SUBGRAPHS WITH ALL COLOURS IN A LINE-COLOURED GRAPH

C. C. Chen, D. E. Daykin and P. Erdös
Reading University

1. INTRODUCTION. Let p, q, r be positive integers with $r \geq 3$ and $2r \geq q \geq r$. We shall denote by $g(p, r, q)$ (resp. $f(p, r, q)$) the least integer $\ell \geq 0$ such that, whenever a graph (resp. complete graph) on p points has each of its lines coloured with one of $r-1$ colours (resp. r colours) in such a way that every colour is on more than ℓ lines, then the graph has a subgraph on $\leq q$ points which contains all the colours. If no such ℓ exists we put $g(p, r, q) = \infty$ (resp. $f(p, r, q) = \infty$). We will prove

Theorem 1. Given $\eta < 1$ for p sufficiently large $f(p,r,r) > \frac{\eta}{r}\binom{p}{2}$. In particular $f(p, 3, 3) = \infty$ for $p \geq 5$.

Theorem 2. $f(p, r, 2r-2) = 0$ for $r \geq 3$.

Theorem 3. $f(p, r, 2r-3) = \binom{\alpha}{2}$ for $r \geq 4$ where $\alpha = \left[\frac{p}{r-1}\right]$.

Theorem 4. $\binom{\alpha}{2} \leq f(p, r, 2r-4) < \binom{\alpha+1}{2}$ for $r \geq 5$ where $\alpha = [\lambda]$ and

$$\lambda = \frac{r-1 + \sqrt{(r-1)^2 + 4(r^2-4r+5)(p^2-p)}}{2(r^2-4r+5)}.$$

Theorem 5. We always have $f \geq g$. In particular we can replace f by g in theorems 2, 3, 4.

Theorem 6. For $p \geq 1$ we have $g(p, 3, 3) = \binom{[\frac{1}{2}p]}{2}$ and $\binom{\alpha}{2} \leq g(p, 4, 4) < \binom{\alpha+1}{2}$ where $\alpha = [\mu]$ and

$$\mu = \frac{3 + \sqrt{9 + 20(p^2-p)}}{10}.$$

Notice that theorem 1 really says there is no sensible theorem for $f(p, r, r)$, but theorem 6 is one for $g(p, r, r)$. This is the only difference that we could find between f and g. Some asymptotic results on f and g are given in section 4. In section 5 we give a best possible theorem on polychromatic trails.

PROC. 5TH BRITISH COMBINATORIAL
CONF. 1975, pp. 101-112.

2. <u>LOWER BOUNDS FOR</u> f <u>AND</u> g . These can be easily evaluated from

<u>Example 1</u>. We partition the complete graph K_p into $r-1$ disjoint subgraphs $V_1, V_2, \ldots, V_{r-1}$. Let $c_1, c_2, \ldots, c_r$ denote r colours and for $1 \leqslant i \leqslant r-1$ colour all lines in V_i with c_i and write U_i for the union of all the points in $V_i, V_{i+1}, \ldots, V_{r-1}$. Now choose an integer z in $0 \leqslant z \leqslant r-3$. For $1 \leqslant i \leqslant z$ we colour every line from a point in V_i to a point in U_{i+1} by c_i . Then for $z < i < j \leqslant r-1$ we colour every line from V_i to V_j with c_r . The resulting graph contains a subgraph of $2r-z-2$ points carrying all r colours, but no smaller one does. Hence if we find the least number of lines of one colour we have a bound for f . If we delete all lines colour c_r , then in the same way we get a bound for g. So let the number of points in U_i and V_i be u_i and v_i respectively, giving $v_1+v_2+ \ldots +v_{r-1} = p$. When the v_i are chosen to yield the largest possible v such that $v_{z+1} = v_{z+2} = \ldots = v_{r-1} = v$ and

$$\binom{v_i}{2} + v_i u_{i+1} \geqslant \binom{v}{2} \quad \text{for } 1 \leqslant i \leqslant z$$

we get the lower bound $\binom{v}{2}$ for both $f(p, r, 2r-z-3)$ and $g(p, r, 2r-z-3)$. We conjecture that f and g are close to this bound for large p .

When the z is chosen to be odd $\geqslant 3$ we can modify example 1 as follows. We colour the V_i and U_{z+1} as before. Then for $1 \leqslant i \leqslant z$ a line from V_i to $V_{i+1} \cup V_{i+2} \cup \ldots \cup V_{i+\frac{1}{2}(z-1)} \cup U_{z+1}$ is colour c_i where suffices are reduced modulo z . We now like $v_1, v_2, \ldots, v_z$ to be nearly equal and $v_{z+1}, v_{z+2}, \ldots, v_{r-1}$ to be nearly equal when calculating bounds.

<u>Example 2</u>. This is like example 1 except we have an extra set V_r and $z=r-1$. No r points carry all r colours.

Suitably choosing the v_i in example 2 proves the first part of theorem 1, but we leave the construction for the second part as an exercise.

Suppose some graph gives a lower bound $\leqslant \frac{1}{r}\binom{p}{2}$ for g . Then by deleting superfluous lines and adding lines of a new colour we get the same bound for f.

3. THE PROOFS OF THEOREMS 2-6. We start by introducing our terminology.

(i) For any positive integers p, r, N we let $G_p(r, N)$ denote a graph on p points whose lines are coloured by $c_1, c_2, \ldots, c_r$ in such a way that each colour c_i is one more than N lines. Also S_i denotes the set of points on at least one line colour c_i .

(ii) The colour of a line xy will be denoted by $c(xy)$.

(iii) For each point x we let $N(x)$ denote the set of all points adjacent to x and $\rho(x)$ be the number of colours in the set $\{c(xy) | y \in N(x)\}$.

(iv) We put $\rho(G) = \mathrm{Sup}\{\rho(x) | x \in G\}$.

(v) Any path abc of length 2 will be called an elbow at b .

(vi) A subgraph H of G is called polychromatic if it contains all colours in G .

(vii) Let S be a set and x an element of S . We shall denote the set of all elements of S other than x by $S-x$.

Lemma 1. *If $r > n \geq 2$ then in each $K_p(r, 0)$ with $\rho(K_p) \geq n$ there is a polychromatic subgraph on $\leq 2r-n$ points.*

Proof. There is a point x on n colours and $r-n$ lines of the other colours. If these r lines do not form the subgraph we get it by considering lines on x .

Proof of theorem 2. Apply lemma 1 with $n = 2$.

Proof of theorem 3. Assume the theorem is false for a certain $K_p(r, \binom{\alpha}{2})$. By lemma 1 with $n = 3$ we must have $\rho(K_p) = 2$. Choose an elbow $y_1x_2y_2$ with $c_1 = c(y_1x_2)$ and $c_2 = c(x_2y_2)$ say. For $i = 3, 4, \ldots, r$ choose a line x_iy_i coloured c_i . Then $x_2 \neq x_3$ and $c(x_2x_3)$ is c_1 or c_2 , say the former. Let $S = \{x_2, \ldots, x_r, y_2, \ldots, y_r\}$ so $|S| = 2r-2$.

Claim 1. $c(x_ix_j) = c(y_iy_j) = c(x_iy_j) = c_1$ for all $i \neq j$ in $2 \leq i, j \leq r$. For example $c(x_2y_3)$ is c_1 or c_2 because $\rho(K_p)=2$, if it was c_2 then $S-y_2$ is polychromatic of size $2r-3$ a contradiction.

<u>Claim 2</u>. If $3 \le i \le r$ and $c(ab) = c_i$ for some line ab then $c(x_2a) = c_1$. Indeed if $c(x_2a) = c_2$ then $(S-\{y_2, x_i, y_i\}) \cup \{a, b\}$ is polychromatic, a contradiction.

<u>Claim 3</u>. $S_2 \cap S_3 = \phi$. Indeed if $a \in S_2 \cap S_3$ then $c(ab_2) = c_2$ and $c(ab_3) = c_3$ for some $b_2, b_3 \in N(a)$. By claim 2 we have $c(x_2a) = c_1$ so $\rho(K_p) > 2$, a contradiction. Similarly we have $S_i \cap S_j = \phi$ for $2 \le i < j \le r$, so $|S_i| \le p/(r-1)$ for some i and the proof is complete.

<u>Proof of theorem 6</u>. The first part is trivial, so assume that a given $G_p(3, \binom{\alpha+1}{2}-1)$ does not contain a polychromatic subgraph of size ≤ 4 . Then $\rho(G) < 3$ and there does not exist a path $abcd$ in G such that $c(ab), c(bc), c(cd)$ are all distinct. Thus G can be partitioned into six sets $T_1, T_{12}, T_2, T_{23}, T_3, T_{31}$, where T_{ij} is the set of points on both colour c_i and c_j , but T_i is the set of points on c_i only. Let the size of the T's be a, b, c, d, e, h as shown in figure 1. Of course some sets or lines may not be in G , but the figure indicates every possibility. Notice that $a+b+h \ge \alpha+1$ for otherwise we would have $<\binom{\alpha+1}{2}$ lines coloured c_1 . Similarly $b+c+d \ge \alpha+1$ and $d+e+h \ge \alpha+1$.

<u>Claim 1</u>. The number m of lines missing from G is less than $(\alpha+1)^2$. Indeed, suppose on the contrary that $m \ge (\alpha+1)^2$. Then $m \ge \mu^2$ and since μ is the positive root of $5\mu^2-3\mu = p^2-p$ the number of lines in G is at most

$$\binom{p}{2}-m \le \binom{p}{2}-\mu^2 = \tfrac{1}{2}(5\mu^2-3\mu)-\mu^2 = \tfrac{3}{2}\mu(\mu-1) < 3\binom{\alpha+1}{2}$$

This contradicts our hypothesis that each colour in G occurs on at least $\binom{\alpha+1}{2}$ lines.

<u>Claim 2</u>. $b > e, d > a, h > c$. Indeed if $b \le e$ say, then $b^2 \le be$, $ba \le ea, bc \le ec$. Hence using our earlier remarks

$$\begin{aligned}(\alpha+1)^2 \le (a+b+h)(b+c+d) &= ab+ac+ad+b^2+bc+bd+hb+hc+hd \\ &\le ae+ac+ad+be+ec+bd+hb+hc+hd = m\end{aligned}$$

a contradiction to claim 1.

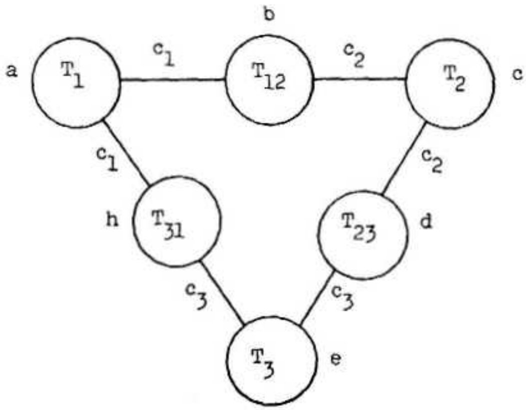

Figure 1

Now since the total number of lines of G is at least $3\binom{\alpha+1}{2}$, one of a+b+c, c+d+e, e+h+a is at least $\alpha+1$, say the first. But then $(\alpha+1)^2 \le (a+b+c)(d+e+h) < m$ because $0 < (d-a)(h-c)$, contradicting claim 1. This proves the right hand inequality for $g(p, 4, 4)$, and example 1 provides the other one.

Proof of theorem 4. The constant λ is the positive solution of the simultaneous equations

(1) $$p = (r-2)\lambda+\beta \quad \text{and} \quad \tfrac{1}{2}\lambda(\lambda-1) = (r-2)\beta\lambda + \tfrac{1}{2}\beta(\beta-1) .$$

Hence the left hand inequality comes from example 1.

Assume that a given $K_p(r, \binom{\alpha+1}{2}-1)$ does not contain a polychromatic subgraph of size $\le 2r-4$. By lemma 1 we have $\rho(K_p) < 4$. We shall next show that $\rho(K_p) < 3$. Indeed if $\rho(K_p) = 3$ then there exists a point x in K_p and $y_1, y_2, y_3 \in N(x)$ such that $c(xy_1)$, $c(xy_2)$, $c(xy_3)$ are distinct (say $= c_1, c_2, c_3$ respectively). For $4 \le i \le r$ choose a line x_iy_i coloured c_i. Then $c(xx_4)$ is c_1, c_2 or c_3, say c_1. Let $S = \{x, x_4, \ldots, x_r, y_2, \ldots, y_r\}$. Then by similar argument as those used in the proof of theorem 3, we have:

Claim 1. $c(xx_i) = c(xy_i) = c_1$ for $i = 4, 5, \ldots, r$.

Claim 2. If $4 \le i \le r$ and $c(ab) = c_i$ for some line ab then $c(xa) = c_1$.

Claim 3. There is no elbow abc such that $c(ab) = c_i$, $c(bc) = c_j$ with $4 \le i < j \le r$.

Claim 4. If $4 \le i < j \le r$ and ab, cd are lines coloured c_i, c_j respectively then $c(ac) = c_1$.

Claim 5. There is no path of the form $abcd$ with $\{c(ab), c(bc), c(cd)\} = \{c_2, c_3, c_i\}$ where $i \ge 4$.

Now by claim 3 we have $S_i \cap S_j = \phi$ for $4 \le i < j \le r$. Also $S_3 \cap (S_4 \cup S_5 \cup \dots \cup S_r) \ne \phi$ for otherwise we would have less than $\binom{\alpha+1}{2}$ lines coloured c_2 , since by claim 5 all lines from S_3 to S_i with $i \ge 4$ are not coloured c_2 . Similarly we have $S_2 \cap (S_4 \cup \dots \cup S_r) \ne \phi$. Let us assume without loss of generality that $S_2 \cap S_4 \ne \phi$.

Claim 6. $S_3 \cap S_i = \phi$ for $5 \le i \le r$. Indeed if false for $i = 5$ say choose elbows abc, def such that $c(ab) = c_2$, $c(bc) = c_4$, $c(de) = c_3$, $c(ef) = c_5$. Then $(S-\{x,y_2,y_3,y_4,y_5,x_4,x_5\}) \cup \{a,b,c,d,e,f\}$ is a polychromatic subgraph of size $\le 2r-4$, a contradiction.

Claim 7. $S_3 \cap S_i = \phi = S_2 \cup S_i$ for $5 \le i \le r$. This follows from claim 6 by symmetry.

Next consider the subgraph $G = S_2 \cup S_3 \cup S_4$ of K_p with all lines colour c_1 omitted therefrom. Then G carries only three colours and if it has w points, by (1) we have

(2) $$w \le p-(r-4)(\alpha+1) \le p-(r-4)\lambda = 2\lambda+\beta .$$

Now from example 1 the μ of theorem 6 is the solution of

(3) $$w = 2\mu+\gamma \quad \text{and} \quad \tfrac{1}{2}\mu(\mu-1) = 2\gamma\mu+\tfrac{1}{2}\gamma(\gamma-1) .$$

Since $r \ge 5$ comparison of (1), (2), (3) shows that $\mu+1 < \lambda$, or in other words that each colour occurs more than $g(w, 4, 4)$ times in G . Hence by theorem 6 there is a polychromatic subgraph H of size ≤ 4 in G . Then $(S-\{x,y_2,y_3,y_4,x_4\}) \cup H$ is a polychromatic subgraph of size

$\leq 2r-4$ of K_p . By this contradiction we have so far proved that $\rho(K_p) < 3$, so it must be 2 .

There must be an elbow in K_p . Suppose it has colours c_1 and c_2 . Since $\rho(K_p) = 2$ we can not have another elbow colour c_3 and c_4 . In general for $3 \leq i < j \leq r$ we have $S_i \cap S_j = \emptyset$ and further, all lines from S_i to S_j are the same colour c_1 say. The last paragraph now gives a contradiction which this time completes the proof.

Proof of theorem 5. Write f for $f(p, r, q)$ and let $G_p(r-1, f)$ be given. The case $f = \infty$ is trivial so assume $f < \infty$. Remove lines arbitrarily until colour c_i occurs exactly f times for $1 \leq i \leq r-1$. Let m be the number of lines missing from G . Then $m = \binom{p}{2}-(r-1)f \geq f$ since obviously $\frac{1}{r}\binom{p}{2} \geq f$. Add to G all the missing lines and colour them c_1 . This gives a $K_p(r, f)$ which by definition of f contains a polychromatic subgraph H of size $\leq q$. Evidently H , as a subgraph of G , is also polychromatic. This proves that $f \geq g$, and the remainder of the theorem follows by taking suitable versions of example 1.

4. SOME ASYMPTOTIC RESULTS. We will need

Lemma 2. Let $\eta \geq 2$ be an integer and $G_p(r; \binom{\alpha}{2})$ be given with $\alpha = \left[\frac{p}{\eta}\right]$. Then no η of the S_i's are pairwise disjoint.

Proof. By hypothesis we have $|S_i| > \alpha$ for $1 \leq i \leq r$. If $S_1, S_2, \ldots, S_\eta$ say are disjoint then $p \geq |S_1| + |S_2| + \ldots + |S_\eta| \geq \eta(\alpha+1) > p$, a contradiction.

Theorem 7. If $r \geq 2t \geq 2$ then $g(p, r, 2r-t-2) \leq \binom{\alpha}{2}$ where $\alpha = \left[\frac{p}{r-2t+1}\right]$.

Proof. Given $G_p(r-1, \binom{\alpha}{2})$ choose the maximum possible number s of elbows such that the $2s$ lines involved carry $2s$ distinct colours. We have $s \geq t$ for otherwise by applying lemma 2 with $\eta = r-2t+1$ to the remaining colours we can increase s by 1. These elbows together with one line for each of the remaining colours form a polychromatic subgraph on $\leq r+2t-2$ points as required.

Theorem 8. *If* $r \geq 5$ *then* $g(p, r, 2r-5) \leq \binom{\alpha}{2}$ *where* $\alpha = \left[\frac{p}{r-4}\right]$, *provided* p *is sufficiently large.*

Proof. Assume the theorem is false for a certain $G_p(r-1, \binom{\alpha}{2})$.

Claim 1. No three elbows carry six colours. Otherwise by choosing a line for each of the other $r-7$ colours we get all colours on $\leq 2r-5$ points, a contradiction.

As in the proof of the last theorem we can get a pair of elbows carrying say c_1, c_2 and c_3, c_4 respectively. Then $S_5, \ldots, S_{r-1}$ are pairwise disjoint by claim 1. Using lemma 2 shows that $S_1 \cap (S_5 \cup \ldots \cup S_{r-1}) \neq \phi$ so $S_1 \cap S_5 \neq \phi$ say. Similarly there is a j in $5 \leq j \leq r-1$ with $S_3 \cap S_j \neq \phi$, and there is a largest i in $5 \leq i \leq r-1$ with $S_2 \cap S_i \neq \phi$. We can not have $i > 5$ as we would then have elbows coloured c_3, c_4 and c_1, c_5 and c_2, c_i contradicting claim 1. By using symmetry claims 2 and 3 below follow easily.

Claim 2. For $i = 1, 2$ the sets S_i, S_k are disjoint for $6 \leq k \leq r-1$, but $S_1 \cap S_5 \neq \phi$ and $S_2 \cap S_5 \neq \phi$.

Claim 3. For $i = 3, 4$ the sets S_i, S_k are disjoint for $5 \leq k \leq r-1$, $k \neq j$, but $S_3 \cap S_j \neq \phi$ and $S_4 \cap S_j \neq \phi$.

Case $j = 5$. Here since $|S_i| \geq \alpha+1 > p/(r-4)$ we have

$$w = |S_1 \cup \ldots \cup S_5| = p - |S_6| - \ldots - |S_{r-1}| < p-(r-6)p/(r-4) = 2p/(r-4)$$

and hence

$$\frac{5}{2}\left(\frac{p}{r-4} - 1\right)\left(\frac{p}{r-4} - 2\right) < 5\binom{\alpha}{2} \leq \binom{w}{2} < \frac{1}{2}\left(\frac{2p}{r-4}\right)\left(\frac{2p}{r-4} - 1\right)$$

which is impossible for large p.

Case $j > 5$ say $j = 6$. In this case

$$w' = |S_1 \cup \ldots \cup S_6| = p - |S_7| - \ldots - |S_{r-1}| < p-(r-7)p/(r-4) = 3p/(r-4) .$$

Now if $Q = S_1 \cup S_2 \cup S_5$ and $R = S_3 \cup S_4 \cup S_6$ by claim 1 we must have $Q \cap R = \emptyset$. Hence we may assume $|R| \le \frac{1}{2}w' < 3p/2(r-4)$ and so

$$\frac{3}{2}\left(\frac{p}{r-4} -1\right)\left(\frac{p}{r-4} -2\right) < 3\binom{\alpha}{2} \le \frac{1}{2}\left(\frac{3p}{2(r-4)}\right)\left(\frac{3p}{2(r-4)} -1\right)$$

which is also impossible for p large, completing the proof.

Conjecture. *If* $0 \le z \le r-3$ *then* $f = g(p, r, 2r-z-2) \sim \binom{\alpha}{2}$ *where* $\alpha = \left[\frac{p}{r-z-1}\right]$, *provided* r *is sufficiently large*.

This conjecture says that when r is large the sets $V_1, V_2, \ldots, V_z$ of example 1 are small. With a little extra work g can be replaced by f in theorems 7 with $r > t+2$ and 8 with $r > 5$. Hence the conjecture is true when z is 0, 1, 2, 3 by theorems 2, 3, 7, 8 respectively. It does not extend to $z = r-2$ by our earlier remarks on theorems 1 and 6. For $2 \le z \le r-3$ if the conjecture is true it is best possible by example 1. We feel it would be the most important result in this subject.

5. POLYCHROMATIC TRAILS. We shall call a sequence T of lines $a_1a_2, a_2a_3, \ldots, a_na_{n+1}$ a *trail* if all lines are distinct. The number n is called the *length* of the trail T.

Lemma 3. *Let* G *be a connected subgraph of* K_p *with* $r \ge 1$ *lines. If* $p > \frac{1}{3}r+2$ *there exists a trail* T *of length* $<\frac{3}{2}r$ *containing all edges of* G. *Moreover* T *contains at most one more arbitrarily chosen point* a *than* G *and if* b, c *are the end points of* T *the lines* ab, ac *are not in* T *and* $a \ne b,c$.

Proof. Let $d_1, d_2, \ldots, d_v$ be the degree sequence of G with d_i odd for $1 \le i \le \alpha$ and d_i even for $\alpha < i \le v$. If $\alpha = 0$ then G has an Eulerian circuit C, and if $\alpha = 2$ then G has an Eulerian trail D. We can then put $T = C$ or D in these cases. Since α is always even, we may assume that $\alpha \ge 4$. If the line joining two points of odd degree of G is not in G, we adjoin it to reduce the number of odd degree points. Repeat this as often as possible to get a graph H with degree sequence $e_1, e_2, \ldots, e_v$ with $e_i = d_i$ for $\alpha < i \le v$ and by renumbering the points, there is a $\beta \ge 2$ such that $e_i = d_i$ for $1 \le i \le \beta$ and $e_i = d_i+1$ for $\beta < i \le \alpha$. From our construction, if $\beta \ge 4$ any two points of H of odd degree are adjacent in H, and in fact in G. Hence we can remove $\binom{\beta-1}{2}$ lines from among these β points and still leave G connected. Since in a connected graph the number of points is at most the number of lines plus one, we have

$$(1) \qquad v \le r - \binom{\beta-1}{2} + 1 \quad \text{for} \quad \beta \ge 2 .$$

If $\beta = 2$, then H has an Eulerian trail T and the number of lines is $r + \frac{1}{2}(\alpha-\beta) \le r + \frac{1}{2}(v-\beta) = r + \frac{1}{2}v - 1 < \frac{3}{2}r$ by (1). Now suppose that $\beta \ge 4$. If a point of H of even degree is not joined to two points of odd degree in H, we join it to reduce the number of odd points by 2. We repeat as often as possible to get a graph I with γ odd points $2 \le \gamma \le \beta$. If $\gamma = 2$, then I contains an Eulerian trail T and at most $r + \frac{1}{2}(\alpha-\beta) + (\beta-\gamma)$ lines. But (1) yields

$$r + \tfrac{1}{2}(\alpha-\beta) + (\beta-\gamma) \le r + \tfrac{1}{2}(v-\beta) + (\beta-2) \le \tfrac{3}{2}r ,$$

with strict inequality if $\beta > 4$. However if $\beta = 4$ inspection of the graph shows we have strict inequality in (1). Hence we get the required result when $\gamma = 2$ and now assume $\gamma \ge 4$. Then in I each point of even degree is adjacent to at least $\gamma-1$ points of odd degree so

$$(2) \qquad r \ge \binom{\gamma}{2} + (v-\gamma)(\gamma-1) .$$

Now $r \ge \binom{\gamma}{2} \ge \frac{3}{2}(\gamma-1)$ and hence $\frac{1}{3}r+2 \ge \frac{r}{\gamma-1} + \frac{\gamma}{2} \ge v$ where the last inequality is (2). Thus there exists a point a of K_p not in G. We join $\gamma-2$ of the odd degree points of I to a, obtaining a graph J with two odd points different from a. Thus there is an Eulerian trail T of J of length at most

$$r + \tfrac{1}{2}(\alpha-\beta) + (\beta-\gamma) + (\gamma-2) \le r + \tfrac{1}{2}(v-\beta) + (\beta-2) < \tfrac{3}{2}r ,$$

completing the proof.

For lemma 3 the example where G is a star shows that $\frac{3}{2}r$ cannot be reduced. Also we need $p > \frac{1}{3}r+2$ because any graph with three vertices having maximum possible degree $p-1$ odd has no T.

Theorem 9. Given $K_p(r, 0)$ with $r \geq 3$ and $p > \frac{1}{3}r+2$ there is a polychromatic trail of length $\leq 2r-3$.

Proof. Let F be a subgraph of K_p consisting of one line of each colour. If each connected component of F is a path the desired trail is easily constructed. Hence suppose G is a connected component of F with s lines which is not a path. Then G has a vertex of degree greater than 2 and $s \geq 3$. If $s = 3$ we easily get all the colours of G in a trail of 3 or 4 different colour lines. If $s > 3$ by lemma 3 we can embed G in a trail T of $<\frac{3}{2}s$ lines. Thus in every case we can embed G in a trail T, with end points b, c say, containing t colours and $\leq 2t-3$ lines. Moreover the t colours include all the s colours of G. We may have a point $a \in T \setminus G$ but $a \neq b,c$ and the lines ab, $ac \notin T$.

Now if $F \setminus T$ is empty we have finished, so suppose it has a connected component G' with s' lines. If G' is not a path, by the method of the last paragraph, we can embed G' in a trail T' with ends b',c' containing t' colours and $\leq 2t'-1$ lines. This weaker result $\leq 2t'-1$ holds also if G' is a trail. We repeat the process on $F \setminus (T \cup T')$ to get T'' and so on. Note carefully that if in applying lemma 3 we need a new point we will always choose the end b of T. We now adjoin the lines cb', $c'b''$, $c''b'''$, ... and this connects $T, T', \ldots$ into the required trail. If we needed a point a in the first paragraph it does not matter where it happens to lie. We have $t+t' + \ldots = r$ and $t \leq 2t-3$ and $t+t'+1 \leq (2t-3) + (2t'-1) + 1 = 2(t+t') - 3$ and so on, so the trail has the correct length.

The example where $r-1$ colours each appear on only one line shows the theorem is best possible. It is clearly stronger than theorem 2.

6. <u>A CLASS OF PROBLEMS</u>. Consider a graph G whose lines or points or both have been coloured with single colours or sets of colours. It is natural to look for global and local conditions on the colouring which ensure that a particular kind or class of coloured graphs exist as subgraphs of G . This paper deals with one such problem, others appear in [1, 2, 3], and there must be many more interesting ones. The conditions on the colouring place restrictions on the histogram for the number of lines of each colour for G and for each point of G .

References

1. D. E. Daykin, Graphs with cycles having adjacent lines different colours, J. Combinatorial Theory (to appear).
2. C. C. Chen and D. E. Daykin, Graphs with Hamiltonian cycles having adjacent lines different colours, J. Combinatorial Theory (to appear).
3. C. C. Chen and D. E. Daykin, Hamiltonian cycles in a line coloured multigraph and a related uncoloured graph, Nanta Math. (to appear).