PROBLEMS AND RESULTS IN GRAPH THEORY
AND COMBINATORIAL ANALYSIS

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Résumé. — Dans cet article nous donnons des résultats sur divers problèmes d'analyse combinatoire et nous en proposons de nouveaux.

In this short paper I mainly discuss the fate of some of the older problems and state only a few new ones. At a recent meeting in Aberdeen, Scotland, I have a paper of the same title to which I will refer as [1]. Many old and new problems are stated there, also there are extensive references to my older problem papers and other relevant literature. First I discuss problems mentioned in [1].

Denote by $\chi(G)$ the chromatic number of G. An old problem of Hajnal and myself states ([1], p. 173). Is there a function $f_r(k)$ so that every G with

$$K(G) \geqslant f_r(k)$$

contains a subgraph G' which contains no C_l, $3 \leqslant l \leqslant r$ and for which $\chi(G') \geqslant k$? (C_l is a circuit of l edges).

Rödl just proved that $f_3(k)$ exists. It would be very desirable to obtain good upper and lower bounds for $f_3(k)$. Rödl's upper bound is probably poor and the theorem of Graver and Yackel gives

$$f_3(k) > ck^2 \log \log k / \log k .$$

In particular it would be interesting to know if $f_3(k) > k^l$ for every l if $k > k_0(l)$.

At present nothing is known if $r > 3$. Both Rödl and I thought that the following graph often used by Hajnal and myself is a candidate for a counterexample. The vertices of G are the pair of integers (a, b), $a < b$. (a, b) is joined to (c, d) if and only if $a < b = c < d$. This graph contains no C_3 and has $\chi(G) = \aleph_0$. Does it have a subgraph of large chromatic number without C_4 ? It is easy to see that every induced (or spanned) subgraph of it has chromatic number $\leqslant 3$.

Hajnal and I further conjectured that if $m \geqslant \aleph_0$ and $\chi(G) = m$ then G has a subgraph G' with $\chi(G') = m$ and G' contains no C_3. Rödl's theorem gives this for $m = \aleph_0$, but for $m > \aleph_0$ the problem remains open. Observe that it is not true that there is a subgraph G' without C_4 and $\chi(G') > \aleph_0$, since by a theorem of

Hajnal and myself if $\chi(G) = \aleph_1$ then G contains a C_4 and in fact it contains for every n a $\chi(n ; \aleph_1)$ (i.e. a complete bipartite graph of n white and $\aleph_1$ black vertices). (See P. Erdös and A. Hajnal, Chromatic number of graphs and set systems, *Acta Math. Acad. Sci.* **17** (1966) 61-99.)

Galvin and Rödl conjectured (independently) that if $\chi(G) = \aleph_0$ then either G contains a $K(n)$ (complete graph of n vertices) for every n or G contains an induced or spanned subgraph G_1 of G for which $\chi(G_1) = \aleph_0$ and G_1 contains no C_3.

I conjectured that there is a function $g(n)$, $g(n) \to \infty$ as n tends to infinity so that if $K(G) \geqslant n$ then G has a subgraph G' so that all but $o(2^{e(G')})$ subgraphs of G' have chromatic number $> g(n)$ where $e(G')$ is the number of edges of G'. In other words : every graph of large chromatic number has a subgraph almost all subgraphs of which have a large chromatic number. Unfortunately I cannot decide about the truth of this conjecture. Rödl independently formulated the following stronger conjecture : There is a $g_\varepsilon(n) \to \infty$ for every $\varepsilon > 0$ as $n \to \infty$ so that if $\chi(G) \geqslant n$ one can associate non negative numbers n_i, $1 \leqslant i \leqslant e(G)$ to the edges of G so that if for the edges of a subgraph G' we have $\sum n_i > \varepsilon$ then with probability tending to 1 (as $e(G) \to \infty$) $\chi(G') > g_\varepsilon(n)$.

Problem 8, p. 175 of [1] stated : Is it true that every graph of girth greater than four can be directed in such a way that it contains no directed circuit and if one reverses the direction of any one of its edges the resulting new digraph should also not contain a directed circuit ?

Nesetril and Rödl showed that the answer is negative ; in fact they showed that for every r there is a graph G_r of girth r (i.e. the shortest circuit of G_r has r edges) so that if we direct the edges of our G in an arbitrary way there always is either a directed circuit or a circuit which becomes directed after reversal of the direction of one of its edges.

Nesetril, Rödl and I formulated the following

strengthening of our original problem with Hajnal : Does there exist for every k and n a graph G with the property that every two subgraphs $K(n)$ of G have at most two vertices in common and for every coloring of the edges of G by k colors these always is a monochromatic $K(n)$? Nesetril and Rödl recently proved this and also extended it to r-graphs. The problem is stated in : P. Erdös, Problems and results on finite and infinite graphs, *Recent advances in graph theory*, *Proc. Symp.* Prague (1974), Academia Prague, p. 186, see also J. Spencer, Restricted Ramsey Configurations, *J. Comb. Theory* ser. **A 19** (1975) 278-286.

In the Prague paper (p. 189) I asked : Let G be a graph with $e(G) = m$ which contains no triangle. Is it true that G contains a bipartite subgraph G_1 with

$$e(G_1) > \frac{m}{2} + m^{1/2} f(m)$$

where $f(m)$ tends to infinity with m.

Lovàsz and I proved that there always is such a bipartite subgraph with

$$e(G_1) > \frac{m}{2} + m^{2/3} f(m), \qquad f(m) \to \infty$$

but that $\frac{2}{3}$ cannot be replaced by $1 - \alpha$ if α is sufficiently small. The best value of α is not known to us (see also problem 6 of I).

Problem 16 (of [1]). Giraud pointed it out to me that I made an oversight in formulating this problem. I repeat the definitions. Let $K^{(r)}(n)$ be the complete hypergraph of n vertices and $\binom{n}{r}$ edges. $f_r(n; l)$ is the smallest integer so that every subgraph of $K^{(r)}(n)$ having $f_r(n; l)$ edges contains a $K^{(r)}(l)$. The determination of $f_r(n; l)$ for $r > 2$ is an old and probably very difficult problem of Turan. $f_2(n; l)$ was as is well known determined by him in 1940, not much progress has been made with $f_r(n; l)$ for $r > 2$, though Turan has some well known and plausible conjectures. It is easy to see that

$$\lim_{n = \infty} f_r(n; l) \Big/ \binom{n}{r} = \alpha(r; l)$$

exists for every r and l. $\alpha(2, l) = 1 - \frac{1}{l}$ is an old result of Turan and he conjectures $\alpha(3, 4) = \frac{5}{9}$. ($\alpha(r, l)$ is not known for $l > r > 2$.)

In 16 [1] I asked the following (foolish) question : Let $G^{(r)}(n; m)$ be any r-graph of n vertices and m edges. Could there be a constant $\beta(r; l) < \alpha(r; l)$ so that every subgraph of $G^{(r)}(n; m)$ having more than $\beta(r; l) m$ edges contains a $K^r(l)$? Giraud by a simple averaging argument showed that the answer is no for every r and l.

In particular I considered the following

$$G^{(3)}(2n; n^2(n - 1)).$$

The vertices are $X_1, ..., X_n$; $Y_1, ..., Y_n$. The edges (X_i, X_j, Y_l) and (Y_i, Y_j, X_l). Is it true that every subgraph having more than $(\frac{1}{2} + \varepsilon) n^2(n - 1)$ edges contains a $K^{(3)}(4)$? Giraud showed that this is nonsense. Consider the triples

$$\{ X_i, X_j, Y_l \} \quad \text{and} \quad \{ X_l, Y_i, Y_j \},$$

$$1 \leqslant i < j < l \leqslant n.$$

It is easy to see that this system intersects every $K^3(4)$. Thus the complementary system does not contain a $K^3(4)$ and the number of its edges is $(\frac{2}{3} + \sigma(1)) n^3$. It is possible that this system is the maximal one without containing a $K^{(3)}(4)$

A family $\{ A_k \}$ of sets is called a strong Δ system if the intersection of any two of these sets is the same i.e. the intersection of any two of them equals the intersection of all of them. An old conjecture of Rado and myself states (3 of [1], p. 171) : Denote by $F(n; r)$ the smallest integer so that if

$$\{ A_k \}, \qquad 1 \leqslant k \leqslant F(n; r), \qquad |A_k| = n$$

is any family of sets then there are always r of them $A_{k_1}, ..., A_{k_r}$ which form a strong Δ system. We conjectured

$$F(n; r) < C_r^n. \tag{1}$$

I offer 500 dollars for a proof or disproof of (1). I am satisfied with a proof for $r = 3$ which probably contains the whole difficulty. J. Spencer recently proved that

$$\lim_{n = \infty} (F(n; r)/n !)^{1/n} \leqslant 1. \tag{2}$$

I would be very glad to see a proof of

$$F(n; r) < n ! \quad \text{for} \quad n > n_0.$$

For further details see [1] and also : P. Erdös and R. Rado, Intersection theorems for systems of sets, I and II, *J. London Math. Soc.* **35** (1960) 85-90 and **44** (1969) 467-479 ; P. Erdös, E. Milner and R. Rado, Intersection theorems for systems of sets III, *J. Australian Math. Soc.* **18** (1974) 22-40.

Now I state a few new problems. Hajnal, Szemerédi and I conjectured that there is an $\varepsilon > 0$ so that if $G(n)$ is an edge critical four chromatic graph of n vertices then it is not the union of a bipartite graph and εn edges. (Dirac calls an r-chromatic graph edge critical if the omission of any of its edges makes it $(r - 1)$-chromatic.)

It is surprising that we could not get anywhere with this seemingly so simple conjecture. In fact let

$$f(n) \to \infty$$

as slowly as we please. We could not prove that $G(n)$ is not the union of a bipartite graph and $\varepsilon f(n)$ edges.

I wanted to characterize the four chromatic edge

critical graphs which are the union of a bipartite graph and disjoint edges. $K(4)$ has this property but Gyarfas found a few other cases and we did not succeed in getting a complete solution of the problem.

I conjectured and Jean Larson proved that if $\chi(G(n)) = 4$ and $G(n)$ does not contain a $K(4)$ then $G(n)$ has an odd circuit with a diagonal. (Her proof will appear in the Journal of Combinatorial theory soon.) I further conjecture that G must contain an odd circuit with two diagonals. The pentagonal wheel shows that if true this is best possible. I am fairly sure though that if the girth of G is large then it must contain an odd circuit with many diagonals. If G has no triangle and chromatic number 4 it probably has an odd circuit with more than two diagonals.

The following pretty problem is due to Nelson : Let G be a graph whose vertices are the points of the plane. Join two points if their distance is one. Prove or disprove : The chromatic number of this graph is 4. It is known to be between 4 and 7. Let the vertices of G be a subset of the plane which does not contain an equilateral triangle of sides 1. Join two points whose distance is 1. Is the chromatic number $\leqslant 3$?

Ringel and Hammer show that if the minimum distance between the vertices of G is one then the chromatic number of G is $\leqslant 4$ and a four chromatic graph of 11 vertices is constructed (*Amer. Math. Monthly* June (1976) 485-86), but all these graphs contain equilateral triangles.

References

[1] P. ERDÖS, Problems and results in graph theory and combinatorial analysis, in : *Proc. of the 5th British Combinatorial Conference Aberdeen* (1975), Utilitas Math. Publ., Congressus Numerantium XV, 169-192.