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Rational Approximation, II

PAUL ERDŐS,* DONALD J. NEWMAN,† AND A. R. REDDY‡,§

* *Hungarian Academy of Sciences, Budapest Hungary,*
† *Temple University, Philadelphia, Pennsylvania 19122,*
‡ *School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540, and*
§ *Rutgers University, New Brunswick, New Jersey 08903*

INTRODUCTION

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function. Denote $M(r) = \max_{|z|=r} |f(z)|$; then the order ρ is defined thus:

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \rho \qquad (0 \leqslant \rho \leqslant \infty). \tag{1}$$

If $0 < \rho < \infty$, then the type τ and the lower type ω are defined as

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^\rho} = \begin{matrix}\tau\\ \omega\end{matrix} \qquad (0 < \omega \leqslant \tau < \infty). \tag{2}$$

Let π_m denote the class of all real polynomials of degree at most m, and $\pi_{m,n}$ similarly denote the collection of all rational functions

$$r_{m,n}(x) = \frac{p(x)}{q(x)}, \qquad p \in \pi_m,\ q \in \pi_n.$$

For convenience we use r_n for $r_{n,n}$, then let

$$\lambda_{m,n} = \lambda_{m,n}(f^{-1}) = \inf_{r_{m,n}\in\pi_{m,n}} \left\| \frac{1}{f(x)} - r_{m,n} \right\|_{L_\infty[0,\infty)}. \tag{3}$$

Throughout our work we use $c_1, c_2, c_3, \ldots$ to denote some positive constants (which may be different on different occasions). $T_n(x)$ denotes the Chebyshev polynomial of degree n. $S_n(x)$ denotes the nth partial sum of $f(x)$.

Recently it has been shown [14] that

$$\lim_{n\to\infty} (\lambda_{0,n}(e^{-x}))^{1/n} = 3^{-1}.$$

In [7], it has been established that for all $n \geqslant 2$

$$\lambda_{n,n}(e^{-x}) \geqslant (1280)^{-n-1},$$

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which clearly shows that the order of magnitude of the error obtained by rational functions of degree n in approximating e^{-x} on the positive real axis is not better than the order of magnitude of the error obtained to e^{-x} by reciprocals of polynomials of degree n on the positive real axis. In [5], we have shown that $e^{-|x|}$ can be approximated by general rational functions of degree n with an error $c_1 \exp(-c_2 n^{1/2})$ on $(-\infty, +\infty)$, but by reciprocals of polynomials of degree n one cannot approximate $e^{-|x|}$ on $(-\infty, +\infty)$ with an error better than $c_3 n^{-1}$, thereby showing that the rational functions of degree n are much better than the reciprocals of polynomials of degree n in approximating $e^{-|x|}$ on $(-\infty, +\infty)$ under the uniform norm. In [8] we have discussed $|x| e^{-|x|}$. For related problems, cf. [5, 8–12].

In this paper we show for certain class functions the error obtained by rational functions of degree n in approximating on $[0, \infty)$ under the uniform norm is much smaller than the error obtained by reciprocals of polynomials of degree n. Most of the methods developed in this paper are new and may be applied successfully to many of the related problems.

Lemmas

Lemma 1. [1, p. 10]. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, be an entire function of order ρ $(0 < \rho < \infty)$ and type τ $(0 \leqslant \tau \leqslant \infty)$. Then*

$$\tau = \limsup_{n \to \infty} n(\rho e)^{-1} \mid a_n \mid^{\rho/n}. \tag{4}$$

Lemma 2 [15, p. 68]. *Let $P(x)$ be any polynomial of degree at most n and satisfies $| P(x)| \leqslant M$ on the segment $[a, b]$, then at any point outside the segment we have*

$$| P(x)| \leqslant M \left| T_n \left(\frac{2x - a - b}{b - a}\right)\right|. \tag{5}$$

Lemma 3 [6, pp. 450–451]. *If $0 < k < 1$, and*

$$\max_{t \in [-1, -k]} | r_n(t)| \leqslant M,$$

then

$$\min_{t \in [k,1]} | r_n(t)| \leqslant M \exp\left(\frac{\pi^2 n}{\log 1/k}\right). \tag{6}$$

Lemma 4 [3, pp. 65–66]. *Let $f(z) = e^{e^z} = \sum_{k=0}^{\infty} a_k z^k$. Then*

$$a_k \sim \left[\exp\left(\frac{k}{\log k}\right)\right] (2\pi k \log k)^{-1/2} (\log k)^{-k}. \tag{7}$$

LEMMA 5. *There exists a sequence $\{P_n(x)\}_{n=0}^{\infty}$ of polynomials of degree n for which for all $n \geqslant 2$,*

$$\left\| x - \frac{1}{P_n(x)} \right\|_{L_\infty[0,1]} \leqslant c_4 n^{-2}. \tag{8}$$

Proof. Choose n even and

$$Q(x) = \frac{(-1)^{n-1}(\cos(\pi/2n) - \cos(\pi/n))}{x + \cos(\pi/2n) - \cos(\pi/n)} T_n(x - \cos(\pi/n)). \tag{9}$$

This is a polynomial, since $T_n(-\cos(\pi/2n)) = 0$. It is easy to see that $Q(0) = 1$, so that

$$P(x) = \frac{1 - Q(x)}{x} \tag{10}$$

is a polynomial.

Set $\delta = \cos(\pi/2n) - \cos(\pi/n)$, then on $[0, 1]$

$$\begin{aligned}
\delta + x - \frac{1}{P(x)} &= \frac{\delta - (\delta + x)\, Q(x)}{1 - Q(x)} \\
&= \delta \frac{1 + (-1)^n\, T_n(x - \cos(\pi/n))}{1 + ((-1)^n/(1 + x/\delta))\, T_n(x - \cos(\pi/n))} \\
&= \frac{1 + t}{1 + st} = \delta M,
\end{aligned}$$

where $t \in [-1, 1]$, $s \in [0, 1]$ and so M is bounded by 2. Hence

$$0 \leqslant \delta + x - \frac{1}{P(x)} \leqslant 2\delta;$$

i.e.,

$$\left\| x - \frac{1}{P(x)} \right\| \leqslant \delta < \frac{\pi^2}{2n^2}.$$

Hence the lemma is completely established.

LEMMA 6. *Let $P(x)$ be a polynomial of degree at most n and $|P(x)| \leqslant e^{2x}$ for $0 \leqslant x \leqslant L$; then*

$$|P(-1)| \leqslant e^{(n+2L)2L^{-1/2}}.$$

Proof. Observe that for $0 \leqslant x \leqslant L$,

$$(1 - x/L)^{2L} \leqslant e^{-2x}.$$

Hence, we have on $[0, L]$,

$$|(1 - x/L)^{2L} P(x)| \leqslant 1.$$

Hence by Lemma 2

$$|P(-1)| \leqslant \left| \left(1 + \frac{1}{L}\right)^{2L} P(-1) \right| \leqslant T_{n+2L}\left(1 + \frac{2}{L}\right) < e^{(n+2L)2L^{-1/2}}.$$

Theorems

THEOREM 1. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order ρ $(0 < \rho < \infty)$ type τ and lower type ω $(0 < \omega \leqslant \tau < \infty)$. Then for all large n*

$$\lambda_{0,n}(x/f(x)) \leqslant c_4 (\log n)^{1/\rho} n^{-2}. \tag{11}$$

Proof. Let $S_n(x)$ denote the nth partial sum of $f(x)$. Then

$$\begin{aligned} &\left| \frac{x}{f(x)} - \frac{1}{Q^*(x)\, S_n(x)} \right| \\ &\qquad \leqslant \frac{1}{f(x)} \left| x - \frac{1}{Q^*(x)} \right| + \frac{1}{|Q^*(x)|} \left| \frac{1}{f(x)} - \frac{1}{S_n(x)} \right|, \end{aligned} \tag{12}$$

where $Q^*(x) = (3\omega^{-1} \log n)^{-1/\rho} P(x(3\omega^{-1} \log n)^{-1/\rho})$, $P(x)$ is defined as in Lemma 5. Let $0 \leqslant x \leqslant (\log n)^{1/\rho} (3\omega^{-1})^{1/\rho}$; then by Lemma 5, we have

$$\frac{1}{f(x)} \left| x - \frac{1}{Q^*(x)} \right| \leqslant \frac{c_5 (3\omega^{-1} \log n)^{1/\rho}}{a_0 n^2}. \tag{13}$$

On the other hand, for sufficiently large n, we get for $x > (3\omega^{-1} \log n)^{1/\rho}$, along with the definition of lower type and the fact that $Q^*(x) > (2x)^{-1}$,

$$\frac{1}{f(x)} \left| x - \frac{1}{Q^*(x)} \right| \leqslant e^{-x^\rho \omega(1-\varepsilon)} 3x \leqslant c_7 (\log n)^{1/\rho} n^{-2}. \tag{14}$$

Similarly, we can show for $0 \leqslant x \leqslant (\log n)^{1/\rho} (3\omega^{-1})^{1/\rho}$, along with Lemmas 1 and 5, that

$$\frac{1}{Q^*(x)} \left| \frac{1}{f(x)} - \frac{1}{S_n(x)} \right| \leqslant \frac{\sum_{k=n+1}^{\infty} a_k x^k}{|Q^*(x)|\, S_n(x) f(x)} \leqslant c_8 (\log n)^{1/\rho} n^{-2}. \tag{15}$$

On the other hand, for all large n we get for $x > (3\omega^{-1} \log n)^{1/\rho}$, along with the fact that $Q^*(x) > (2x)^{-1}$,

$$\begin{aligned} \frac{1}{|Q^*(x)|} \left| \frac{1}{f(x)} - \frac{1}{S_n(x)} \right| &\leqslant \frac{1}{|Q^*(x)|\, f(x)} + \frac{1}{|Q^*(x)|\, S_n(x)} \\ &\leqslant \frac{2x}{f(x)} + \frac{2x}{S_n(x)} \\ &< \frac{1}{n^2}. \end{aligned} \tag{16}$$

Hence, result (11),

$$\left\| \frac{x}{f(x)} - \frac{1}{q_n(x)} \right\|_{L_\infty[0,\infty)} \leqslant c_4 \frac{(\log n)^{1/\rho}}{n^2},$$

follows from (13)–(16).

THEOREM 2. *There is an entire function of order ρ $(0 < \rho < \infty)$ and type $\tau = \infty$ for which, for all large n,*

$$\lambda_{0,n} \leqslant c_9 (\log n)^{1/\rho} n^{-2} (\log \log n)^{-1/\rho}. \tag{17}$$

Proof. Let $f(z) = 1 + \sum_{k=2}^{\infty} ((\log k)/k)^{k/\rho} z^{k/\rho}$ $(0 < \rho < \infty)$. Clearly $f(z)$ is an entire function of order ρ and type $\tau = \infty$. We consider here for simplicity $\rho = 1$ only. As earlier, we write

$$\left| \frac{x}{f(x)} - \frac{1}{S_n(x)\, q^*(x)} \right|$$
$$\leqslant \frac{1}{f(x)} \left| x - \frac{1}{q^*(x)} \right| + \frac{1}{q^*(x)} \left| \frac{1}{f(x)} - \frac{1}{S_n(x)} \right|,$$

where

$$q^*(x) = c_{11} \frac{(\log \log n)}{\log n}\, p\left(\frac{x \log \log n}{c_{11} \log n} \right), \tag{18}$$

$p(x)$ defined as in Lemma 5.

Now for $0 \leqslant x \leqslant c_{11} (\log n)(\log \log n)^{-1}$,

$$\frac{1}{f(x)} \left| x - \frac{1}{q^*(x)} \right| \leqslant c_{12} (\log n)(\log \log n)^{-1} n^{-2}. \tag{19}$$

On the other hand, for $x \geqslant c_{11}((\log n)/\log \log n)$, we get by using the relation that

$$f(x) \sim e^{x \log x},$$

$$\frac{1}{f(x)} \left| x - \frac{1}{q^*(x)} \right| \leqslant \frac{3x}{f(x)} \leqslant 4x e^{-(x \log x)/2} \leqslant n^{-3}. \tag{20}$$

Similarly, we can show as in the case of Theorem 1 for $x \in [0, \infty)$ that

$$\frac{1}{q^*(x)} \left| \frac{1}{f(x)} - \frac{1}{S_n(x)} \right| \leqslant c_{13} (\log n)(\log \log n)^{-1} n^{-2}. \tag{21}$$

Hence, result (17) follows from (18)–(21).

THEOREM 3. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order ρ $(0 < \rho < \infty)$ type τ $(0 < \tau < \infty)$. Then for all large n*

$$\lambda_{0,n} \geqslant (\log n)^{1/\rho}(10n^2(2\tau)^{1/\rho})^{-1} \left(f\left[\left(\frac{\log n}{2\tau}\right)^{1/\rho} n^{-2}\right]\right)^{-1}. \tag{22}$$

Proof. Let us assume, on the contrary, that

$$\lambda_{0,n} < (\log n)^{1/\rho}(10n^2(2\tau)^{1/\rho})^{-1} \left[f\left(\left(\frac{\log n}{2\tau}\right)^{1/\rho} n^{-2}\right)\right]^{-1},$$

and assume that $[P_n(x)]^{-1}$ deviates least from $x/f(x)$; then we get on $[((2\tau)^{-1}\log n)^{1/\rho}n^{-2} = \delta_n n^{-2}, \delta_n]$,

$$\frac{1}{P_n(x)} \geqslant \frac{x}{f(x)} - \lambda_{0,n} \geqslant \frac{\delta_n n^{-2}}{\theta_n} - \frac{\delta_n}{10n^2\theta_n} \geqslant \frac{9}{10\theta_n}\,\delta_n n^{-2},$$

i.e.,

$$\max_{[\delta_n n^{-2}, \delta_n]} | P_n(x)| < (10/9)\, n^2\delta_n^{-1}\theta_n\,, \qquad \theta_n = f(\delta_n n^{-2}). \tag{23}$$

Now, by applying lemma 2 to (23), we get

$$| P_n(0)| < (10/9)\,\theta_n\delta_n^{-1}n^2 8 < 9n^2\theta_n\delta_n^{-1}. \tag{24}$$

On the otherhand, we have

$$\lambda_{0,n}^{-1} < | P_n(0)|. \tag{25}$$

Therefore, from (24) and (25) we get

$$\lambda_{0,n} > (\log n)^{1/\rho}(2\tau)^{-1/\rho}((9n^2)^{-1}\theta_n^{-1}),$$

which contradicts our earlier assumption that

$$\lambda_{0,n} < (10n^2)^{-1}((2\tau)^{-1}\log n)^{1/\rho}\theta_n^{-1}\,.$$

Hence, the result is proved.

THEOREM 4. *There is an entire function of order ρ $(0 < \rho < \infty)$ and type $\tau = 0$ for which for all large n,*

$$\lambda_{0,n} \geqslant (10n^2)^{-1}[(\log n)(\log\log\log n)]^{1/\rho}(f(\log n(\log\log n)n^{-2}))^{-1}. \tag{26}$$

Proof. Let

$$f(z) = 1 + \sum_{k=2}^{\infty} z^k(k\log k)^{-k/\rho} \qquad (0 < \rho < \infty).$$

This is an entire function of order ρ and type $\tau = 0$. We consider here only the case $\rho = 1$, and all the other values of ρ can be treated in the same way. As earlier, let us assume, on the contrary, that (26) is false; then

$$\lambda_{0,n} < (10n^2)^{-1}(\log n)(\log\log n)(f[(\log n)(\log\log n)n^{-2}])^{-1}, \tag{27}$$

for a sequence of values of n.

Let us suppose that $[P_n(x)]^{-1}$ deviates least from $x/f(x)$ on $[0, \infty)$; then by definition

$$\left\| \frac{x}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \leqslant \lambda_{0,n}\,. \tag{28}$$

From (28), we get over the interval

$$[(\log n)(\log\log n)n^{-2} = \alpha_n n^{-2}, \alpha_n]$$

along with (27), by using the fact that

$$f(x) \sim \exp\left(\frac{x}{\log x}\right), \qquad \text{and} \qquad \psi_n = f((\log n)(\log\log n)n^{-2}),$$

$$\frac{1}{P_n(x)} \geqslant \frac{x}{f(x)} - \lambda_{0,n} \geqslant (\alpha_n n^{-2} - \alpha_n(10n^2)^{-1})\psi_n^{-1}$$

$$\geqslant \frac{9\alpha_n n^{-2}\psi_n^{-1}}{10},$$

i.e.,

$$\max_{[\alpha_n n^{-2}, \alpha_n]} | P_n(x)| \leqslant (10/9)\, n^2\alpha_n^{-1}\psi_n\,. \tag{29}$$

Now by applying lemma 2 over the interval $[0, \alpha_n]$, we get

$$| P_n(0)| < 9n^2\alpha_n^{-1}\psi_n\,. \tag{30}$$

On the otherhand, it is known that

$$\lambda_{0,n}^{-1} \leqslant | P_n(0)|. \tag{31}$$

Equations (30) and (31) flatly contradict (27); hence (26) is established.

THEOREM 5. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of infinite order. Then for infinitely many n,*

$$\lambda_{0,n}\left(\frac{x}{f(x)}\right) \leqslant c_{13}\left(\frac{\log n}{n}\right)^2 | a_n |^{-1/n}. \tag{32}$$

Proof. By assumption, $f(z)$ is entire; hence

$$\lim_{n\to\infty} | a_n |^{1/n} = 0. \tag{33}$$

Let $t_n = a_n^{-1/n}$. Then $t_n \to \infty$. Now it is easy to note from the convergence of

$$\prod_{k=4}^{\infty} (1 + \{k \log k(\log\log k)^2\}^{-1}) \tag{34}$$

that there exist arbitrarily large values of n for which for each $l > 0$,

$$t_{n+l} > t_n \prod_{s=1}^{l} (1 + [A(n+s)]^{-1}), \tag{35}$$

where

$$A(n) = n \log n(\log\log n)^2.$$

From (35) it follows, with $l = n - 1$, that

$$t_{2n-1} > t_n(1 + 2(\log n)^{-1}(\log\log n)^{-2}). \tag{36}$$

Let

$$P_1(x) = \sum_{k=0}^{2n-1} a_k x^k,$$

$$P_2(x) = \frac{a_0 T_{2n-1}(1 + \delta - xB)}{T_{2n-1}(1+\delta)},$$

$$\delta = \left(\frac{4 \log n}{n}\right)^2, \qquad B = (2 + \delta) \mid a_n |^{1/n} (1 + (\log n)^{-1-\epsilon})^{-1},$$

$$xP(x) = P_1(x) - P_2(x).$$

Now we consider the values of $x \in [0, \delta B^{-1}]$. It is easy to see that

$$P(x) \geqslant [P_1(x) - P_2(x)]x^{-1} \geqslant c_{14} \mid a_n |^{1/n} \delta^{-1}.$$

Hence, over $[0, \delta B^{-1}]$, $x/f(x)$ and $1/| P(x)|$ are less than

$$c_{15}(\log n)^2 n^{-2} \mid a_n |^{-1/n}. \tag{37}$$

Now let

$$\delta B^{-1} \leqslant x \leqslant (2 + \delta)B^{-1}.$$

For these values of x, it is easy to verify that

$$| P_2(x)| \leqslant a_0 \mid T_n(1 + \delta)|^{-1} \leqslant 2a_0 e^{-(n/2)\delta^{1/2}} = 2a_0 n^{-2}.$$

On the otherhand, we get from (34) and (35) for all $k \geqslant 2n-1$,

$$a_k < t_n^{-k}(1 + [2 \log n(\log \log n)^2]^{-1})^{-k}. \tag{38}$$

Therefore, we get over $[\delta B^{-1}, (2+\delta)B^{-1}]$, along with (38),

$$|f(x) - P_1(x)| \leqslant \sum_{k=2n}^{\infty} a_k x^k \leqslant \sum_{k=2n}^{\infty} \left(\frac{1 + (\log n)^{-1-\epsilon}}{1 + [2 \log n(\log \log n)^2]^{-1}}\right).$$

A simple calculation gives us

$$|f(x) - P_1(x)| \leqslant \exp(-n(\log n)^{-2}).$$

Hence

$$\left| \frac{x}{f(x)} - \frac{1}{P(x)} \right| \leqslant c_{16} n^{-2} \, | a_n |^{-1/n}. \tag{39}$$

Finally we consider

$$(2+\delta)B^{-1} \leqslant x < \infty.$$

On this interval

$$P_1(x) \geqslant a_n x^n \geqslant (1 + (\log n)^{-1-\epsilon})^n$$
$$\geqslant \exp\left(\frac{n c_{17}}{(\log n)^{1+\epsilon}}\right) > n^4.$$

$P_2(x) < 0$, since $2n-1$ is odd. Therefore $x/f(x)$ and $1/|P(x)|$ are bounded by

$$c_{18} n^{-2} \, | a_n |^{-1/n}. \tag{40}$$

Result (32) follows from (37). (39), and (40).

Remark. For $f(z) = \sum_{k=0}^{\infty} a_k z^k = \exp(e^z)$,

$$\lambda_{0,n}\left(\frac{x}{f(x)}\right) \leqslant c_{19}(\log \log n) n^{-2}. \tag{41}$$

The proof of (41) is somewhat similar to the proof of Theorem 1 (except for the fact that here we use Lemma 4), that

$$a_n \sim \left[\exp\left(\frac{n}{\log n}\right)\right] (2\pi n \log n)^{-1/2} (\log n)^{-n}.$$

The rest of the details are left to the reader.

THEOREM 6. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* ρ $(0 < \rho < \infty)$ *type* τ $(0 < \tau < \infty)$. *Then for all large n and every a* $(0 < a \leqslant \ )$,

$$\lambda_{0,n}(x^a f^{-1}(x)) \geqslant c_{20}(\log n)^a n^{-2a}. \tag{42}$$

The proof of this theorem is very similar to the proof of Theorem 2; hence the details are omitted.

THEOREM 7. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* ρ $(\frac{1}{2} \leqslant \rho < \infty)$ *type* τ *and lower type* ω $(0 < \omega \leqslant \tau < \infty)$. *Then for all large* n,

$$\left\| \frac{x}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[1,\infty)} \leqslant \exp(-c_{21} n^{1/2}). \tag{43}$$

Proof. Set $P_1(x) = \sum_{k=0}^{n} a_k x^k$,

$$P_2(x) = \frac{a_0 T_n(1 + \delta - x\delta)}{T_n(1+\delta)}, \qquad n \text{ odd},$$

$$\delta = 4 \left(\frac{4\rho\tau}{n^{1/2}}\right)^{1/\rho}, \qquad xP(x) = P_1(x) - P_2(x).$$

For $x \in [1, (2+\delta)\delta^{-1}]$,

$$\begin{aligned} |P_2(x)| &\leqslant a_0 \,|\, T_n(1+\delta)|^{-1} \leqslant 2a_0 e^{-(n/2)\delta^{1/2}} \\ &\leqslant 2a_0 \exp(-(n^{1-1/4\rho})(4\rho\tau)^{1/2\rho}), \end{aligned}$$

and

$$\begin{aligned} |f(x) - P_1(x)| &\leqslant \sum_{k=n+1}^{\infty} a_k x^k \leqslant \sum_{k=n+1}^{\infty} \left(\frac{\rho e\tau(1+\epsilon)\, k^{-1}(2+\delta)^\rho n^{1/2}}{4^\rho(4\rho\tau)}\right)^{k/\rho} \\ &\leqslant \sum_{k=n+1}^{\infty} (3/4)^k \leqslant c(3/4)^n. \end{aligned}$$

Hence

$$\left| \frac{x}{f(x)} - \frac{1}{P(x)} \right| \leqslant \exp(-c_{22} n^{1-1/4\rho}). \tag{44}$$

On $[(2+\delta)\delta^{-1}, \infty)$

$$\begin{aligned} P_1(x) > P_1(\delta^{-1}(2+\delta)) &\geqslant 4^{-1} f(2\delta^{-1}) \\ &\geqslant 4^{-1} \exp\left(\frac{2^\rho n^{1/2}\omega(1-\epsilon)}{4^\rho(4\rho\tau)}\right), \end{aligned} \tag{44'}$$

and $P_2(x) < 0$.

Similarly, we can show that on $[\delta^{-1}(2+\delta), \infty)$

$$\frac{x}{f(x)} \leqslant \exp(-c_{23} n^{1/2}). \tag{45}$$

The required result (43) follows from (44), (44'), and (45).

THEOREM 8. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* ρ $(0 < \rho < \frac{1}{2})$ *type* τ *and lower type* ω $(0 < \omega \leqslant \tau < \infty)$. *Then for all large* n,

$$\left\| \frac{x}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[1,\infty)} \leqslant \exp\left(-c_{24}\left(\frac{n}{\log n}\right)^{\rho}\right). \tag{46}$$

Proof. Let $\delta = n^{-1}\log n$, n odd,

$$P_1(x) = \sum_{k=0}^{n} a_k x^k, \qquad P_2(x) = a_0 \frac{T_n(1 + \delta - x\delta)}{T_n(1 + \delta)}.$$

As usual, on $[1, (2 + \delta)\delta^{-1}]$,

$$\begin{aligned} |P_2(x)| &\leqslant a_0 \,|\, T_n(1 + \delta)|^{-1} \leqslant 2a_0 e^{-n\delta^{1/2}} \\ &\leqslant 2a_0 e^{-(n\log n)^{1/2}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |f(x) - P_1(x)| &\leqslant \sum_{k=n+1}^{\infty} a_k x^k \\ &\leqslant \sum_{k=n+1}^{\infty} \{\rho e \tau(1 + \epsilon)\, k^{-1} n^{\rho} e(\log n)^{-\rho}\}^{k/\rho} \\ &\leqslant c_{25}(\log n)^{-n}. \end{aligned}$$

Therefore on $[1, \delta^{-1}(2 + \delta)]$, we get

$$\left\| \frac{x}{f(x)} - \frac{1}{P_n(x)} \right\| \leqslant \exp(-c_{26}(n \log n)^{1/2}). \tag{47}$$

On $[(2 + \delta)\delta^{-1}, \infty)$

$$\begin{aligned} P_1(x) &\geqslant P_1((2 + \delta)\delta^{-1}) \geqslant P_1(2\delta^{-1}) \geqslant \frac{f(2n(\log n)^{-1})}{4} \\ &\geqslant \exp\left(2^{\rho}\left(\frac{n}{\log n}\right)^{\rho} \omega(1 - \epsilon)\right), \end{aligned}$$

$P_2(x) < 0$, and hence on $[(2 + \delta)\delta^{-1}, \infty)$, $1/P(x)$ and $x/f(x)$ are bounded by

$$\exp\left(-c_{27}\left(\frac{n}{\log n}\right)^{\rho}\right). \tag{48}$$

The required result (46) follows from (47) and (48).

THEOREM 9. *Let $f(z) = e^{e^z}$. Then for all large n,*

$$\left\| \frac{x}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[1,\infty)} \leqslant \exp(-n(\log n)^{-2}). \tag{49}$$

Proof. $f(x) = e^{e^x} = \sum_{k=0}^{\infty} a_k x^k$,

$$P_1(x) = \sum_{k=0}^{n} a_k x^k, \qquad P_2(x) = \frac{eT_n(1+\delta-x\delta)}{T_n(1+\delta)}, \qquad n \text{ odd}.$$

$$xP(x) = P_1(x) - P_2(x), \qquad \delta = (2+\delta)\left[\log\left(\frac{n}{(\log n)^2}\right)\right]^{-1}.$$

Then for $x \in [1, (2+\delta)\delta^{-1}]$,

$$|P_2(x)| < e\,|T_n(1+\delta)|^{-1} < 2ee^{-(n/2)\delta^{1/2}} = c_{28}\exp\left(\frac{-2^{-1}n}{[\log(n/(\log n)^2)]^{1/2}}\right)$$

and by using Lemma 4, we get

$$\begin{aligned} |e^{e^x} - P_1(x)| &\leqslant \sum_{k=n+1}^{\infty} \left\{e^{1/\log k}\log\left(\frac{n}{(\log n)^2}\right)(\log n)^{-1}\right\}^k \\ &\leqslant \sum_{k=n+1}^{\infty}\left[(1+2(\log n)^{-1})\left(1-\frac{2\log\log n}{\log n}\right)\right]^k \\ &\leqslant \sum_{k=n+1}^{\infty}\left(1-\frac{4\log\log n}{(\log n)^2}\right)^k \\ &\leqslant c_{29}\exp(-n(\log n)^{-2}). \end{aligned}$$

Hence

$$\left\| \frac{x}{f(x)} - \frac{1}{P(x)} \right\|_{L_\infty[1,(2+\delta)\delta^{-1}]} \leqslant c_{30}\exp\left(\frac{-n}{(\log n)^2}\right). \tag{50}$$

On the otherhand, for

$$x \in [(2+\delta)\delta^{-1}, \infty),$$

$$\begin{aligned} P_1(x) \geqslant P_1((2+\delta)\delta^{-1}) &\geqslant 4^{-1}\exp(e^{(2+\delta)\delta^{-1}}) \\ &\geqslant c_{31}\exp\left(\frac{n}{(\log n)^2}\right), \end{aligned} \tag{51}$$

$P_2(x) < 0$ and

$$x/f(x) \leqslant c_{32}\exp(n(\log n)^{-2}). \tag{52}$$

Hence the required result follows from (50), (51), and (52).

THEOREM 10. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, and $f(-1) \geqslant c > 0$ be an entire function of order ρ $(1 \leqslant \rho < \infty)$ type τ $(0 \leqslant \tau < \infty)$. Then for every polynomial $P_n(x)$ of degree n and all large n, there exist positive constants c_{33} and c_{34} for which*

$$\left\| \frac{x+1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \geqslant c_{33} \exp(-c_{34} n^{1-1/3\rho}). \tag{53}$$

Proof. Let us assume that

$$\left\| \frac{x+1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,n^{(\rho^{-1}-(3\rho^2)^{-1})}]} \leqslant c_{35} \exp(-c_{36} n^{1-1/3\rho}). \tag{54}$$

From (54) along with the assumption that $f(z)$ is an entire function of order ρ $(1 \leqslant \rho < \infty)$ and type τ $(0 \leqslant \tau < \infty)$, we get

$$\left| (x+1) P_n(x) - \sum_{k=0}^{n} a_k x^k \right|_{L_\infty[0,n^{(\rho^{-1}-3^{-1}\rho^{-2})}]} \leqslant c_{37} \exp(-c_{38} n^{1-1/3\rho}). \tag{55}$$

Now by applying Lemma 2 to (55) over the interval $[-1, n^{(\rho^{-1}-3^{-1}\rho^{-2})}]$ we get at $x = -1$,

$$|f(-1)| \leqslant c_{38} \exp(-c_{39} n^{1-1/3\rho}),$$

which obviously is false for all large n; hence (53) is proved.

THEOREM 11. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, and $s_n(-1) \geqslant c > 0$ be an entire function of order ρ $(1 \leqslant \rho < \infty)$. Then for every polynomial $P_n(x)$ of degree n, we have for all large n*

$$\left\| \frac{x+1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \geqslant c_{40} \exp(-c_{41} n^{1-1/3(\rho+\epsilon)}).$$

The proof of this theorem is very similar to the proof of Theorem 10; hence the details are omitted.

THEOREM 12. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order ρ $(0 < \rho < 1/5)$ and type τ $(0 \leqslant \tau < \infty)$. Then for all large n*

$$\lambda_{0,n} \left(\frac{1+x}{f(x)} \right) \geqslant c_{42} \exp(-c_{43} n^{1/2-\rho/2}). \tag{56}$$

Proof. Let us assume (56) is false. Then for a sequence of values of n,

$$\lambda_{0,n} < c_{42} \exp(-c_{43} n^{1/2-\rho/2}).$$

In other words, there exists a sequence of polynomials $P_n(x)$ for which

$$\left\| \frac{1+x}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0, n^{1/2\rho - 1/2}]} \leqslant c_{42} \exp(-c_{43} n^{1/2-\rho/2}). \tag{57}$$

From (57), we get

$$|(1+x) P_n(x) - f(x)|_{L_\infty[0, n^{1/2\rho-1/2}]} \leqslant c_{44} \exp(-c_{45} n^{1/2-\rho/2}). \tag{58}$$

As earlier, we get from (58), by using the assumption that $f(z)$ is an entire function of order ρ $(0 < \rho < 1/5)$ and type τ $(0 \leqslant \tau < \infty)$ for $0 \leqslant x \leqslant n^{1/2\rho-1/2}$,

$$\left| (1+x) P_n(x) - \sum_{k=0}^{n} a_k x^k \right| \leqslant c_{46} \exp(-c_{47} n^{1/2-\rho/2}). \tag{59}$$

Now applying Lemma 2 to (59) we get at $x = -1$,

$$| S_n(-1)| < c_{48} \exp(-c_{49} n^{1/2-\rho/2}),$$

which is false; hence (56) is established.

THEOREM 13. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$, and $S_n(-1) \geqslant c > 0$ be an entire function of order ρ $(0 < \rho < 1/5)$. Then every $\epsilon > 0$, satisfying the further assumption that $\rho + \epsilon < 1$, there exist infinitely many n for which*

$$\lambda_{0,n} \left(\frac{x+1}{f(x)} \right) \geqslant c_{50} \exp(-c_{51} n^{1/2-(\rho+\epsilon)/2}). \tag{60}$$

The proof of this theorem is very similar to the proof of Theorem 12 and hence is omitted.

THEOREM 14. *Let $f(z) = e^{e^z} = \sum_{k=0}^{\infty} a_k z^k$. Then for every polynomial $P_n(x)$ of degree at most n, we have for all large n*

$$\left\| \frac{1+x}{e^{e^x}} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \geqslant c_{52} \exp\left(\frac{-nc_{53}}{(\log n)^{1/2}} \right). \tag{61}$$

Proof. Let us assume

$$\left\| \frac{1+x}{e^{e^x}} - \frac{1}{P_n(x)} \right\|_{L_\infty[0, 2^{-1}\log n]} < c_{52} \exp\left(\frac{-nc_{53}}{(\log n)^{1/2}} \right). \tag{62}$$

From (62), we get

$$\|(1+x) P_n(x) - e^{e^x}\|_{L_\infty[0, 2^{-1}\log n]} \leqslant c_{54} \exp\left(\frac{-nc_{55}}{(\log n)^{1/2}} \right). \tag{63}$$

A simple manipulation based on (63) gives us, along with Lemma 4, for $0 \leqslant x \leqslant 2^{-1} \log n$,

$$\left\| (1+x)P(x) - \sum_{k=0}^{n} a_k x^k \right\| \leqslant c_{56} \exp\left(\frac{-nc_{57}}{(\log n)^{1/2}}\right). \tag{64}$$

Now by applying Lemma 2 to (64) over the interval $[-1, \log n/2]$, we get

$$| S_n(-1)| < c_{58} \exp\left(\frac{-nc_{59}}{(\log n)^{1/2}}\right), \tag{65}$$

(65) is obviously false, hence the result is proved.

Remark. The method adopted in proving Theorem 2 of [2] may also be used to prove (61).

THEOREM 15. *Let* $f(z) = e^{e^z} = \sum_{k=0}^{\infty} a_k z^k$. *Then for all large* n,

$$\lambda_{n,n}\left(\frac{1+x}{f(x)}\right) \geqslant \exp(-12n(\log\log n)^{-1}). \tag{66}$$

Proof. Let us assume (66) is false. Then there must exist an infinite sequence of natural numbers n for which

$$\lambda_{n,n} < \exp(-12n(\log\log n)^{-1}). \tag{67}$$

In other words, there is a sequence of rational functions $\{r_n(x)\}$ for which

$$\left\| \frac{1+x}{f(x)} - r_n(x) \right\|_{L_\infty[0,\infty)} < \exp(-12n(\log\log n)^{-1}). \tag{68}$$

Let $g(x) = (1+x)^{-1} \exp(e^x)$.

$$x = (1+t)\log n, \qquad -1 \leqslant t \leqslant 1, \qquad 0 \leqslant x \leqslant 2\log n.$$

Now set $t = -k$, $k = (\log\log\log n)(\log n)^{-1}$. Then at

$$x = x_1 = (1-k)\log n,$$

$$g(x_1) = \frac{e^{n^{1-k}}}{1+(1-k)\log n} \leqslant e^{n^{1-k}} = \exp\left(\frac{n}{\log\log n}\right). \tag{69}$$

It is easy to verify that

$$\max_{[0,x_1]} \left| \frac{1}{r_n(x)} \right| < \exp\left(\frac{n}{\log\log n} + \log n\right). \tag{70}$$

If (70) were not true, then

$$\max_{[0,x_1]} \left| \frac{1}{r_n(x)} \right| \geqslant \exp(n(\log \log n)^{-1} + \log n). \tag{71}$$

Let us assume that the maximum is attained in (71) at $x = x_2$; then from (69) and (71), we get at $x = x_2$, by noting the fact that $g(x_2) \leqslant g(x_1)$,

$$\begin{aligned} \exp\left(\frac{-12n}{\log \log n}\right) &< \exp\left(\frac{-n}{\log \log n}\right) - \exp\left(\frac{-n}{\log \log n} - \log n\right) \\ &\leqslant \frac{1}{g(x_2)} - r_n(x_2). \end{aligned} \tag{72}$$

Equation (72) clearly contradicts (68); hence (70) is valid. Now set $t = k$; then $x_3 = (1 + k) \log n$ and

$$g(x_3) \geqslant \exp(2^{-1}n^{1+k}) \geqslant \exp\left(\frac{n \log \log n}{2}\right). \tag{73}$$

But according to Lemma 3, we have

$$\min_{[x_3, 2 \log n]} \left| \frac{1}{r_n(x)} \right| \leqslant \exp\left(\frac{n}{\log \log n} + \log n + \frac{\pi^2 n}{\log 1/k}\right). \tag{74}$$

Let us suppose for $x_4 \in [x_3, 2 \log n]$, that $[r_n(x)]^{-1}$ attains minimum value. If $[r_n(x)]^{-1}$ assumes minimum value at more than one point (which is very unlikely), then we pick the one which is closest to $2 \log n$. Now we get from (73) and (74)

$$\begin{aligned} r_n(x_4) - \frac{1}{g(x_4)} &\geqslant \exp\left(\frac{-n}{\log \log n} - \log n - \frac{\pi^2 n}{\log((\log n)/\log \log n)}\right) \\ &\quad - \exp\left(\frac{-n \log \log n}{2}\right) \\ &> \exp(-12(\log \log n)^{-1}). \end{aligned}$$

This contradicts (68); hence the theorem is proved.

THEOREM 16. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* ρ $(0 < \rho < \infty)$ *type* τ *and lower type* ω $(0 < \omega \leqslant \tau < \infty)$. *Then for all large* n, *there is an* α $(0 < \alpha < 1)$, *such that*

$$\lambda_{1,n}(x/f(x)) \leqslant \alpha^n. \tag{75}$$

Proof. For $0 \leqslant x \leqslant (n/(\rho e\tau + \rho\omega))^{1/\rho}$, along with Lemma 1, we have for all large n,

$$0 \leqslant \frac{x}{\sum_{k=0}^{n} a_k x^k} - \frac{x}{f(x)} \leqslant a_0^{-2} x \sum_{k=n+1}^{\infty} a_k x^k$$

$$\leqslant a_0^{-2} x \sum_{k=n+1}^{\infty} \left(\frac{\rho e\tau(1+\epsilon)n}{k(\rho\tau e + \rho\omega)} \right)^{k/\rho}$$

$$\leqslant \alpha_1^{\,n} \qquad (0 < \alpha_1 < 1). \tag{76}$$

On the otherhand, for $x \geqslant (n(\rho\tau e + \rho\omega)^{-1})^{1/\rho}$, along with (2), we get for all large n

$$0 \leqslant \frac{x}{\sum_{k=0}^{n} a_k x^k} - \frac{x}{f(x)} \leqslant \frac{x}{\sum_{k=0}^{n} a_k x^k} \leqslant \frac{(n(\rho\tau e + \rho\omega)^{-1})^{1/\rho}}{\sum_{k=0}^{n} a_k (n(\rho\tau e + \rho\omega)^{-1})^{k/\rho}}$$

$$\leqslant 4 \left(\frac{(n(\rho\tau e + \rho\omega)^{-1})^{1/\rho}}{f((n(\rho\tau e + \rho\omega)^{-1})^{1/\rho})} \right)$$

$$\leqslant \alpha_2^{\,n} \qquad (0 < \alpha_2 < 1). \tag{77}$$

Equation (75) follows from (76) and (77).

THEOREM 17. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order ρ $(0 < \rho < \infty)$ type τ and lower type ω $(0 < \omega \leqslant \tau < \infty)$. Then for all large n, there is a β $(0 < \beta < 1)$ such that*

$$\lambda_{1,n} \left(\frac{1+x}{f(x)} \right) \leqslant \beta^n. \tag{78}$$

The proof of (78) is very similar to the proof of Theorem 16; hence the details are omitted.

THEOREM 18. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order ρ $(0 \leqslant \rho < \infty)$ type τ and lower type ω $(0 < \omega \leqslant \tau < \infty)$. Then for all large n, there is a $c_{60} > 1$, such that*

$$\lambda_{n,n}(x/f(x)) \geqslant c_{60}^{-n}. \tag{79}$$

If $f(z)$ satisfies the assumptions of Theorem 18, then with the help of Lemma 3, we have established in [12], *for a $\xi > 1$,*

$$\lambda_{n,n}(1/f(x)) \geqslant \xi^{-n}. \tag{79'}$$

The proof of (79) is very similar to (79'); hence we omit the details here.

THEOREM 19. *Let $f(x) = e^x$. Then*

$$\lim_{n\to\infty}(\lambda_{0,n}(1+x)e^{-x})^{1/(2n)^{2/3}} = 1/e. \tag{80}$$

Proof. First we get a lower bound. Now suppose for an $\epsilon > 0$,

$$\left\| \frac{1}{P(x)} - \frac{1+x}{e^x} \right\|_{L_\infty[0,\infty)} < \exp(-(1+\epsilon)(2n)^{2/3}). \tag{81}$$

Hence it must be obviously true even for $[0, (2n)^{2/3}]$. Thus on this interval

$$\frac{1}{P_n(x)} \geqslant \frac{1+x}{e^x} - \exp(-(1+\epsilon)(2n)^{2/3}) > (x+1/2)e^{-x} \geqslant \frac{1}{2e^x}. \tag{82}$$

Hence

$$\| e^x - (1+x)\, P(x) \| \leqslant 2\exp(2x - (1+\epsilon)(2n)^{2/3}). \tag{83}$$

Set

$$S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}\,;$$

then we obtain on $[0, (2n)^{2/3}]$,

$$\| S_n(x) - (1+x)\, P(x) \| \leqslant 3e^{2x} \exp(-(1+\epsilon)(2n)^{2/3}). \tag{84}$$

Now by using Lemma 6 to (84), we get

$$\tfrac{1}{3} \leqslant | S_n(-1) | \leqslant 3\exp((2n)^{2/3} + 4(2n)^{1/3} - (1+\epsilon)(2n)^{2/3}), \tag{85}$$

which is false for each $\epsilon > 0$ and all large n. Hence for all large n,

$$\lambda_{0,n}((1+x)e^{-x}) \geqslant \exp(-(1+\epsilon)(2n)^{2/3}). \tag{86}$$

Now we get an upper bound. Let us assume n odd and set

$$t(x) = T_n\left(\frac{2x}{(2n)^{2/3}} - 1\right),$$

$$P(x) = \frac{S_n(x) + ct(x)}{(1+x)}.$$

We choose c such that $P(x)$ is a polynomial,

$$c = \frac{S_n(-1)}{T_n(1 + 2/(2n)^{2/3})} \approx \exp(-1 - (2n)^{2/3}).$$

Therefore, we have

$$\left\| \frac{1}{P(x)} - \frac{(1+x)}{e^x} \right\| = (x+1) \left| \frac{e^x - S_n(x)}{e^x(S_n(x) + ct(x))} - \frac{ct(x)}{e^x(S_n(x) + ct(x))} \right|. \tag{87}$$

For $0 \leqslant x \leqslant (2n)^{2/3}$,

$$\left\| \frac{1}{P(x)} - \frac{x+1}{e^x} \right\| \leqslant (x+1) \frac{e^x - S_n(x)}{e^x S_n(x)} + \frac{c}{1-c} \leqslant 2c. \tag{88}$$

For $x > (2n)^{2/3}$,

$$\begin{aligned} \left\| \frac{1}{P(x)} - \frac{x+1}{e^x} \right\| &\leqslant (x+1) \left(\frac{e^x - S_n(x)}{e^x S_n(x)} + e^{-x} \right) \\ &\leqslant \frac{(x+1)}{S_n(x)} \leqslant 6c(2n)^{2/3}. \end{aligned} \tag{89}$$

Hence the result

$$\left\| \frac{1}{P(x)} - \frac{1+x}{e^x} \right\|_{L_\infty[0,\infty)} \leqslant 6 \exp(-1 - (2n)^{2/3})(2n)^{2/3} \tag{90}$$

follows from (88) and (89). Our result (80) follows from (86) and (90).

THEOREM 20. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of zero order satisfying the further assumptions that*

$$1 < \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = A + 1 < \infty$$

and

$$0 < \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log M(r)}{(\log r)^{A+1}} = \begin{matrix}\tau_1 \\ \omega_1\end{matrix} < \infty. \tag{91}$$

Then there exists a sequence of polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ for which for all large n,

$$\left\| \frac{x}{f(x)} - \frac{1}{Q_n(x)} \right\|_{L_\infty[0,\infty)} \leqslant \exp(c_{61}(\log n)^{1/A+1} - 2\log n). \tag{92}$$

Proof. As earlier, for $0 \leqslant x < \infty$,

$$\begin{aligned} &\left\| \frac{1}{f(x)} - \frac{1}{S_n(x)\,P_n(x)} \right\| \\ &\qquad \leqslant \left\| \frac{1}{f(x)} - \frac{1}{f(x)\,P_n(x)} \right\| + \left\| \frac{1}{f(x)} - \frac{1}{S_n(x)} \right\| \frac{1}{P_n(x)}. \end{aligned} \tag{93}$$

Let $0 \leqslant x \leqslant \exp(4\omega^{-1} \log n)^{1/A+1}$; then by Lemma 5, we have

$$\left\| \frac{x}{f(x)} - \frac{1}{f(x)\, P_n(x)} \right\| \leqslant \frac{1}{f(x)} \left\| x - \frac{1}{P_n(x)} \right\| \leqslant \frac{a_0^{-1} \exp(c_{62} \log n)^{1/A+1}}{n^2}. \tag{94}$$

For all sufficiently large n and $x > \exp(4\omega^{-1} \log n)^{1/A+1}$ we get by using (91) and the fact that $\{P_n(x)\}^{-1} \leqslant 2x$,

$$\begin{aligned} \frac{1}{f(x)} \left\| x - \frac{1}{P_n(x)} \right\| &\leqslant \exp(-(\log x)^{A+1} \omega_l (1 - \epsilon)) \\ &\leqslant \exp\left(-\frac{(\log x)^{A+1} \omega_l}{2}\right) \leqslant n^{-2}. \end{aligned} \tag{95}$$

Similarly we can show very easily by using the known fact [13, p. 499] that

$$\limsup_{n\to\infty} \frac{n^{A+1}}{[\log |1/a_n|]^A} = \tau_l \frac{(A+1)^{A+1}}{A^A}.$$

$$\frac{1}{P_n(x)} \left\| \frac{1}{f(x)} - \frac{1}{S_n(x)} \right\|_{L_\infty[0,\infty)} \leqslant \exp(c_{63}(\log n)^{1/A+1} - 2 \log n). \tag{96}$$

Hence the result (92) follows from (94), (95) and (96).

THEOREM 21. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function satisfying the assumptions of Theorem 20. Then for every polynomial $Q_n(x)$ of degree n, and all large n,*

$$\left\| \frac{x}{f(x)} - \frac{1}{Q_n(x)} \right\|_{L_\infty[0,\infty)} \geqslant (16)^{-1} \exp\left(\left(\frac{\log n}{2\tau}\right)^{1/A+1} - 3 \log n\right) = \frac{\psi(n)}{16n^3}. \tag{97}$$

Proof. Let us assume, on the contrary, that

$$\left\| \frac{x}{f(x)} - \frac{1}{Q_n(x)} \right\|_{L_\infty[0,\infty)} < \frac{\psi(n)}{16n^3}. \tag{98}$$

Then for

$$\exp\left(\left(\frac{\log n}{2\tau}\right)^{1/A+1} - 2 \log n\right) \leqslant x \leqslant \exp\left(\left(\frac{\log n}{2\tau}\right)^{1/A+1}\right) = \psi(n),$$

we get from (98) for all large n,

$$\frac{1}{Q_n(x)} \geqslant \frac{x}{f(x)} - \frac{\psi(n)}{n^3} \geqslant \frac{\psi(n)}{n^3} - \frac{\psi(n)}{16n^3} \geqslant \frac{\psi(n)}{2n^3}.$$

i.e.,

$$\max_{[n^{-2}\psi(n),\, \psi(n)]} |Q_n(x)| \leqslant 2n^3[\psi(n)]^{-1}. \tag{99}$$

By applying Lemma 2 to (99) we obtain

$$|Q_n(0)| \leqslant 15n^3[\psi(n)]^{-1} = 15n^3 \exp\left(-\left(\frac{\log n}{2\tau}\right)^{1/A+1}\right). \tag{100}$$

On the otherhand, we have by (98)

$$16n^3 \exp\left(-\left(\frac{\log n}{2\tau}\right)^{1/A+1}\right) \leqslant |Q_n(0)|. \tag{101}$$

Clearly (100) and (101) contradict (98); hence (97) is valid.

THEOREM 22. *Let $f(z)$ satisfy the assumptions of Theorem 20. Then there exist a sequence of rational functions $r_n(x)$ of degree at most n for which for all large n,*

$$\left\| \frac{x}{f(x)} - r_n(x) \right\|_{L_\infty[0,\infty)} \leqslant \exp(-c_{66}n^{1+1/A}). \tag{102}$$

Proof. Choose

$$r_n(x) = x/S_n(x),$$

where $S_n(x)$ as usual denotes the nth partial sum of $f(x)$. Then by adopting the proof used in [13, Theorem 7′], we get the required result (102).

CONCLUDING REMARKS

It is interesting to note that

$$\lim_{n\to\infty}[\lambda_{0,n}(xe^{-x})]^{1/2\log n} = e^{-1} = \lim_{n\to\infty}[\lambda_{0,n}((1+x)e^{-x})]^{1/(2n)^{2/3}}.$$

From a comparison of Theorems 3 and 16, it is obvious that rational functions of degree n approximate certain functions much better then reciprocals of polynomials of degree n. Similarly, Theorems 10 and 17 give us the same information. By comparing Theorems 9 and 15, one easily notes that for $f(z) = \exp(e^z)$, there is little difference between the errors obtained by rational functions and the errors obtained by reciprocals of polynomials.

One can see very easily from Theorems 20, 21, and 22 that one can approximate certain class functions better by rational functions than by reciprocals of polynomials on $[0, \infty)$.

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