SYSTEMS OF FINITE SETS HAVING A COMMON INTERSECTION

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1. Introduction.

We say that a collection of finite sets has a common intersection of size t provided that the intersection of each pair of these sets is equal to the intersection of all of them and this intersection has exactly t elements. A family of sets with this property is therefore a strong Δ-system [6;7]. Denote by f(n;r,k,t) the smallest integer with the property that if F is any family of subsets each of size r of a set of size n, and if $|F| > f(n;r,k,t)$, then some k members of F have a common intersection of size t.

The values of f for $t = 0$ or for $r = 2$ can be deduced from various well known results. A few theorems and conjectures have also appeared dealing with certain cases in which $k = 2$ and $t \geq 1$. It appears, however, that almost no attention has been given to any of the cases where $k > 2$ and $t \geq 1$. Here we investigate the general behavior of f(n;r,k,t) for large n, obtaining specific values or bounds for certain r and t and proposing a conjecture about the size of f for all $r \geq 2$, $k \geq 2$, and $0 \leq t \leq r-1$.

2. Older Results.

The value of f(n;r,k,0) is the size of the largest possible collection of r-sets, no k of which are pairwise disjoint, that can be chosen from a set of size n. As such, the value of f(n;r,2,0) for $r \geq 2$ and $n \geq 2r$ can be deduced directly from the Erdös-Ko-Rado Theorem [5]. A generalization of that theorem [3] states that for each $r \geq 2$ there exists a constant c(r) such that

$$(1) \qquad f(n;r,k,0) = \binom{n}{r} - \binom{n-k-1}{r} \text{ for } n > c(r)k.$$

For $r = 2$ this result is contained in older theorems [4] dealing with sets of independent edges in graphs. The values $f(n;2,k,1) = \left[\frac{(k-1)n}{2}\right]$ can also be obtained from well known graph-theoretic results.

3. r = 3; Families of Triples.

The first result we mention for $r = 3$ is due to Brown, Erdös, and Sós [1]. It concerns conditions for the existence of a pair of triples having exactly

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two elements in common, and states that

$$\lim_{n\to\infty} n^{-2} f(n;3,2,2) = 1/6. \tag{2}$$

The lower bounds for f in this case follow from the existence of Steiner triple systems for $n \equiv 1$ or 3 (mod 6). By using collections of disjoint triple systems, (2) has been generalized [2] yielding the following for a family of k triples having a common intersection of size 2.

$$\lim_{n\to\infty} n^{-2} f(n;3,k,2) = \frac{k-1}{6}. \tag{3}$$

For pairs of triples having exactly one element in common the next result was obtained by Erdös and Sós [1] and independently in [2].

$$f(n;3,2,1) = \begin{cases} n-2 & \text{for } n \equiv 2 \text{ or } 3 \pmod 4 \\ n-1 & \text{for } n \equiv 1 \pmod 4 \\ n & \text{for } n \equiv 0 \pmod 4. \end{cases} \tag{4}$$

It was the question of determining the correct analogue of (4) for $k > 2$ which led to the present work. The following construction yields a lower bound for $f(n;3,k,1)$ for all $k \geq 2$. Given an n-element set S and an odd integer k, we form a graph G consisting of two disjoint copies of the complete graph K_k and having all of its vertices in S. Let F be the collection of all triples which can be obtained by taking the union of an edge of G with an element of S not in G. F contains $k(k-1)(n-2k)$ such triples, no k of which have a common intersection of size 1. For k even we replace G by a graph consisting of one copy of K_k and one copy of K_{k-1}. It follows that

$$f(n;3,k,1) \geq \begin{cases} k(k-1)(n-2k) & \text{for } k \text{ odd} \\ (k-1)^2(n-2k+1) & \text{for } k \text{ even.} \end{cases} \tag{5}$$

Peter Frankl has pointed out that the bound can be improved somewhat when k is even by taking G to be a graph on $2k-1$ vertices having degree sequence $k-1, k-1, \ldots, k-1, k-2$.

In the other direction one can show that for each $k \geq 2$ there exist constants $c(k)$ and $n_o(k)$ such that

$$f(n;3,k,1) \leq c(k)n \qquad \text{for } n > n_o(k). \tag{6}$$

This result can be derived from the proof of the theorem given in the next section, and can also be established by an argument along the following lines. Let F be a family of triples chosen from a set S with $|S| = n$ and $|F| > c(k)n$. We may assume that the elements of S which are contained in few triples of F (say less than $\frac{1}{2}c(k)$) have been deleted. Each element of S therefore has a large "valence" with respect to the triples of F. Either there exist k triples of F having a common intersection of size 1 or each element of S is contained in a pair which in turn is contained in many members of F. In the latter case there exist many such pairs of high valence, so either k such pairs share a common element or there exists a large collection of these pairs which are independent. In either event the result follows.

Frankl has now given an argument involving Δ-systems of edges in graphs which shows that $f(n;3,k,1) < (5/3)k(k-1)n$ for sufficiently large n. He also informs us that he has been able to use this technique to show that $f(n;3,3,1) = 6(n-6) + 2$ for $n > 54$, thus settling in the affirmative a conjecture which we made just a few months ago.

4. r = 4; Families of Quadruples.

Let F be a family of quadruples chosen from a set S. By the link of an element x in S we mean the collection of all triples whose union with x yields a member of F. If $|S| = n$ and $|F| \geq cn^2$, for a given constant c, then there exists an element of S whose link contains at least cn triples. It follows from (6) that there exist constants $c_1(k)$ and $n_1(k)$ such that

$$(7) \qquad f(n;4,k,2) \leq c_1(k)n^2 \qquad \text{for } n > n_1(k).$$

This argument can also be used in conjunction with (3) to show that there exist $c_2(k)$ and $n_2(k)$ such that

$$(8) \qquad f(n;4,k,3) \leq c_2(k)n^3 \qquad \text{for } n > n_2(k).$$

As in the case of (3), lower bounds can be obtained in some of these cases from the study of designs. The first, for $t = 2$, comes from a result on disjoint pairwise balanced designs due to Poucher [10]. The second, for $t = 3$ and $k = 2$, is from a result of Hanani [8] concerning sparse designs.

$$(9) \qquad f(n;4,k,2) \geq c_3(k)n^2 \qquad \text{for } n > n_3(k).$$

(10) $f(n;4,2,3) \geq c_4 n^3$ for $n > n_4$.

Katona (unpublished) proved that $f(n;4,2,1) = \binom{n-2}{2}$, for n sufficiently large. A lower bound for $f(n;4,k,1)$ for all $k \geq 2$ can be obtained by using the following inequality which holds for all $r \geq 2$ and $0 \leq t \leq r-1$ and sufficiently large n.

(11) $f(n;r,k,t) \geq \binom{n-t-1}{r-t-1}$.

This inequality, which has been used many times elsewhere, can be seen as follows. Let S be a set with $|S| = n$, $A \subseteq S$, and $|A| = t+1$. If F is the family of all r-element subsets of S which contain A, then no k members of F have a common intersection of size exactly t.

It follows that there are constants $c_5(k)$ and $n_5(k)$ such that

(12) $f(n;4,k,1) \geq c_5(k)n^2$ for $n > n_5(k)$.

An upper bound is given by the following general result.

<u>Theorem</u>. There exist constants $c_6(k)$ and $n_6(k)$ such that when $k \geq 2$ we have

(13) $f(n;4,k,1) \leq c_6(k)n^2$ for $n > n_6(k)$.

<u>Proof</u>. Suppose F is a family of quadruples chosen from a set S with $|S| = n$, $|F| \geq c(k)n^2$. Let x be an element of S for which the valence $v(x)$ (that is, the number of quadruples of F containing x) is as large as possible. If $v(x) \geq \frac{1}{2}(k-1)n^2$, then by (1) the link of x contains at least k disjoint triples and the result follows. Thus we may assume that $v(x) = \alpha n$, where $\alpha \leq \frac{1}{2}(k-1)n$, and that no collection of mutually disjoint triples in the link of x has more than k-1 members. It follows that some element y of S is contained in at least $\frac{\alpha n}{3(k-1)}$ triples which are in the link of x. The pair $\{x,y\}$ is then contained in at least $\frac{\alpha n}{3(k-1)}$ quadruples of F. Hence there is a triple $\{x,y,z\}$ which is contained in at least $\frac{\alpha}{3(k-1)}$ members of F.

Delete x, y, and z and those quadruples of F which contain one or more of these three elements. At most $3\alpha n$ quadruples are thus removed, and at least $\frac{\alpha}{3(k-1)}$ of these quadruples contain the triple $\{x,y,z\}$.

For c(k) sufficiently large we can repeat this procedure (at least k times) until $\frac{1}{2}c(k)n^2$ quadruples have been deleted. Note that on the average at least one out of every $9(k-1)n$ quadruples which were removed contained one of the triples which was deleted, and that these triples were mutually disjoint. It follows that if $c(k) \geq 18k(k-1)$, then some element of S forms a quadruple of F with each of at least k of these triples, and this completes the proof.

5. The General Case.

The proof of the theorem in the last section can be modified to yield the following for all $k \geq 2$ and $r \geq 3$.

There exist constants $c(r)$ and $n(k,r)$ such that

$$(14) \qquad f(n;r,k,1) \leq c(r)k(k-1)n^{r-2} \qquad \text{for } n > n(k,r).$$

The bounds given in (11) and (14) for $t = 1$, $r \geq 3$ and those obtained for the remaining cases when $r = 2,3$, or 4 suggest the following which we conjecture to be true for all $k \geq 2$, $r \geq 2$, and $0 \leq t \leq r-1$.

Conjecture. There exist constants $c_1(k,r)$ and $c_2(k,r)$ such that for all sufficiently large n we have

$$(15) \qquad c_1(k,r)n^{\max(r-t-1,t)} \leq f(n;r,k,t) \leq c_2(k,r)n^{\max(r-t-1,t)}.$$

P. Frankl has just recently informed us that he has been able to establish (15) for all $r \leq 8$. He further states that he has obtained $f(n;r,k,t) \leq c(k,r)n^{r-t-1}$ for $[r/3] > t$, which, together with (11), establishes the conjecture for these values as well.

It would also be interesting to know whether k always enters as a multiplicative constant. That is, does there always exist a constant $c(k)$ such that $f(n;r,k,t) < c(k)f(n;r,2,t)$?

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