, $P(n) \| n$ (this last by Lemma 2). Therefore

$$T(x) \ll x \exp(-Cy) + T_1(x) + T_2(x), \tag{3.10}$$

where

$$T_1(x) = \sum_{mqp \leqq x,\, p \log^{-A} x < q \leqq p,\, P(m) < p} 1,$$

and if n is counted by $T_2(x)$, then there is a prime power $q^a|n$, $a \geqq 2$ such that $q^a > P(n)\log^{-A} x > \exp(y/2)$. Therefore

$$T_2(x) \ll x \sum_{n,\, a \geqq 2,\, n^a \geqq \exp(y/2)} n^{-a} \ll x \sum_{n \geqq \exp(y/4)} n^{-2} \ll x/\log x. \tag{3.11}$$

To estimate $T_1(x)$ we use (3.4) and

$$\sum_{A<p\leqq B} 1/p = \log(\log B/\log A) + O(1/\log A), \quad A \leqq B \tag{3.12}$$

so that

$$\begin{aligned} T_1(x) &\leqq \sum_{p\leqq x} \sum_{p\log^{-A}x<q\leqq p} \psi(x/pq, p) \leqq \\ &\leqq x \sum_{p\leqq x} \frac{1}{p} \sum_{p\log^{-A}x<q\leqq p} \frac{1}{q} \exp\left(-C\frac{\log x/pq}{\log p}\right) \ll \\ &\ll x \sum_{p\leqq x} \frac{1}{p} \exp(-C\log x/\log p) \cdot \frac{\log\log x}{\log p} \ll \\ &\ll x \int_2^x \frac{\log\log x}{t\log t} \exp(-C\log x/\log t)\, d\pi(t) \ll \end{aligned} \tag{3.13}$$

$$\ll x\log\log x \int_2^x \exp\left(-C\frac{\log x}{\log t}\right) \cdot t^{-1}\log^{-2} t\, dt \ll x\frac{\log\log x}{\log x} \int_1^{\log x/\log 2} e^{-Cu}\, du \ll$$

$$\ll x\frac{\log\log x}{\log x},$$

after substituting $u = \log x/\log t$. The lemma follows then from (3.10), (3.11), (3.13).

Lemma 4. *Let* $U(x)$ *denote the number of integers* $n \leqq x$ *for which*

$$B_1(n) = P(n)\big(1 + O(\log\log n/\log n)\big)$$

does not hold. Then

$$U(x) \ll (x\log\log x)/\log x. \tag{3.14}$$

Proof. Let for brevity $g(x) = \log\log x/\log x$. From Lemma 2 and Lemma 3 it is seen that for $x + O\big(xg(x)\big)$ integers $n \leqq x$ we have $P(n)\| n$ and $q^a \leqq P(n)\log^{-3} x$ if $q^a | n$, $q < P(n)$, so that for these n's

$$\begin{aligned} B_1(n) = P(n) + \sum_{q^a\|n,\, q<P(n)} q^a &\leqq P(n) + \omega(n)P(n)\log^{-3} x \leqq \\ &\leqq P(n)\big(1 + O(\log^{-2} x)\big) \leqq P(n)\big(1 + O(g(n))\big). \end{aligned}$$

Since $B_1(n) \geqq P(n)$ we have $B_1(n) = P(n)\big(1 + O(g(n))\big)$ for $x + O\big(xg(x)\big)$ integers $n \leqq x$, hence the lemma.

Lemma 5. *The assertion of Lemma 4 remains true when* $B_1(n)$ *is replaced by* $\beta(n)$ *or* $B(n)$.

Proof. Follows from the proof of Lemma 4 and

$$P(n) \leqq \beta(n) = \sum_{p|n} p \leqq \sum_{p^a\|n} ap = B(n) \leqq \sum_{p^a\|n} p^a = B_1(n).$$

LEMMA 6. *If $f(n)$ is any additive function and $y \leqq x$, then*

$$\sum_{n \leqq x,\, P(n) \leqq y} f(n) = \sum_{p^a \leqq x,\, p \leqq y} \big(f(p^a) - f(p^{a-1})\big)\psi(xp^{-a}, y). \tag{3.15}$$

PROOF.

$$\sum_{n \leqq x,\, P(n) \leqq y} f(n) = \sum_{n \leqq x,\, P(n) \leqq y} \sum_{p^a \| n} f(p^a) = \sum_{p^a m \leqq x,\, (p,\, m)=1,\, p \leqq y,\, P(m) \leqq y} f(p^a) =$$

$$= \sum_{p^a \leqq x,\, p \leqq y} \big(f(p^a) - f(p^{a-1})\big) \sum_{m \leqq xp^{-a},\, P(m) \leqq y} 1 = \sum_{p^a \leqq x,\, p \leqq y} \big(f(p^a) - f(p^{a-1})\big)\psi(xp^{-a}, y).$$

4. Proofs of theorems

Theorem 1 (and also (1.1)) follows easily from Lemma 4 and Lemma 5. To prove (2.1) note that $P(n) \leqq B_1(n)$, so that using Lemma 4 one obtains

$$\sum_{2 \leqq n \leqq x} P(n)/B_1(n) = O\big(U(x)\big) + \sum_{2 \leqq n \leqq x} \frac{P(n)}{P(n)\big(1 + O(\log\log n/\log n)\big)} =$$

$$O(x \log\log x/\log x) + \sum_{2 \leqq n \leqq x} \big(1 + O(\log\log n/\log n)\big) = x + O(x \log\log x/\log x),$$

and similarly one derives (2.2) and (2.3).

The proofs of the remaining theorems are more difficult and will be carried out in three steps. The first step consists in proving

$$\sum_{2 \leqq n \leqq x} B_1(n)/P(n) = \sum_{p^r m \leqq x,\, P(m) < p} p^{r-1} + O(x), \tag{4.1}$$

$$\sum_{2 \leqq n \leqq x} B_1(n)/B(n) = \sum_{p^r m \leqq x,\, P(m) < p} r^{-1} p^{r-1} + O(x \log^{-1/3} x), \tag{4.2}$$

$$\sum_{2 \leqq n \leqq x} B_1(n)/\beta(n) = \sum_{p^r m \leqq x,\, P(m) < p} p^{r-1} + O(x). \tag{4.3}$$

The sums on the right-hand sides of the above formulas will be transformed into sums involving the function $\varrho(u)$ of Lemma 1, and the second step of the proof will be to show that

$$\sum_{p^r m \leqq x,\, P(m) < p} p^{r-1} = x \sum_{p \leqq x} p^{-1} \sum_{s=0}^{\infty} \varrho\left(\frac{\log x}{\log p} - \left[\frac{\log x}{\log p}\right] + s\right) + O(x), \tag{4.4}$$

$$\sum_{p^r m \leqq x,\, P(m) < p} r^{-1} p^{r-1} = x \sum_{p \leqq x} \frac{1}{p} \sum_{0 \leqq s \leqq \log x/\log p - 1} \frac{\varrho\left(\frac{\log x}{\log p} - \left[\frac{\log x}{\log p}\right] + s\right)}{[\log x/\log p] - s} + \tag{4.5}$$

$$+ O(x \log^{-1/3} x).$$

Finally it remains to simplify the expressions containing $\varrho(u)$, and our results will then follow from

$$(4.6)\qquad \sum_{p\leqq x} p^{-1}\sum_{s=0}^{\infty}\varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right)=e^{\gamma}\log\log x+O(1),$$

$$(4.7)\qquad \sum_{p\leqq x} p^{-1}\sum_{0\leqq s\leqq \log x/\log p-1}\frac{\varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right)}{[\log x/\log p]-s}=D+O(1/\log x),$$

where

$$(4.8)\qquad D=\int_1^{\infty}u^{-1}\sum_{0\leqq s\leqq u-1}\frac{\varrho(u-[u]+s)}{[u]-s}\,du>1.$$

After sketching this plan of our proofs, we begin with the proof of (4.1). By additivity of $B_1(n)$ we have

$$(4.9)\qquad \sum_{2\leqq n\leqq x}B_1(n)/P(n)=\sum_{p^r m\leqq x,\ P(m)<p}p^{r-1}+\sum_{p^r m\leqq x,\ P(m)<p}B_1(m)/p,$$

and (4.1) follows from

$$(4.10)\qquad S=\sum_{p^r m\leqq x,\ P(m)<p}B_1(m)/p\ll x.$$

Noting that $B_1(p^a)=p^a$ and using Lemma 6 we obtain

$$(4.11)\ S=\sum_{p^r\leqq x}p^{-1}\sum_{m\leqq x/p^r,\ P(m)<p}B_1(m)\ll\sum_{p^r\leqq x}p^{-1}\sum_{q^s\leqq x/p^r,\ q\leqq p}\psi(xp^{-r}q^{-s},p)q^s.$$

For some integer $k\geqq 1$ we have

$$(4.12)\qquad x/p^{k+r}<q^s\leqq x/p^{k+r-1},$$

which implies

$$(4.13)\qquad \log x/p^{k+r}<s\log q\leqq \log x/p^{k+r}+\log p,$$

so that there are at most

$$(4.14)\qquad \log p/\log q+1\leqq 2\log p/\log q$$

values of s for which (4.12) holds. Moreover, if (4.12) holds, then we can use (3.3) of Lemma 1 to obtain

$$(4.15)\qquad \psi(xp^{-r}q^{-s},p)\ll x/\big(p^r q^s(k-1)!\big),$$

and therefore

$$S \ll \sum_{p^r \leqq x} \sum_{q^s \leqq x/p^r,\, q \leqq p} \sum_{k \leqq 1} \frac{x}{p^r q^s (k-1)!} \frac{q^s}{p} \ll \sum_{p^r \leqq x} x p^{-r-1} \sum_{q^s \leqq x/p^r,\, q \leqq p} 1 \ll$$

$$\ll \sum_{p^r \leqq x} x p^{-r-1} \sum_{q \leqq p} \log p / \log q \ll x \sum_{p^r \leqq x} p^{-r-1} \log p \int_2^p \frac{d\pi(t)}{\log t} \ll \tag{4.16}$$

$$\ll x \sum_{p^r \leqq x} p^{-r-1} p \log p \cdot \log^{-2} p \ll x \sum_{p^r \leqq x} 1/\log p \sum_{r=1}^{\infty} p^{-r} \ll x \sum_{p \leqq x} 1/(p \log p) \ll x,$$

since the last sum converges. This proves (4.10), and therefore (4.1).

We turn now to the proof of (4.2). By additivity of $B(n)$ and $B_1(n)$ we have

$$\sum_{2 \leqq n \leqq x} B_1(n)/B(n) = \tag{4.17}$$
$$= \sum_{p^r m \leqq x,\, P(m) < p} p^r/(rp + B(m)) + \sum_{p^r m \leqq x,\, P(m) < p} B_1(m)/(rp + B(m)).$$

First we show

$$S_1 = \sum_{p^r m \leqq x,\, P(m) < p} B_1(m)/(rp + B(m)) \ll x \log^{-1/3} x. \tag{4.18}$$

Let now for brevity $w = \log^{1/3} x$ throughout the proof of (4.2). In the above sum we may suppose $p \leqq \exp w$, for otherwise following the reasoning given in (4.16) we obtain

$$\sum_{p^r m \leqq x,\, P(m) < p,\, p > \exp w} B_1(m)/(rp + B(m)) \ll \sum_{p^r m \leqq x,\, P(m) < p,\, p > \exp w} p^{-1} B_1(m) \ll \tag{4.19}$$
$$\ll x \sum_{p > \exp w} 1/(p \log p) \ll x/w = x \log^{-1/3} x,$$

since by the prime number theorem and integration by parts we have, as $y \to \infty$,

$$\sum_{p > y} 1/(p \log p) \ll \int_y^{\infty} t^{-1} \log^{-1} t \, d\pi(t) \ll 1/\log y. \tag{4.20}$$

Next we observe that $1/(rp + B(m)) \leqq \min\,(1/rp,\, 1/B(m))$ and

$$B_1(n)/B(n) = \Big(\sum_{p^a \| n} p^a\Big) \Big/ \Big(\sum_{p^a \| n} ap\Big) \leqq \sum_{p^a \| n} a^{-1} p^{a-1} = f(n), \tag{4.21}$$

so that $f(n)$ is additive. Therefore by Lemma 6

$$\sum_{p^r m \leqq x,\, P(m) < p \leqq \exp w} \frac{B_1(m)}{rp + B(m)} \ll \tag{4.22}$$
$$\ll \sum_{p^r \leqq x,\, p \leqq \exp w} \sum_{q^s \leqq x p^{-r},\, q \leqq p} \min\left(\frac{1}{pr}, \frac{1}{sq}\right) q^s \psi\left(\frac{x}{p^r q^s}, p\right).$$

This is the fundamental inequality in the proof of (4.2). Denoting by Σ the expression on the right-hand side of (4.22) we may write

$$(4.23)\quad \sum \leqq \sum_{r+s\leqq \log x/(2\log p)} + \sum_{r>\log x/(4\log p)} + \sum_{s>\log x/(4\log p)} = \sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3,$$

since if $r+s>\log x/(2\log p)$, then either $r>\log x/(4\log p)$ or $s>\log x/(4\log p)$.

In Σ_1 we have

$$(4.24)\quad \log xp^{-r}q^{-s}/\log p \geqq \log x/\log p-r-s \geqq \log x/(2\log p) \geqq \log x/(2w),$$

whence by (3.4)

$$(4.25)\qquad \psi(xp^{-r}q^{-s}, p) \ll xp^{-r}q^{-s}\exp(-C\log x/w),$$

and since trivially $s\ll\log x$ we obtain then

$$(4.26)\ \sum\nolimits_1 \ll x\exp(-C\log x/\log^{1/3}x)\sum_{p^r\leqq x} p^{-r-1}\log x\sum_{q\leqq p}1 \ll x\exp(-C_1\log^{2/3}x).$$

Now we come to the estimation of Σ_2 and Σ_3 in (4.23). In Σ_2 it is seen that r is large, so that we shall take $\min(1/rp, 1/sq)\leqq 1/rp$, and in Σ_3 we shall take $\min(1/rp, 1/sq)\leqq 1/sq$. In the estimation of Σ_2 and Σ_3 we repeat the reasoning given by (4.12)—(4.16), taking also into account that $p\leqq\exp w$. Since in Σ_2 we have $1/r\ll\log p/\log x$, we obtain

$$(4.27)\quad \begin{aligned}\sum\nolimits_2 &= \sum_{r>\log x/(4\log p)} \ll \frac{x}{\log x}\sum_{p^r\leqq x,\ p\leqq\exp w} p^{-r-1}\log^2 p\sum_{q\leqq p}1/(\log q) \ll \\ &\ll \frac{x}{\log x}\sum_{p\leqq\exp w}\sum_{r\geqq 1}p^{-r} \ll x\log^{-1}x\sum_{p\leqq\exp w}1/p \ll x\log\log x\cdot\log^{-1}x.\end{aligned}$$

Similarly for Σ_3 we obtain analogously as in (4.16)

$$(4.28)\quad \begin{aligned}\sum\nolimits_3 &= \sum_{s>\log x/(4\log p)} \ll \\ &\ll \sum_{p^r\leqq x,\ p\leqq\exp w}\ \sum_{q^s\leqq xp^{-r},\ q\leqq p,\ s>\log x/(4\log p)} s^{-1}q^{s-1}\psi\left(\frac{x}{p^rq^s}, p\right) \ll \\ &\ll x\log^{-1}x\sum_{k\geqq 1}1/(k-1)!\sum_{p^r\leqq x,\ p\leqq\exp w}p^{-r}\log p\sum_{q^s\leqq xp^{-r},\ q\leqq p}1/q \ll \\ &\ll x\log^{-1}x\sum_{p^r\leqq x,\ p\leqq\exp w}p^{-r}\log^2 p\sum_{q\leqq p}1/(q\log q) \ll \\ &\ll x\log^{-1}x\sum_{p\leqq\exp w}\log^2 p\sum_{r\geqq 1}p^{-r} \ll x\log^{-1}x\int_2^{\exp w}t^{-1}\log^2 t\,d\pi(t) \ll \\ &\ll xw^2\log^{-1}x = x\log^{-1/3}x,\end{aligned}$$

since by (4.14) there are $O(\log p/\log q)$ choices for s.

We have shown that (4.18) holds, and to finish the proof of (4.2) it remains to prove

$$(4.29)\qquad \sum_{p^r m\le x,\ P(m)<p} p^r/\big(rp+B(m)\big) = \sum_{p^r m\le x,\ P(m)<p} r^{-1}p^{r-1}+O(x\log^{-1/3}x),$$

which after subtraction follows from

$$(4.30)\qquad S_2 = \sum_{p^r m\le x,\ P(m)<p} r^{-2}p^{r-2}B(m) \ll x\log^{-1/3}x.$$

The sum S_2 is easier to estimate than S_1 of (4.18), and by Lemma 6 we have

$$(4.31)\qquad S_2 \ll \sum_{p^r\le x} r^{-2}p^{r-2} \sum_{q^s\le xp^{-r},\ q\le p} q\psi(xp^{-r}q^{-s}, p),$$

since $B(p^a)-B(p^{a-1})=ap-(a-1)p$. We estimate first the subsum Σ' of the sum on the right-hand side of (4.31) with $p>\exp w$, $w=\log^{1/3}x$. Trivially we have

$$\sum{}' \ll \sum_{p^r\le x,\ p>\exp w} r^{-2}p^{r-2} \sum_{q^s\le xp^{-r},\ q\le p} qxp^{-r}q^{-s} \ll$$

$$(4.32)\quad \ll x \sum_{p^r\le x,\ p>\exp w} r^{-2}p^{-2}\sum_{q\le p}\sum_{j\ge 0} q^{-j} \ll x \sum_{p^r\le x,\ p>\exp w} r^{-2}p^{-1}\log^{-1}p \ll$$

$$\ll x\sum_{r\ge 1} r^{-2} \sum_{p>\exp w} 1/(p\log p) \ll x/w = x\log^{-1/3}x,$$

where we have used again (4.20). In the remaining subsum Σ'' we have $p\le \exp w$, and we split it analogously as in the case of S_1 in (4.18), i.e.

$$(4.33)\quad \sum{}'' = \sum_{r+s\le \log x/(2\log p)} + \sum_{r>\log x/(4\log p)} + \sum_{s>\log x/(4\log p)} = \sum{}''_1+\sum{}''_2+\sum{}''_3.$$

As for Σ_1 of (4.23) we obtain similarly

$$(4.34)\qquad \sum{}''_1 \ll x\exp(-C\log^{2/3}x).$$

Since $r\ll \log x$ we further have

$$\sum{}''_2 = \sum_{r>\log x/(4\log p)} \ll \log^{-2}x \sum_{p^r\le x,\ p\le \exp w} p^{r-2}\log^2 p \sum_{q^s\le xp^{-r},\ q\le p} qxp^{-r}q^{-s} \ll$$

$$(4.35)\qquad \ll x\log^{-2}x \sum_{p^r\le x,\ p\le\exp w} p^{-2}\log^2 p \sum_{q\le p}\sum_{j\ge 0} q^{-j} \ll$$

$$x\log^{-2}x \sum_{p^r\le x,\ p\le \exp w} p^{-1}\log p \ll x\log^{-2}x\cdot\log x\cdot\log(\exp w) = x\log^{-2/3}x.$$

Finally we have

$$(4.36)\ \sum{}''_3 = \sum_{s>\log x/(4\log p)} \ll \sum_{p^r\le x,\ p\le\exp w} r^{-2}p^{r-2} \sum_{q^s\le xp^{-r},\ q\le p,\ s>\log x/(4\log p)} qxp^{-r}q^{-s} \ll$$

$$\ll x\cdot 2^{-\log x/(4w)} \sum_{p^r<x,\ p\le \exp w} p^{-1}r^{-2}\sum_{q\le p}\log x \ll x\exp(-C\log^{2/3}x),$$

since in Σ''_3 we have $q^{-s}\le 2^{-\log x/(4w)}$, $q\le p$ and $s\ll \log x$. Therefore we have proved (4.30), completing the proof of (4.2).

Up to now we have proved (4.1) and (4.2), and now we move to the proof of (4.3), viz.

$$\sum_{2\leqq n\leqq x} B_1(n)/\beta(n) = \sum_{p^r m\leqq x,\ P(m)<p} p^{r-1}+O(x).$$

By additivity of $B_1(n)$ and $\beta(n)$ we have

$$(4.37)\quad \sum_{2\leqq n\leqq x} B_1(n)/\beta(n) = \sum_{p^r m\leqq x,\ P(m)<p} p^r/\big(p+\beta(m)\big)+\sum_{p^r m\leqq x,\ P(m)<p} B_1(m)/\big(p+\beta(m)\big).$$

By (4.10) the last sum above is $O(x)$, and so it remains to show

$$(4.38)\qquad \sum_{p^r m\leqq x,\ P(m)<p}\big(p^{r-1}-p^r/(p+\beta(m))\big) \ll \sum_{p^r m\leqq x,\ P(m)<p} p^{r-2}\beta(m) \ll x.$$

The first inequality in (4.38) is obvious, and for the second we note that $\beta(p^a)-\beta(p^{a-1})=p-1$ for $a=1$ and zero for $a>1$, so that Lemma 6 gives

$$(4.39)\qquad \sum_{p^r m\leqq x,\ P(m)<p} p^{r-2}\beta(m) \ll \sum_{p^r\leqq x} p^{r-2}\sum_{q\leqq xp^{-r},\ q\leqq p} q\psi(xp^{-r}q^{-1},\ p).$$

Using (3.4) we have

$$\psi(xp^{-r}q^{-1},\ p) \ll xp^{-r}q^{-1}\exp\big(-C\log x/(\log p)+Cr\big),$$

which gives then

$$(4.40)\qquad \sum_{p^r m\leqq x,\ P(m)<p} p^{r-2}\beta(m) \ll x\sum_{p^r\leqq x} p^{-2}\sum_{q\leqq p}\exp\left(-C\frac{\log x}{\log p}+Cr\right) \ll$$

$$\ll x\sum_{p\leqq x}(p\log p)^{-1}\exp(-C\log x/\log p)\sum_{r\leqq \log x/\log p}\exp(Cr) \ll x\sum_{p\leqq x} 1/(p\log p) \ll x,$$

as asserted.

Now we shall pass to the proof of (4.4) and (4.5), but first we need to clear a technical point. Since the function $\psi(x, y)$ is defined as

$$\psi(x,\ y) = \sum_{n\leqq x,\ P(n)\leqq y} 1,$$

we remark that

$$(4.41)\qquad \sum_{p^r m\leqq x,\ P(m)<p} p^{r-1} = \sum_{p^r m\leqq x,\ P(m)\leqq p} p^{r-1}+O(x),$$

$$(4.42)\qquad \sum_{p^r m\leqq x,\ P(m)<p} r^{-1}p^{r-1} = \sum_{p^r m\leqq x,\ P(m)\leqq p} r^{-1}p^{r-1}+O(x\log\log x/\log x),$$

which will facilitate later transformations of our sums. To obtain (4.41) note that

$$\sum_{p^r m\leqq x,\ P(m)\leqq p} p^{r-1}-\sum_{p^r m\leqq x,\ P(m)<p} p^{r-1} = \sum_{p^{r+1}n\leqq x,\ P(n)\leqq p} p^{r-1} =$$

$$= \sum_{p^{r+1}\leqq x} p^{r-1}\psi\left(\frac{x}{p^{r+1}},\ p\right),$$

since if $P(m)=p$, then $m=np$ with $P(n)\leqq p$. With (3.4) we obtain

$$\sum_{p^{r+1}\leqq x} p^{r-1}\psi(xp^{-r-1}, p) \ll x \sum_{p^{r+1}\leqq x} p^{-2}\exp(-C\log x/\log p)\exp(Cr) \ll$$
$$\ll x\sum_{p\leqq x} p^{-2}\exp(-C\log x/\log p) \sum_{r\leqq \log x/\log p} \exp(Cr) \ll x\sum_{p\leqq x} p^{-2} \ll x,$$

and the proof of (4.42) is analogous, when we consider separately the cases $r\leqq \log x/(2\log p)$ and $r>\log x/(2\log p)$.

To prove (4.4) note first that we may take $r\geqq 2$ in (4.1), since the sum with $r=1$ is trivially $O(x)$. Suppose now $a=[\log x/\log p]$, or equivalently

$$p^a \leqq x < p^{a+1}. \tag{4.43}$$

Writing $r=a-s$ we have $s=0, 1, \ldots, a-2$, so that s can take at most $O(\log x)$ values. Therefore we can write

$$\sum = \sum_{p^r m\leqq x,\ P(m)\leqq p,\ r\geqq 2} p^{r-1} = \sum_{p^{a-s}\leqq x} p^{a-s-1}\psi(xp^{s-a}, p). \tag{4.44}$$

If $s<\log^{1/2} p$ then

$$u=\log xp^{s-a}/\log p=\log x/\log p-[\log x/\log p]+s<s+1<2\log^{1/2} p, \tag{4.45}$$

so that for $s<\log^{1/2} p$ we may use the asymptotic formula (3.2) to estimate $\psi(xp^{s-a}, p)$ in (4.4). Writing

$$\sum = \sum_{s<\log^{1/2} p} + \sum_{s\geqq \log^{1/2} p} = \sum\nolimits_1 + \sum\nolimits_2, \tag{4.46}$$

we obtain then

$$\begin{aligned}\sum\nolimits_1 = x\sum_{p\leqq x} p^{-1} &\sum_{0\leqq s<\min(\log x/\log p,\ \log^{1/2} p)} \varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right)+\\ &+O\big(x\sum_{p\leqq x} p^{-1}\sum_{0\leqq s<\log^{1/2} p} \log^{-1} p\cdot\log\log(xp^{s-a})\cdot\varrho(s)\big).\end{aligned} \tag{4.47}$$

But in view of (4.43) $\log\log(xp^{s-a})\ll\log\log(s+1)p$, and from the defining properties of $\varrho(u)$ it is seen that $\varrho(u)$ is nonincreasing and that $\varrho(s)\ll 1/s!$, which gives for the error term in (4.47)

$$O(x\sum_{p\leqq x} p^{-1}\log^{-1} p\cdot\log\log p) = O\big(x\int_z^x t^{-1}\log^{-1} t\cdot\log\log t\, d\pi(t)\big)=O(x), \tag{4.48}$$

after integrating by parts and using the prime number theorem. Next we have

$$\begin{aligned}\sum_{p\leqq x} p^{-1} \sum_{s\geqq \log x/\log p} \varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right) &\leqq \sum_{p\leqq x} p^{-1}\sum_{s\geqq \log x/\log p}\varrho(s) \ll\\ \ll \sum_{p\leqq x} p^{-1}\sum_{s\geqq \log x/\log p} 1/s! &\ll \log^{-1} x\sum_{p\leqq x} p^{-1}\log p \ll 1,\end{aligned} \tag{4.49}$$

and similarly

$$\sum_{p\le x} p^{-1} \sum_{s\ge \log^{1/2} p} \varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right) \ll 1. \tag{4.50}$$

This implies

$$\sum\nolimits_1 = x\sum_{p\le x} p^{-1}\sum_{s=0}^{\infty} \varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right)+O(x). \tag{4.51}$$

Using (3.3) we obtain

$$\begin{aligned}\sum\nolimits_2 &= \sum_{s\ge \log^{1/2} p} p^{a-s-1}\psi(xp^{s-a}, p) \ll x \sum_{p^{a-s}\le x} 1/p \sum_{s\ge \log^{1/2} p} 1/s! \ll \\ &\ll x\sum_{p\le x} p^{-1}\exp(-\log^{1/2} p) \ll x\sum_{p} 1/(p\log p) \ll x,\end{aligned} \tag{4.52}$$

which finishes the proof of (4.4) in view of (4.41), i.e.

$$\sum_{p^r m\le x,\ P(m)<p} p^{r-1} = x\sum_{p\le x} p^{-1}\sum_{s=0}^{\infty} \varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right)+O(x).$$

To prove (4.5) we shall need (4.42), and writing again $r=a-s$, $a=[\log x/\log p]$, we obtain

$$\begin{aligned}\sum_{p^r m\le x,\ P(m)\le p} r^{-1}p^{r-1} &= \sum_{p^{a-s}\le x} (a-s)^{-1}p^{a-s-1}\psi(xp^{s-a}, p) = \\ &= \sum_{p\le \exp(\log^{2/3} x)} + \sum_{\exp(\log^{2/3} x)<p\le x} = S_1+S_2.\end{aligned} \tag{4.53}$$

We observe that for $u\ge 1$ we have $[u]\ge u/2$, so that

$$\sum_{0\le s\le a-1} 1/((a-s)s!) \ll 1/a, \tag{4.54}$$

and with $a=[\log x/\log p]$ we obtain using (3.3)

$$S_1 \ll x\log^{-1} x \sum_{p\le \exp(\log^{2/3} x)} p^{-1}\log p \ll x\log^{-1/3} x. \tag{4.55}$$

For S_2 in (4.53) we have $s<\log x/\log p<\log^{1/2} p$, and so as in the proof of (4.4) we may use the asymptotic formula (3.2) to evaluate $\psi(xp^{s-a}, p)$. Therefore

$$\begin{aligned}S_2 = x &\sum_{\exp(\log^{2/3} x)<p\le x} p^{-1} \sum_{0\le s\le \log x/\log p-1} \varrho\left(\frac{\log x}{\log p}-\right. \\ &\left.-\left[\frac{\log x}{\log p}\right]+s\right)\left(\left[\frac{\log x}{\log p}\right]-s\right)^{-1}+ \\ &+O\left(x\sum_{p\le x}\sum_{0\le s\le \log x/\log p-1} \frac{\log\log xp^{s-a}}{p\log p}\varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right)\left(\left[\frac{\log x}{\log p}\right]-s\right)^{-1}\right).\end{aligned} \tag{4.56}$$

As in the proof of (4.4) we have $\log\log(xp^{s-a}) \ll \log\log(s+2)p$, so that the error term above is

$$O\big(x\log^{-1}x \sum_{p\le x} p^{-1}\log\log p\big) = O\big(x(\log\log x)^2/\log x\big). \tag{4.57}$$

Using (4.54) and $\varrho(s)\ll 1/s!$ we have

$$x \sum_{p\le \exp(\log^{2/3}x)} p^{-1}\varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right)\left(\left[\frac{\log x}{\log p}\right]-s\right)^{-1} \ll x\log^{-1/3}x, \tag{4.58}$$

which completes the proof of (4.5).

Now we finally come to the simplification of sums which appear in (4.4) and (4.5) and involve $\varrho(u)$. If one wants only to show that the sums in (4.6) and (4.7) are asymptotically equal to $C\log\log x$ and C, respectively, with ineffective C's and without error terms, this can be obtained by elementary methods using only the continuity of $\varrho(u)$. To prove (4.6), however, we shall use the prime number theorem and Stieltjes integral representation to obtain

$$\begin{aligned}
&\sum_{p\le x} p^{-1}\sum_{s=0}^{\infty}\varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right) = \\
&= \sum_{s=0}^{\infty}\int_{2-0}^{x} t^{-1}\varrho\left(\frac{\log x}{\log t}-\left[\frac{\log x}{\log t}\right]+s\right)d\pi(t) = \\
&= \sum_{s=0}^{\infty}\int_{2}^{x}\varrho\left(\frac{\log x}{\log t}-\left[\frac{\log x}{\log t}\right]+s\right)\frac{dt}{t\log t}+O\Big(\sum_{s=0}^{\infty}e^{-s}\int_{2}^{x}t^{-1}d\big(O(t\log^{-2}t)\big)\Big),
\end{aligned} \tag{4.59}$$

since $\varrho(u)\ll e^{-u}$, which follows from $u\varrho'(u)=-\varrho(u-1)$. The second integral above is $O(1)$, and in the first integral we make the change of variable $u=\log x/\log t$ and obtain

$$\begin{aligned}
&\sum_{p\le x} p^{-1}\sum_{s=0}^{\infty}\varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right) = \\
&= \sum_{s=0}^{\infty}\int_{1}^{\log x/\log 2} u^{-1}\varrho(u-[u]+s)\,du+O(1).
\end{aligned} \tag{4.60}$$

We transform the series on the right-hand side above as follows:

$$\begin{aligned}
&\sum_{s=0}^{\infty}\int_{1}^{\log x/\log 2} u^{-1}\varrho(u-[u]+s)\,du = \\
&= \log\log x+O(1)+\sum_{s=1}^{\infty}\varrho(s)\int_{1}^{\log x/\log 2}u^{-1}du+ \\
&+\sum_{s=1}^{\infty}\int_{1}^{\log x/\log 2}u^{-1}\big(\varrho(u-[u]+s)-\varrho(s)\big)du = \\
&= \sum_{s=0}^{\infty}\varrho(s)\log\log x+O(1)-\sum_{s=1}^{\infty}\sum_{n\le\log x/\log 2}\int_{n}^{n+1}u^{-1}\int_{s}^{u-[u]+s}t^{-1}\varrho(t-1)\,dt\,du,
\end{aligned} \tag{4.61}$$

where we have used $\varrho(u)=1-\int\limits_1^x \varrho(t-1)t^{-1}dt$, which follows from $u\varrho'(u) = -\varrho(u-1)$. Changing the order of integration gives

$$\int\limits_n^{n+1} u^{-1} \int\limits_s^{u-[u]+s} t^{-1}\varrho(t-1)\,dt\,du = \int\limits_s^{s+1} t^{-1}\varrho(t-1) \int\limits_{t+n-s}^{n+1} u^{-1}du =$$

$$= \int\limits_s^{s+1} t^{-1}\varrho(t-1)\log\left(1+\frac{s+1-t}{t+n-s}\right)dt =$$

(4.62)

$$= \int\limits_s^{s+1} t^{-1}\varrho(t-1)\frac{s+1-t}{t+n-s}\,dt+O(e^{-s}n^{-2}) =$$

$$= n^{-1}\int\limits_s^{s+1} \varrho(t-1)t^{-1}(s+1-t)\,dt+O(e^{-s}n^{-2}).$$

Therefore from (4.61) and (4.62) we obtain

$$\sum_{s=1}^{\infty}\sum_{n\le\log x/\log 2}\int\limits_n^{n+1} u^{-1}\int\limits_s^{u-[u]+s} t^{-1}\varrho(t-1)\,dt\,du =$$

$$= \sum_{s=1}^{\infty}\big(\log\log x+O(1)\big)\int\limits_s^{s+1} t^{-1}\varrho(t-1)(s+1-t)\,dt+O(1) =$$

(4.63) $$= \log\log x\Big(\sum_{s=1}^{\infty}(s+1)\int\limits_s^{s+1} t^{-1}\varrho(t-1)\,dt-\sum_{s=1}^{\infty}\int\limits_s^{s+1}\varrho(t-1)\,dt\Big)+O(1) =$$

$$= -\int\limits_0^{\infty}\varrho(t)dt\cdot\log\log x+O(1)+\log\log x\sum_{s=1}^{\infty}(s+1)\big(\varrho(s)-\varrho(s+1)\big) =$$

$$= -\int\limits_0^{\infty}\varrho(t)\,dt\cdot\log\log x+O(1)+\log\log x\big(1+\sum_{s=1}^{\infty}\varrho(s)\big) =$$

$$= \Big(\sum_{s=0}^{\infty}\varrho(s)-\int\limits_0^{\infty}\varrho(t)\,dt\Big)\log\log x+O(1).$$

Putting (4.63) into (4.61) we obtain (4.6), which completes then the proof of Theorem 2 and Theorem 4. It remains yet to prove (4.7), which will give then

Theorem 3. We have similarly as in the proof of (4.6)

$$\sum_{p\le x} p^{-1} \sum_{0\le s\log x/\log p-1} \varrho\left(\frac{\log x}{\log p}-\left[\frac{\log x}{\log p}\right]+s\right)([\log x/\log p]-s)^{-1} =$$

$$\int_{2-0}^{x} t^{-1}d\pi(t) \sum_{0\le s\le\log x\,\log t-1} \varrho\left(\frac{\log x}{\log t}-\left[\frac{\log x}{\log t}\right]+s\right)([\log x/\log t]-s)^{-1} =$$

$$= \int_{2}^{x} t^{-1}\log^{-1} t \sum_{0\le s\le\log x/\log t-1} \varrho\left(\frac{\log x}{\log t}-\left[\frac{\log x}{\log t}\right]+s\right)([\log x/\log t]-s)^{-1}dt+$$

$$+O\Big(\int_{2}^{x}\frac{\log t}{t\log x}d\big(O(t\log^{-2}t)\big)\Big) = \tag{4.64}$$

$$= \int_{2}^{\log x/\log 2} u^{-1}\sum_{0\le s\le u-1}\varrho(u-[u]+s)([u]-s)^{-1}du+O(1/\log x) =$$

$$= \int_{1}^{\infty} u^{-1}\sum_{0\le s\le u-1}\varrho(u-[u]+s)([u]-s)^{-1}du+O\Big(\int_{\log x/\log 2}^{\infty}u^{-2}du\Big)+O(1/\log x) =$$

$$= D+O(1/\log x),$$

where D is given by (4.8). This completes the proof of all of our theorems, when we note (see [2], p. 314) that

$$\int_{0}^{\infty}\varrho(t)dt = e^{\gamma} = 1.78107\ldots. \tag{4.65}$$

In concluding we shall show how the error term in (1.1), (2.7) and (2.8) can be improved to $O(x/\log x)$. We shall only sketch the proof of

$$\sum_{2\le n\le x}\beta(n)/P(n) = x+O(x/\log x), \tag{4.66}$$

since the proof of the analogous formula with $B(n)$ in place of $\beta(n)$ is only technically more complicated. Using Lemma 2 we have

$$\begin{aligned}\sum_{2\le n\le x}\beta(n)/P(n) &= \sum_{2\le n\le x,\ P(n)\| n}\beta(n)/P(n)+O\Big(\sum_{2\le n\le x,\ P^2(n)|n}\omega(n)\Big) =\\ &\sum_{2\le n\le x,\ P(n)\| n}\beta(n)/P(n)+O\big(x\exp(-C(\log x\cdot\log\log x)^{1/2})\big),\end{aligned} \tag{4.67}$$

since $\beta(n)\le P(n)\omega(n)\ll P(n)\log x$ for $n\le x$. Further using Lemma 2 and Lemma 6 we have

$$\begin{aligned}\sum_{2\le n\le x,\ P(n)\| n}\beta(n)/P(n) &= \sum_{pm\le x,\ P(m)<p}(\beta(p)+\beta(m))/p =\\ &= x+O\left(\frac{x}{\log x}\right)+\sum_{p\le x}p^{-1}\sum_{m\le x/p,\ P(m)<p}\beta(m) =\\ &= x+O\left(\frac{x}{\log x}\right)+O\Big(\sum_{p\le x}p^{-1}\sum_{q\le x/p,\ q\le p}^{\bullet} q\psi(x/qp,p)\Big),\end{aligned} \tag{4.68}$$

since $\beta(q^r)=q$. With (3.4) and the prime number theorem we finally have

$$\sum_{p\leq x} p^{-1} \sum_{q\leq p} q\psi(x/qp, p) \ll \sum_{p\leq x} xp^{-2} \sum_{q\leq p} \exp\left(-C(\log x/qp)/\log p\right) \ll$$

$$\ll x \sum_{p\leq x} (p\log p)^{-1} \exp(-C\log x/\log p) =$$

$$(4.69) \qquad = x \int_{2-0}^{x} (t\log t)^{-1} \exp(-C\log x/\log t)\cdot d\pi(t) \ll$$

$$\ll x \int_{2}^{x} t^{-1}\log^{-2} t\cdot \exp(-C\log x/\log t)\,dt =$$

$$= x \int_{1}^{\log x/\log 2} \log^{-1} x\cdot \exp(-Cu)\,du \ll x/\log x,$$

when we substitute $u=\log x/\log t$.

REFERENCES

[1] ALLADI, K. and ERDŐS, P., On an additive arithmetic function, *Pacific J. Math.* **71** (1977), 275—294. *MR* **56** # 5401.

[2] ALLADI, K., and ERDŐS, P., On the asymptotic behavior of large prime factors of integers, *Pacific J. Math.* **82** (1979), 295—315. *MR* **81c**: 10049.

[3] BRUIJN, N. G. DE, On the number of positive integers $\leq x$ and free of prime factors $>y$, *Nederl. Akad. Wetensch. Proc. Ser. A* **54** = *Indag. Math.* **13** (1951), 50—60. *MR* **13**—724; II. *Nederl. Akad. Wetensch. Proc. Ser. A* **69** = *Indag. Math.* **28** (1966), 239—247. *MR* **34** # 5770.

[4] DE KONINCK, J.-M., ERDŐS, P. and IVIĆ, A., Reciprocals of large additive functions, *Canad. Math. Bull.* **24** (1981), 225—231. *Zbl.* **463** 10032.

[5] DE KONINCK, J.-M. and IVIĆ, A., *Topics in arithmetical functions,* Notas de Matemática 72, North-Holland Publishing Co., Amsterdam, 1980. *MR* **82a**: 10047.

[6] IVIĆ, A., Sum of reciprocals of the largest prime factor of an integer, *Arch. Math. (Basel)* **36** (1981), 57—61. *MR* **82g**: 10067.

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