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STRONGLY ANNULAR FUNCTIONS WITH SMALL COEFFICIENTS, AND RELATED RESULTS

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ABSTRACT. A technique of Bagemihl and Seidel is applied to two problems in annular functions. It is shown that there exists a strongly annular function with Maclaurin coefficients tending to zero, and that there exist annular functions that are far from being strongly annular.

0. Introduction. We show that there is a function

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \qquad |z| < 1, \tag{0.1}$$

such that

$$\lim_{\nu \to \infty} a_\nu = 0 \tag{0.2}$$

and

$$\sup_{0<r<1} \min_{|z|=r} |f(z)| = \infty. \tag{0.3}$$

A function f, holomorphic in the unit disk D (briefly, $f \in \mathcal{H}(D)$), is said to be *strongly annular* if (0.3) holds; an f in $\mathcal{H}(D)$ is *annular* if

$$\lim_{n \to \infty} \min\{|f(z)|: z \in J_n\} = \infty \tag{0.4}$$

for some sequence of Jordan curves J_n in D with 0 in their interiors. An example of an annular function for which (0.2) holds was known previously [**4**, p. 100], [**2**, p. 21].

While it is known that not every annular function is strongly annular [**3**], one might speculate that every annular function enjoys some of the special properties of the strongly annular functions. For example, given an annular function f, can the $\{J_n\}$ satisfying (0.4) always be chosen so that the sequence of lengths $l(J_n)$ remains bounded? Can the $\{J_n\}$ be chosen so that the ratio of the distances to $|z| = 1$ from the closest and farthest points of J_n is bounded away from zero as n increases? In §2, we construct a counterexample to these conjectures.

Both constructions make use of a technique of Bagemihl and Seidel [**1**, pp. 188–190].

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1. Strongly annular functions with small Maclaurin coefficients. Let $H(D)$ be provided with the topology of uniform convergence on compact subsets of D. We use the methods of [1] to obtain the following lemma.

LEMMA. *Let $\mathfrak{B}$ be a family of functions holomorphic in the closed unit disk. Suppose that, given any number $M > 0$ and any neighborhood $\mathfrak{N}$ of 0 in $H(D)$, there is a function g in $\mathfrak{B} \cap \mathfrak{N}$ such that $|g(z)| > M$ on $|z| = 1$. Then there is a sequence $\{f_k\}$ in $\mathfrak{B}$ such that the function*

$$f(z) = \sum_{k=1}^{\infty} f_k(z), \qquad |z| < 1, \tag{1.1}$$

is strongly annular.

PROOF. Choose f_1 in $\mathfrak{B}$ so that $|f_1(z)| > 1$ for $|z| = 1$, then choose r_1, $0 < r_1 < 1$, so close to 1 that the inequality holds on $|z| = r_1$. Next, choose f_2 in $\mathfrak{B}$, so that: (i) $|f_2(z)| < 2^{-2}$ in $|z| \leqslant r_1$ and $|f_1(z) + f_2(z)| > 1$ on $|z| = r_1$, and (ii) $|f_2(z)| > 2 + |f_1(z)|$ on $|z| = 1$. Choose r_2, $r_1 < r_2 < 1$, so that the last inequality continues to hold on $|z| = r_2$. Continue choosing the functions f_k and the numbers r_k, inductively, in the obvious way.

THEOREM 1. *There exists a strongly annular function* (0.1) *such that* (0.2) *holds. More explicitly, f is of the form* (1.1), *each f_k being a polynomial; the coefficients are small and noninterfering*:

$$\begin{aligned} &\text{(i)} \quad f_k(z) = \sum_{\nu} \alpha(k, \nu) z^{\nu}, \\ &\text{(ii)} \quad |\alpha(k, \nu)| \leqslant 1/k, \\ &\text{(iii)} \quad \alpha(k, \nu)\alpha(j, \nu) = 0 \quad \text{for } \nu = 0, 1, \ldots, \text{ whenever } j \neq k. \end{aligned} \tag{1.2}$$

Let δ be the operator on the set of nonconstant complex polynomials defined by

$$(\delta P)(z) = P(z)P(z^{d+1}), \quad d = \text{degree of } P, \tag{1.3}$$

and let $\delta^p = \delta(\delta^{p-1})$, $p = 2, 3 \ldots$. We consider the particular polynomial $Q(z) = 1 - z + z^2 + z^3 + z^4$. One may verify that we have

$$|Q(e^{i\theta})|^2 = 5 + 2\cos 2\theta + 2\cos 4\theta = \frac{11}{4} + 4\left(\cos 2\theta + \tfrac{1}{4}\right)^2,$$

so that the minimum modulus of $Q(z)$ on $|z| = 1$ is $\mu = \sqrt{11}/2 \doteq 1.66$. It is clear from (1.3) that the coefficients of $\delta^p Q$ are ± 1, and that its minimum modulus on $|z| = 1$ is at least $\exp(2^p \log \mu)$.

DEFINITION. $\mathfrak{B}_1$ is the sequence of polynomials

$$g_p(z) = p^{-1} z^{m(p)} \delta^p Q(z), \qquad p = 1, 2, \ldots,$$

where $m(1) = 0$ and $m(p+1)$ is one greater than the degree of g_p.

PROOF OF THEOREM 1. Clearly the g_p have small, noninterfering coefficients in the sense of (1.2). For $|z| \leqslant r$, we have

$$|g_p(z)| \leqslant r^{m(p)} p^{-1} (1-r)^{-1},$$

while on $|z| = 1$, we have

$$|g_p(z)| > p^{-1}\exp(2^p \log \mu).$$

Hence, the sequence $\mathfrak{B}_1$ satisfies the hypotheses of the Lemma, and we can extract a subsequence $f_k = g_{p(k)}$ such that (1.1) is strongly annular.

2. Functions far from strongly annular.

THEOREM 2. *There exists an annular function f with the following property: If $\{J_n\}$ is any sequence of Jordan curves about* 0 *in D for which* (0.4) *holds, then $l(J_n)$ approaches infinity as n increases.*

PROOF. Choose $0 < r_1 < r_2 < \cdots < 1$. For each n, form a closed Jordan curve I_n in D which coincides with $|z| = r_n$ in the left semidisk, while in the right semidisk it is a perturbation of $|z| = r_n$ by a sinusoidal function of large frequency and small amplitude. These are chosen so that $l(I_n)$ is greater than n and I_n lies in the interior of I_{n+1}. The set $N(n, \varepsilon_n)$ of points of D that lie less than ε_n from I_n is open, and we may choose ε_n so small that for each Jordan curve J about 0 that lies in $N(n, \varepsilon_n)$, we have $l(J) > n$. We require further that $N(n, \varepsilon_n) \cap N(n + 1, \varepsilon_{n+1})$ is empty.

For $n = 1, 2, \ldots,$ we define a compact set K_n. It is the portion of the region between I_n and I_{n+1} that lies in the closed right semidisk and meets neither $N(n, \varepsilon_n)$ nor $N(n + 1, \varepsilon_{n+1})$. The set K_n does not disconnect the plane.

Let $f_1(z) = 2$ for all z. Suppose that, for some $n \geq 1$, we have found an entire function f_n such that

$|f_n(z)| > j$ for all z on I_j and for $j = 1, \ldots, n$,

$|f_n(z)| < 1$ for all z in $\bigcup_{j=1}^{n-1} K_j$.

We then define an entire function $\eta_n(z)$ that has small modulus on I_n (and hence in I_n), approximates $-f_n(z)$ on K_n, and has large modulus on I_{n+1}; such a function exists (cf. Remark 1). We choose the tolerances so that we have

$|f_n(z) + \eta_n(z)| > j$ for all z on $I_j, j = 1, \ldots, n + 1$,

$|f_n(z) + \eta_n(z)| < 1$ for z in $\bigcup_{j=1}^{n} K_j$,

and so that, if $f_{n+1} = f_n + \eta_n$, the sequence $\{f_n\}$ converges almost uniformly in the unit disk. The limit function f is annular, and has modulus at most 1 on $\bigcup K_j$. Hence, each sequence $\{J_n\}$ for which (0.4) holds meets only finitely many of the K_j, so that the lengths $l(J_n)$ must grow without bound.

REMARK 1. We add a few words about the existence of η_n. Take $g_n(z) = 0$ on J_n and its interior and $g_n(z) = -f_n(z)$ on K_n. By Runge's theorem, some entire function h_n approximates g_n on these two sets. Let ψ_n be the continuous extension of a conformal map of the interior of J_{n+1} onto $|w| < 1$, and let M be a number larger than $n + 1 + \max\{|f_n(z) + h_n(z)|\colon |z| \leq 1\}$. For k sufficiently large, the function $M\psi_n^k$ has modulus M on J_{n+1} but is small on $J_n \cup K_n$. Approximate $M\psi_n^k$ by an entire function φ_n, and take $\eta_n = h_n + \varphi_n$.

REMARK 2. Instead of the sequence $|z| = r_n$, one may use a sequence

$|z - a_n| = r_n$, with r_n increasing to 1 and a_n decreasing to zero, so that the circles do not intersect, and so that

$$\lim_{r\to\infty} \frac{1-(r_n+a_n)}{1-(r_n-a_n)} = 0.$$

If the ε_n are taken small enough, the construction will give a function f that is far from strongly annular in an additional sense. That is, for each sequence $\{J_n\}$ for which (0.4) holds, the ratio of the distances to $|z| = 1$ from the closest point of J_n and from the farthest point approaches zero.

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