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COMBINATORIAL PROBLEMS IN GEOMETRY AND NUMBER THEORY

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In a previous paper (given at the International Congress of Mathematicians at Nice 1970) entitled "On the application of combinatorial analysis to number theory geometry and analysis", I discussed many combinatorial results and their applications. I will refer to this paper as I (as much as possible I will try to avoid overlap with this paper).

First I discuss those problems mentioned in I where significant progress has been made.

1. Let $f(n)$ be the smallest integer so that if there are $f(n)$ points in the plane no three on a line, then there are always n of them which form the vertices of a convex n-gon. This problem of E. Klein (Mrs Szekeres) was discussed in I. Szekeres and I proved that

$$2^{n-2} + 1 \le f(n) \le \binom{2n-4}{n-2} \quad . \tag{1}$$

Probably $f(n) = 2^{n-2} + 1$, but this has only been proved for $n = 4$ and $n = 5$. No progress has been made here but recently I asked the following related question: Let $g(n)$ be the smallest integer so that if there are $g(n)$ points in the plane, no three on a line, then there are always n of them which form the vertices of a convex polygon which has none of the other points in its interior. Trivially $g(4) = 5$, but I could not prove the existence of $g(n)$ for $n \ge 5$. Ehrenfeucht a few days ago found a simple proof for the existence of $g(5)$, unfortunately his proof does not seem to work for $n > 5$ and the general problem is still open. By the way, I just heard that independently and simultaneously Harborth proved $g(5) = 10$.

Now I outline the ingenious proof of Ehrenfeucht. By the result of Szekeres and myself we can assume that if the number of points is $\geq \binom{12}{6}$ there is a convex 8-gon. If the number of points in the interior is ≤ 1 we clearly get an empty convex pentagon. Thus we can assume that the number of points in the interior is ≥ 2. Let (x_1, x_2) be a side of the least convex polygon of the points in the interior. The line (x_1, x_2) divides the convex 8-gon into two parts, one of them has no point in its interior. If this part contains three or more vertices of our 8-gon, we have our convex pentagon. If it cuts off one or two vertices, then omitting these and replacing them by x_1, x_2 we obtain a convex 8-gon or 9-gon which has fewer points in its interior. Repeating this proves we clearly obtain an empty convex pentagon.

Ehrenfeucht then asked: Let $x_1, \ldots, x_n$ be n points in the plane no three on a line. Denote by $k(n)$ the largest integer so that for every choice of n such points there are at least $k(n)$ triangles with no point in their interior. Determine or estimate $k(n)$. I proved $(1 + o(1))n^2 < k(n) < cn^2 \log n$. I cannot decide whether $k(n) < cn^2$ is true. Further problem: How many empty convex t-gons $(t \geq 4)$ must we have?

Delsarte, Goethals, Seidel and Larman, Rogers, Seidel nearly completely solved the problem of two distance sets in n-dimensional space. Every set of $\frac{1}{2}(n + 1)(n + 4) + 1$ points in n-dimensional space determines at least three distinct distances, but there always is a set of $\frac{n(n+1)}{2}$ points which determine only two distances. On the other hand, if $k_3(n)$ is the smallest integer so that $k_3(n)$ points in n-dimensional space always contains a non-isoseles triangle, then we have no good upper bound for $k_3(n)$. Probably $k_3(n) < n^c$ and perhaps even $k_3(n) = (\frac{1}{2} + o(1))n^2$. We only have an exponential upper bound for $k_3(n)$.

The 2^n vertices of the n-dimensional cube determine n distinct distances. Perhaps any 2^n points in n-dimensional space determine more than cn distinct distances.

Seidel just informed me that if S is a set of n elements on the sphere in Euclidean d-space which has $\leq s$ distances, then S contains at most $\binom{d+s-1}{d-1} + \binom{d+s-2}{d-1}$ points.

D.G. Larman, C.A. Rogers and J.J. Seidel, On two distance sets in Euclidean space, Bull. London Math. Soc. 9 (1977), 261-277; P. Delsarte, J.M. Goethals and J.J. Seidel, Spherical codes and designs, to appear in Geometriae Dedicata.

For further problems and literature, see Paul Erdős, On some problems of elementary and combinatorial geometry, Annali di Math. 103 (1975), 99-108.

2. The conjecture of Graham and Rothschild has been proved first by Hindman and then by Baumgartner and finally by Glazer.

The conjecture stated: Divide the integers into two classes. Then there always is an infinite sequence of integers $a_1 < a_2 < \ldots$ so that all the sums $\sum_i \varepsilon_i a_i$, $\varepsilon_i = 0$ or 1 belong to the same class.

Various modifications and extensions are open: Let S be a sequence of integers of positive density (upper density?) Is there always an infinite sequence $a_1 < a_2 \ldots$ and an integer t so that all the integers $\{a_i, a_i + a_j + t\}$ belong to the same class?

There is a problem due to J. Owings: Divide the integers into two classes. Then there always is an infinite sequence $a_1 < a_2 \ldots$ so that all the sums $a_i + a_j$ ($i = j$ permitted) are in the same class. It is surprising that this harmelsss looking problem causes so much difficulty. Though Hindman has some partial results, his paper will soon appear in the Journal of Combinatorial Theory.

I conjectured that if S is any infinite set, then its subsets can always be divided into two classes so that if $\{A_n\}$, $n = 1,2,\ldots$ is any infinite family of disjoint subsets of S there are two sets S_1 and S_2 in different classes both of which are unions of infinitely many A's. On the other hand, I conjecture that if m is any cardinal number, then if the power of S is sufficiently large and we divide the countable subsets of S into two classes there are always m disjoint sets $\{A_\alpha\}$, $\alpha < \Omega_m$

so that all finite unions of the A_α's belong to the same class.

A few years ago I had the following fascinating conjecture: Divide the integers into two classes. Then there is an infinite sequence $a_1 < a_2 \ldots$ so that all the multilinear expressions formed from the a's are in the same class. I (and others) first tried to find a counterexample but so far no success. A weaker conjecture which also seems inaccessible at present states that if we divide the integers into two classes there always is an infinite $a_1 < a_2 \ldots$ where all the a_i, $a_i + a_j$, $a_i a_j$ belong to the same class. It is not even known that there are three a's, a_1, a_2, a_3 with this property.

R.L. Graham proved that if one divides the integers $1 \leq t \leq 252$ into two classes there are always $a_1, a_2, a_1 + a_2$, $a_1 a_2$ $(a_1 \neq a_2)$ all in the same class, and this is false for 251. $a_1 = 1$ cannot be excluded. Hindman proved that if $1 \leq t \leq 969$ then we can assume $a_1 > 1$, 969 is best possible. As far as I know this is all that is known at present.

The proof of Glazer is given in a survey paper by W.W. Comfort, Ultrafilters: some old and new results, Bull. Amer. Math. Soc. 83 (1977), 417-455; references to Hindman and Baumgartner can be found in this paper.

3. In I, I state that Kleitman proved the following conjecture of mine: Let $z_1, \ldots, z_n$ be n vectors in a linear vector space satisfying $\|z_i\| \geq 1$. Consider the 2^n sums $\sum_{i=1}^{n} \varepsilon_i z_i$, $\varepsilon_i = \pm 1$ which are in the interior of a sphere of radius 1. The number of these sums is $\leq \binom{n}{\left[\frac{n}{2}\right]}$. Kleitman's proof appeared in the meantime.

On a lemma of Littlewood and Offord on the distribution of linear combinations of vectors, Advances of Math. 5(1970), 1-3. For further problems, see Lee Jones, On the distribution of sums of vectors, SIAM J. Applied Math. 39 (1978), 1-6.

4. An old problem of Rado and myself states: Let $a > 1$, $b > 1$ be integers. $f(a,b)$ is the smallest integer so that if we have $f(a,b) + 1$ sets each having at most b elements, there are always $a + 1$ of them

which have pairwise the same intersection. Rado and I conjectured that for some absolute constant c

$$f(a,b) < c^{b+1} a^{b+1} \quad . \tag{1}$$

(1) was stated in I. Little progress has been made. Abbott proved that $f(3,3) = 20$ and Spencer proved that for fixed a and $b > b_0(\varepsilon)$

$$f(a,b) < (1+\varepsilon)^b \, b! \quad .$$

I offer 500 dollors for a proof or disproof of (1).

P. Erdös, and R. Rado, Intersection theorems for systems of sets I and II, J. London Math. Soc. 35 (1960), 85-90 and 44 (1969), 467-479. For further reference, see P. Erdös, E. Milner and R. Rado, Intersection theorems for systems of sets III, J. Australian Math. Soc. 18 (1969), 22-41.

5. Denote by $r_k(n)$ the smallest integer ℓ so that if $1 \le a_1 < \ldots < a_\ell \le n$, $\ell = r_k(n)$ is any sequence of integers, then the a's contain an arithmetic progression of k terms. Turán and I conjectured 45 years ago that $r_k(n) = o(n)$. This was finally proved by E. Szemerédi in 1973. His proof is a masterpiece of combinatorial reasoning. One of his lemmas is a purely combinatorial theorem on decomposition of graphs which already had several applications to various problems and there is no doubt that it will have further applications in the future. Since this "lemma" (or rather theorem) is not sufficiently well known, I restate it here. First we need a few definitions: Let $G(n;\ell)$ be a graph of n vertices and ℓ edges. A and B are two disjoint sets of vertices of G and $e(A,B)$ denotes the number of edges one endpoint of which is in A and the other in B. Define

$$d(A,B) = \frac{e(A,B)}{|A||B|} \quad .$$

$d(A,B)$ is the "density" of the edges between A and B. The pair (A,B) is called ε - regular if

$$X \subset A \ , \ Y \subset B \ , \ |X| \geq \varepsilon|A|,$$

imply

$$|d(X,Y) - d(A,B)| < \varepsilon \ ,$$

otherwise the pair is called ε - irregular. By an equitable partition of a set V , we shall mean a partition of V into pairwise disjoint sets $C_0, C_1, \ldots, C_k$ such that all the C_i's with $1 \leq i \leq k$ have the same cardinality. The set C_0 may be empty. It is called the exceptional set. Let $G(n ; \ell)$ be a graph. An equitable partition of the vertex set into sets $C_0, C_1, \ldots, C_k$ will be called ε - regular if the cardinality of the exceptional set C_0 does not exceed εn and if at most εk^2 of the pairs (C_s, C_t) with $1 \leq s < t \leq k$ are ε - irregular.

Szemerédi's theorem states as follows: for every $\varepsilon > 0$ there is an integer M so that if $n > n_0(M, \varepsilon)$, then every $G(n ; \ell)$ admits an ε - regular partition into k classes with $k < M$ (In fact, we may also prescribe a lower bound m on the number of classes; then of course, M becomes a function of ε and m).

Using this "Lemma", Ruzsa and Szemerédi partially settled a problem of W. Brown, V.T. Sós and myself. Denote by $f(n ; k)$ the smallest integer so that if $|S| = n$ and $A_i \subset G$, $|A_i| = 3$, $1 \leq i \leq f(n ; k)$ then there is an $S_1 \subset S$, $|S_1| = k$ which is the union of $k - 3$ $(k > 3)$ of the A's. We conjectured that

$$f(n ; k) = o(n^2)$$

holds for every k . The conjecture is trivial for $k < 6$ and Ruzsa and Szemerédi proved it for $k = 6$. They also showed

$$f(n ; 6) > cnr_3(n) > n^{2 - \varepsilon}$$

which is, in my opinion, a very surprising and deep result. The conjecture is still open for $k > 6$.

Szemerédi's proof uses Van der Waerden's theorem and thus gets a very poor upper bound for $r_k(n)$. About two years ago Furstenberg proved

Szemerédi's theorem by methods of ergodic theory. Furstenberg does not use Van der Waerden's theorem but his proof of $r_k(n)/n \to 0$ is an existence proof and does not give any explicit bound for $r_k(n)$. Very recently, Katz Nelson and Ornstein simplified Furstenberg's proof.

I offered 1000 dollars for $r_k(n) = o(n)$, which Szemerédi collected. I offer 3000 dollars for the proof or disproof of

$$r_k(n) = o\left(\frac{n}{(\log n)^{\ell}}\right) \tag{2}$$

for every ℓ if $n > n_0(\ell)$. (2) would imply that for every k there are k primes in an arithmetic progression. An attractive but slightly weaker conjecture states: Let $\Sigma \frac{1}{a_i} = \infty$. Then for every k there are k a's in an arithmetic progression. I offer 3000 dollars for the proof or disproof of this conjecture. I do not even have a guess what the true order of magnitude of $r_k(n)$ is. It is not even known that $r_3(n)/r_4(n) \to 0$.

E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, Acta Arithmetica 27 (1975), 199-245.

H. Furstenberg, Ergodic behaviour of diagonal measures and a theorem of Szemerédi, J. Analyse Math. 31 (1977), 204-

For further problems, see P. Erdös, Problems and results on combinatorial number theory III, Number theory day, Proc. Conference Rockefeller Univ. 1976, 43-73, Lectures Notes in Math 626, Springer-Verlag.

6. Let $x_1,\ldots,x_n$ be n distinct points in k dimensional space. Denote by $d_k(n)$ the maximum number of pairs (x_i,x_j) whose distance is 1. This problem was discussed in I. The most important progress is that Szemerédi and Józsa proved that $d_2(n) = o(n^{3/2})$. The proof is surprisingly complicated especially if one compares it with the simple proof of $d_2(n) < cn^{3/2}$. I am sure that $d_2(n) < n^{1+\varepsilon}$ for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$ and offer 100 dollars for a proof or disproof. (I suspect this will be a very difficult method to earn 100 dollars.) The right order of magnitude of $d_2(n)$ is probably $n^{1+c/\log\log n}$.

A possible approach to these problems is perhaps the problem of forbidden subgraphs; this method is often applied successfully. A graph $G(m)$ of m vertices is forbidden if there is no set of m distinct points in the plane so that $G(x_1,\ldots,x_m)$ contains $G(m)$ as a subgraph. It is called minimal if the omission of any edge of $G(m)$ ensures the existence of such a set of m points. $K(3,2)$ is such a minimal forbidden graph. It would certainly be interesting to give a complete list of minimal forbidden graphs. Hopefully a graph of n vertices which does not contain any of the forbidden graphs has fewer than $n^{1+\varepsilon}$ edges for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$.

Hadwiger and independently Nelson raised the following problem: Join two points in the plane if their distance is 1. Determine the chromatic number of this graph. It seems likely that the chromatic number is greater than four. By a theorem of de Bruijn and myself this would imply that there are n points $x_1,\ldots,x_n$ in the plane so that if we join any two of them whose distance is 1, then the resulting graph $G(x_1,\ldots,x_n)$ has chromatic number > 4. I believe such an n exists but its value may be very large. The chromatic number of the plane is known to be at most 7.

Let our n points be such that they do not contain an equilateral triangle of side 1. Then their chromatic number is probably at most 3, but I do not see how to prove this. If the conjecture would unexpectedly turn out to be false, the situation can perhaps be saved by the following new conjecture: There is a k so that if the girth of $G(x_1,\ldots,x_n)$ is greater than k, then its chromatic number is at most three -- in fact, it will probably suffice to assume that $G(x_1,\ldots,x_n)$ has no odd circuit of length $\le k$.

Larman and Rogers investigated the chromatic number $k(n)$ of n-dimensional space. Probably the chromatic number tends to infinity exponentially. V.T. Sós and I observed that if $|S| = n$, and $A_i \subset S$, $1 \le i \le n+1$, $|A_i| = 3$, then for some $1 \le i_1 < i_2 \le n+1$, $|A_{i_1} \cap A_{i_2}| = 1$. This easily implies that $k(n) \ge cn^2$. We conjectured that if $n > n_0(k)$ and $|A_i| = k$, $1 \le i \le \binom{n-2}{k-2} + 1$, then for some $1 \le i_1 < i_2 \le \binom{n-2}{k-2} + 1$, $|A_{i_1} \cap A_{i_2}| = 1$.

Katona proved this for $k = 4$ and Frankl for all k, from his results one can deduce that $k(n) > n^c$ for every c if $n > n_0(c)$.

Let $f(n,r)$ be the smallest integer for which if $|S| = n$, $A_i \subset S$, $1 \le i \le f(n,r)$ there are two A's say A_{i_1} and A_{i_2} with $|A_{i_1} \cap A_{i_2}| = r$. Trivially $f(n,0) = 2^{n-1} + 1$. It would be of interest to determine $f(n,r)$ for $r > 0$ at least for $n > n_0(r)$. I conjectured that for every $\eta n < r < (\frac{1}{2} - \eta)n$

$$f(n,r) < (2 - \varepsilon)^n, \ \varepsilon = \varepsilon(\eta).$$

(1), if true, easily implies $k(n) > (1 + \delta)^n$ for some fixed $\delta > 0$.

The following question just occurred to me: Let $\alpha_1,\dots,\alpha_k$ be k positive numbers. Let $x_1,\dots,x_n$ be the n points in the plane. Join x_i and x_j if $d(x_i,x_j) = \alpha_i$ for some $1 \le i \le k$. ($d(x_i,x_j)$ is the distance of x_i and x_j). Denote the resulting graph by $G_{\alpha_1,\dots,\alpha_k}(x_1,\dots,x_n)$. Let $f(k)$ be the maximum of the chromatic numbers of these graphs. Estimate or determine $f(k)$; at the moment I do not know how fast $f(k)$ tends to infinity, but perhaps I overlook a simple argument.

Finally, I wish to state the classical conjecture of Borsuk: Is it true that a set S of diameter 1 in n-dimensional space is the union of $n+1$ sets of diameter less than 1. This conjecture is trivial for $n = 1$, easy for $n = 2$, difficult for $n = 3$ and unsolved for $n \ge 4$.

S. Józsa and E. Szemerédi, The number of unit distance on the plane, Infinite and finite sets, Coll. Math. Soc. J. Bolyai 1973, North Holland, 939-950.

D.E. Larman and C.A. Rogers, The realisation of distances within sets in Euclidean space, Mathematika 19 (1972), 1-24.

B. Grunbaum, Borsuk's problem and related questions, Convexity, Proc. Symp. Pure Math Vol. 7 (1963), Amer. Math. Soc. 271-284. This book contains many other articles of great combinatorial interest.

7. George Purdy and I plan to write a book on some of the problems and their higher dimensional analogues - which we considered in 6. Here I just

give a short and not very systematic outline of some of our results. Denote by $g_k^{(r)}(n)$ the largest integer for which there are n points $x_1,\ldots,x_n$ in k-dimensional space for which there are $g_k^{(r)}(n)$ sets $\{x_{i_1},\ldots,x_{i_{r+1}}\}$ which have the same non zero r-dimensional volume. We first of all proved

$$c_1 n^2 \log\log n < g_2^{(2)}(n) < 4n^{5/2} \quad . \tag{1}$$

We believe that in (1) the lower bound is closer to the "Truth" - in fact, perhaps it gives the right order of magnitude. Purdy proved that for some $\varepsilon > 0$

$$g_5^{(2)}(n) < n^{3-\varepsilon} \quad . \tag{2}$$

In the proof of (2) Purdy used the following purely combinatorial theorem of mine on hypergraphs. Let $G^{(r)}(n\,;m)$ be a hypergraph of n vertices and m edges (i.e., m r-tuples). Then if $m > n^{r-\varepsilon(\ell,r)}$ our $G^{(r)}(n\,;m)$ contains a $K_r(\ell,\ldots,\ell)$, where $K_r(n_1,\ldots,n_r)$ is the hypergraph $\sum_{i=1}^{r} n_i$ vertices and $\prod_{i=1}^{r} n_i$ edges, where the vertices are divided into r disjoint sets $|A_i| = n_i$, $i = 1,\ldots,r$ and the edges are all the r-sets which meet each A_i in exactly one point.

An example of Lenz gives $g_4^{(1)}(2n) \geq n^2$ and I proved $g_4^{(1)} < n^2(1+o(1))$. Oppenheim observed that the idea of Lenz gives $g_{2k+2}^{(k)}((k+1)n) \geq n^{k+1}$ and we conjectured $g_{2k+2}^{(k)}((k+1)n) < (1+o(1))n^{k+1}$ but the proof seems to present difficulties even for $k = 2$.

We also investigate the maximum number of isosceles or equilateral triangles one can obtain from n points in r-dimensional space. We almost never succeed in getting asymptotic formulas but have to be satisfied with more or less crude upper and lower bounds. A typical problem states as follows: Let there be given n distinct points in the plane. Let $A(n)$ be the largest number of four-tuples one can form so that not all the six distances should be distinct. We proved

$$c_2 n^3 \log n < A(n) < c_1 n^{7/2} \quad .$$

We are fairly sure that $A(n) < n^{3+\varepsilon}$ holds but have not been able to prove it.

Many further related questions could be discussed, but not to make this chapter too long, I only state some results of Bollobas, Purdy and myself. Let T be a triangle. $f(n\,;T)$ is the largest integer so that for every $\varepsilon > 0$ there are n distinct points in the plane which determine $f(n\,;T)$ triangles which are congruent to T with a possible error of less than ε (or more precisely the sides of the triangles differ from the corresponding sides of T by less than ε). Clearly $f(3n\,,T) \geq n^3$ for every triangle T and we prove that if T is equilateral, then there is equality. For which T is $f(n\,;T)$ maximal? We outlined a proof that T must be isosceles with a right angle. Here $f(4n\,;T) = 4n^3$. We did not complete all the details which were quite messy. Many further questions could be raised here and if we live we hope to return to them.

P. Erdös and G. Purdy, Some extremal problems in geometry, J. Combinatorial Theory, 10 (1969), 246-252.

G. Purdy, Some extremal problems in geometry, Discrete Math. 7 (1974), 305-314.

P. Erdös and G. Purdy, Some extremal problems in geometry III, IV and V, Proc. Sixth Conference on Combinatorics Graph Theory and Computing, Florida Atlantic Univ. 1975, 291-308 (1976), 307-322 and (1977).

P. Erdös, On extremal problems of graphs and generalised graphs, Israel J. Math. 2 (1964), 183-190.

For further (I hope) interesting problems and results, see P Erdös and G. Purdy, Some combinatorial problems in the plane, will appear in J. Combinatorial Theory.

8. Graham, Montgomery, Rothschild, Spencer, Straus and I in several papers investigated the following problems: A well-known theorem of Gallai states as follows: Let S be any finite set in r-dimensional Euclidean

space. Color the points of the space by k colors in an arbitrary way. Then there always is a monochromatic set which is similar to S. This theorem is a generalisation of the well known theorem of van der Waerden on arithmetic progressions. What happens if we require a monochromatic set congruent to S?. One of the first non-trivial results which we proved with the help of S. Burr states: There is a set of 15 points in 6-dimensional space so that if we color the points by two colors, there always is a monochromatic unit square. It is quite probable that the dimension of the space can be reduced to three (at the expense of perhaps increasing the number of points). More generally a set S in a Euclidean space is called Ramsey if for every k there is an n_k so that if we color the points of n_k - dimensional space by k colors there always is a monochromatic set congruent ot S. We proved that every brick (i.e., rectangular parallelepid) is Ramsey and that every Ramsey set must be spherical, i.e., it must be situated on the surface of a sphere. We do not know any set which is Ramsey and is not a brick and we do not know if there is a spherical finite set which is not Ramsey. The simplest open problem states: Is it true that every non-degenerate triangle is Ramsey? If all angles are $\leq \frac{\pi}{2}$ this is obvious since the triangle can be imbedded into a brick. I offer 100 dollars for a clarification of these questions.

Color the plane by two colors. $\{x,y,z\}$ is any triangle. Then, with the possible exception of a single equilateral triangle - there always is a monochromatic triangle congruent to $\{x,y,z\}$. Many special cases have been proved by us and others but the general case is still open and I offer 100 dollars for the proof or disproof.

Let S be a set of the plane no two points of which are at distance 1. We proved that $\bar{S}$ contains four points x_1,x_2,x_3,x_4 on a line so that the distance between two consecutive points is 1. We conjectured that $\bar{S}$ contains the vertices of a unit square. Ms. R. Juhasz proved this - in fact, she proved that $\bar{S}$ contains a congruent copy of any four points. ($\bar{S}$ is the complement of S). It would be of great interest to characterize the set of points which always can be imbedded into $\bar{S}$. The paper of R. Juhasz will be published soon.

Several further problems for finite and infinite sets are stated in our papers.

P. Erdös, R.L. Graham, P. Montgomery, B.L. Rothschild, J. Spencer, E.G. Straus, Euclidean Ramsey theorems I, II, and III, Journal of Combinatorial Theory Ser. A 14(1973), 341-363, Infinite and finite sets, Coll. Math. Soc. J. Bolyai, 1973 North Holland, 529-557 and 559-583.

L.E. Shader, All right triangles are Ramsey in E_2 , J. Combinatorial Theory, Ser. A. 20 (1976), 385-389.

9. Concluding remarks. Here I state a few problems of a combinatorial nature which do not seem hopeless to me and which perhaps have been unduly neglected.

1. Let $1 \le a_1 < \dots$ be an infinite sequence of integers and denote by $f(n)$ the number of solutions of $n = a_i + a_j$. Is there a sequence for which

$$\lim f(n)/\log n = 1 \quad ? \tag{1}$$

If $\log n$ is replaced by $g(n)\log n$ where $g(n)$ is a monotonic function tending to infinity the answer is easily seen to be affirmative. The probability method does not seem to apply to (1). For further details, see the well known book of Halberstam and Roth, "Sequences".

2. Let there be given n points in the plane at most $n-k$ on a line. Join every two of them and prove that these points determine at least ckn lines where c is an absolute constant independent of k and n . If $k^2 < c_1 n$ then L.M. Kelly and W.O.J. Moser proved a stronger result.

3. Conjecture of Szemerédi: Let there be given n points in the plane, no three on a line. Then they determine at least $\left[\frac{n}{2}\right]$ distinct distances. Szemerédi proved only $\left[\frac{n}{3}\right]$. I conjectured and Altman proved $\left[\frac{n}{2}\right]$ if the n points form a convex polygon.

For references to 2. and 3. and also for other problems in geometry, see my paper in Annali di Math. referenced in 1; for combinatorial problems in number theory, see P. Erdös, Problems and results in combinatorial number theory III, Number Theory Day, meeting held at Rockefeller Univ., March, 1976, Lecture Notes in Math 626, Springer Verlag.

For many interesting problems in combinatorial geometry, see Hadwiger, Delrunner, Klee, Combinatorial geometry in the plane, Holt, Rinehart, and Winston. For problems in combinatorial geometry of a different kind, see B. Grünbaum, Arrangements and spreads, Amer. Math. Soc., 1972, also the well known books and papers of Fejes Toth on discrete geometry.

I neglected to discuss the various applications of Ramsey's theorem to the distance distribution of point sets in a complete metric space and its applications to geometry, potential theory and other subjects.

P. Erdös, A. Meir, V.T. Sós and P. Turna, On some applications of graph theory I, II, and III, Discrete Math. 2 (1972), 207-228; Studies in pure math. Papers presented to Richard Rado, Acad. Press, London, 1971, pp. 89-99; Canad. Math. Bull 15 (1972), 27-32.