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SOME RECENT PROBLEMS AND RESULTS IN GRAPH THEORY, COMBINATORICS AND NUMBER THEORY.

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I have written many papers of a similar title during my very long life and to avoid repetitions as much as possible I will restrict myself to recent problems. As is usual in my papers of this sort the choice of problems is purely subjective; I include a problem if it seems interesting to me and if I spent some time thinking about it.

I

In the first chapter I discuss some problems in graph theory. I start with some questions on chromatic graphs.

In October 1975 Rival, in a lecture at the University of Calgary, stated the following interesting problem: Let L be a finite lattice with elements $x_1,\dots,x_n$. Join x_i and x_j by an edge if $x_i < x_j$ but there is no element x with $x_i < x < x_j$. Prove that the chromatic number of this graph is less than four.

Early in March 1976 Bollobás wrote me that he disproved this conjecture and in fact found lattices for which the chromatic number of the corresponding graph is arbitrarily large.

When I heard the problem of Rival the following question occured to me. Let G be a graph which contains no $k(4)$ ($k(n)$ denotes the complete graph of n vertices) and no odd circuit with a diagonal. Prove that G has chromatic number <4. At first sight this seems trivial but I could not prove it.

Bollobás and I in trying to prove this came to the following problem: Is there a finite graph which has no cut point, no $k(4)$, no odd circuit with a diagonal, each vertex of valency ≥ 3, having an odd circuit? We thought that such a graph does not exist, and this would imply my

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conjecture.

Jean Larson proved in February 1976 that our conjectures are true. Her proof will appear soon.

Perhaps every four chromatic graph without a $k(4)$ has an odd circuit with two diagonals. The pentagonal wheel shows that three diagonals are not needed. On the other hand probably the following result holds: There is an $f(r) \to \infty$ as $r \to \infty$ so that every four chromatic graph which has no four chromatic subgraph of fewer than r vertices contains an odd circuit with at least $f(r)$ diagonals.

A. Hajnal, E. Szemeredi and I recently formulated the following conjecture: Is it true that there is an absolute constant $c > 0$ with the following properties? Let $G(n)$ be a graph of n vertices. Assume that for every $m \leq n$ every subgraph $G(m)$ of $G(n)$ is the union of a bipartite graph and a graph of fewer than cm edges. Then if c is a sufficiently small absolute constant, $G(n)$ has chromatic number ≤ 3. If this conjecture is proved (we are convinced that it is true) one should try to determine the best value of c.

More generally the following result should be true: There is a $c_r > 0$ so that if every subgraph $G(m)$ of $G(n)$ $(1 \leq m \leq n)$ is the union of an r-chromatic graph and a graph of fewer than $c_r m$ edges, then $G(n)$ has chromatic number not exceeding $r + 1$.

Now I discuss some very recent problems in generalised Ramsey theory. In a forthcoming paper with Faudree, Rousseau and Schelp we study among others the following question: Denote by $\hat{r}(G)$ the smallest integer for which there is a graph H having $\hat{r}(G)$ edges so that if we color the edges of H by two colors then there is a monochromatic copy of G in one of the colors. Perhaps the most interesting unsolved problem is: Determine or estimate $\hat{r}(P_n)$ where P_n is a path of n edges. We can not even prove: If $n \to \infty$, then

$$(1) \qquad \frac{\hat{r}(P_n)}{n} \to \infty \quad \text{and} \quad \frac{\hat{r}(P_n)}{n^2} \to 0$$

In fact we are really not even sure whether (1) holds. Another rather annoying problem is that we could not prove that

$$(2) \qquad \lim_{n\to\infty} \hat{r}(nG)/n = c(G)$$

holds, where $\hat{r}(nG)$ is the smallest integer for which there is a graph of $\hat{r}(nG)$ edges with the property that if we color the edges by two colors at least one of the colors always contains n vertex disjoint copies of G. On the other hand in a recent paper Burr, Spencer and I prove that

$$(3) \qquad \hat{r}(nG) = (2\ v\ (G) - I(G))\ n + 0(1)$$

where $v(G)$ denotes the number of vertices and $I(G)$ the cardinality of the largest independent set of G. $r(nG)$ is the smallest integer m such that, if we colour the edges of $k(m)$ with two colors, at least one color contains n vertex disjoint monochromatic copies of G (A. Burr, P. Erdös and J. Spencer, Ramsey theorems for multiple copies of graphs, Trans. Amer. Math. Soc. 209(1975), 87-99.)

Burr, Faudree, Schelp and I further conjecture that $r(G_{3n+1}) = 4n + 1$, where G_{3n+1} has the vertices $x;\ y_1,\dots,y_n;\ z_1,\dots,z_n;\ w_1,\dots,w_n$, where x is joined to the y's and z's and y_i is joined to w_i $(1 \le i \le n)$.

We know that our conjecture would follow if we could show that every graph of $4n + 1$ vertices, each vertex of which has valency $\ge 2n$, contains our G_{3n+1}. Further perhaps every

$$G(4n + 1;\ \binom{3n}{2} + \binom{n+1}{2} + 1)$$

contains our G_{3n+1} ($G(k;\ell)$ denotes a graph of k vertices and ℓ edges).

Denote by C_m a circuit of m edges. Several months ago Burr and I investigated the following question: Let $m \equiv a \pmod{b}$ be a congruence which contains infinitely many even numbers. Is it then true that there is a constant $c_{a,b}$ so that every $G(n;[c_{a,b}n])$ contains a C_m, $m \equiv a \pmod{b}$? $c_{0,2} = \frac{3}{2}$ is easy. Burr and I showed that $c_{2,b} \leq b + 2$ and Robertson showed that $c_{0,b}$ exists for every b. It is rather annoying that we have not been able to prove that $c_{1,3}$ exists.

The proof of $c_{2,b} \leq b + 2$ uses an idea of Czipszer and is easy. First of all we can assume that the valency of every vertex is $\geq b + 2$ (for if not, use induction). Take now a longest path $(x_1,\ldots,x_t)$ of our graph ((x_i,x_{i+1}) are edges of the path). Since x_1 has valency $\geq b + 2$ there are at least $b + 1$ edges (x_1,y_j), $y_j \neq x_2$. But since our path $(x_1,\ldots,x_t)$ was longest all the y_j must in fact be x's say $x_{i_1},\ldots, x_{i_{b+1}}$. Two of these indices are congruent mod b which gives a C_m, $m \equiv 2 \pmod{b}$. $c_{2,b} \leq b + 2$ could be improved, but it is not clear how to get the best possible value of $c_{2,b}$.

The proof of Robertson for the existence of $c_{0,b}$ is much deeper. A result of Mader states: Let $g(n;k)$ be the smallest integer so that every $G(n; g(n;k))$ contains a topological k - gon (i.e. a set of k points $x_1,\ldots,x_k$ every two of which can be joined by a path, no two of which have a common vertex except their endpoints). Then $g(n; k) < c_k n$. $g(n;3) = n$ is trivial and G. Dirac first proved $g(n; 4) = 2n - 2$ and he conjectured $g(n; 5) = 3n - 5$. This pretty conjecture is still open.

Robertson now argues as follows: Let $k = k(b)$ be sufficiently large. By Mader's theorem our $G(n; c_k n)$ contains a topological complete k-gon with the vertices $\{x_1,\ldots,x_k\}$. Join x_i to x_j with an edge of color u if the length of the path joining x_i to x_j has length $\equiv u \pmod{b}$. By Ramsey's theorem there is a monochromatic circuit (in fact a monochromatic complete graph) of length b. having the vertices say $x_{i_1}, \dots, x_{i_b}$ (if $k > k_0(b)$). Now in our $G(n; c_k n)$, x_{i_j} is joined to $x_{i_{j+1}}$ by a path of length $\equiv u \pmod b$ and hence the length of the whole circuit is $\equiv 0 \pmod b$ as stated. The bounds for $c_{0,b}$ obtained by Robertson's method are probably very far from being best possible.

Very recently Bollobás proved our original conjecture $c_{a,b}$ exists whenever $a \pmod b$ contains even numbers, and I just heard that somewhat earlier H. Jochens of Hamburg proved the existence of $c_{1,3}$. In fact he showed that if every vertex of G has valency ≥ 3 then, if G is not the Petersen graph, it contains a C_m, $m \equiv 1 \pmod 3$.

Finally I state a few older problems of mine about which I have not thought for some time.

Let $f(n)$ be the largest integer for which one can color the edges of $k(n)$ by $f(n)$ colors so that every Hamiltonian circuit contains all the $f(n)$ colors. It is easy to see that $f(2^n - 1) \geq n$. As an upper bound I could only prove $f(n) < cn^{1/2}$.

Let $G(n)$ denote a graph on n vertices. Assume that every induced subgraph of $G(10n)$ having $5n$ vertices has more than $2n^2$ edges. Is it true that our graph contains a triangle ? If true this is easily seen to be best possible. Clearly many generalizations are possible.

Is it true that if a $G(5n)$ contains more than n^5 C_5's then it must contain a triangle ? If true, n^5 is best possible. Here also many generalizations are possible. Let $G(5n)$ be a graph which has no triangles. Is it true that by omitting at most n^2 of its edges we can make it bipartite? Here also many generalizations are possible.

Replace the vertices of a pentagon by sets of n elements. Thus we obtain a $G(5n, 5n^2)$. It is easy to see that this graph shows that our last three conjectures, if true, are best possible.

Let $g(n)$ be the smallest integer so that every $G(n;g(n))$ contains two circuits of the same number of vertices. Bondy and I showed

$$c_1 n^{1/2} < g(n) < c_2 n^{1/2} \log n .$$

Perhaps $g(n) < c_3 n^{1/2}$.

Now I state some problems on set systems.

Let S be a set and let $A_k \subset S$, $|A_k| = n$, $A_{k_1} \cap A_{k_2} \neq \phi$ be a family of subsets of S. Assume that every element of S is contained in the same number of A's (i.e. the uniform hypergraph $\{A_k\}$ is regular). What is the maximum possible value of $|S|$? We would of course guess that $|S| \leq n^2 - n + 1$, equality if there is a finite geometry of n element sets (i.e. certainly if $n = q + 1$ where q is a power of a prime). Frankl and others showed $|S| \leq n^2 - cn$ but I do not know of a proof of $|S| \leq n^2 - n + 1$. It would be very interesting to determine or estimate the maximum of $|S|$ if no such finite geometry exists. In fact I just found out that this problem was posed in a more general form earlier by B. Bollobás - see his forthcoming paper: "Disjoint triples in a 3-graph with given maximal degree".

The following problem is due to Lovász and myself. This has already been published, but in view of the (in my subjective opinion) extremely challenging nature of the problem I can not resist the opportunity of stating it once more): Let $t(n)$ be the smallest integer for which there is a family $|A_k| = n$, $1 \leq k \leq t(n)$ of sets satisfying $A_{\ell_1} \cap A_{\ell_2} \neq \phi$ and for every $|S| \leq n - 1$ there is an $A_k \cap S = \emptyset$ i.e. the system $\{A_k\}$, $1 \leq k \leq t(n)$ can not be represented by fewer than n elements. We only proved

$$2.7\, n < t(n) < n^{3/2 + \epsilon} .$$

The upper bound could almost certainly be improved to cn log n. The really interesting question is: Is it true that

$$\lim_{n\to\infty} t(n)/n = \infty ?$$

I offer 100 dollars for a proof or disproof. ("For problems and results in three-chromatic hypergraphs and some related questions, see P. Erdos and L. Lovasz, Proc. Colloquium Keszthely, Infinite and Finite Sets, 1973, 609-627").

A few weeks ago P. Frankl sent me the following very interesting problem: Let $\{A_k\}$, $|A_k| = n$, $A_{k_1} \cap A_k \neq \phi$ be a two chromatic clique. Let $f(n)$ be the smallest integer for which there always is a subset $S \subset \cup A_k$ with $|S| \leq f(n)$, $S \cap A_k \neq \phi$, $S \not\supset A_k$, for every k. Frankl proved $f(n) < c\, n^2 \log n$, but only knows that $f(n) > n + c\sqrt{n}$. Determine or estimate $f(n)$ as exactly as possible. Here also the main problem is to decide whether $f(n) < cn$ holds.

B. Grunbaum and I considered the following problem: Let $|S| = n$, $A_k \subset S$, and assume that every pair of elements x, y of S are contained in one and only one A_k. Define a graph whose vertices are the A's. Two vertices are joined if the A's have a non-empty intersection. Prove that the chromatic number of this graph is at most n. If the points are all in the plane and the A's are the lines joining any two of them, then the chromatic number probably can not be small. We think it must be greater than $n - c$. As Bollobás remarked this is certainly not true in general. Just consider a finite geometry and omit a line and all points on it - the chromatic number of this set system is only $\sqrt{n}$. Grünbaum and I also wondered if one could get an asymptotic formula for the number of such set systems and also for those which originate from points in the plane (the sets are the lines).

Here I would like to state some older problems ("older means 2-3 years old) which seem to me to be particularly attractive. Faber, Lovász and I conjectured that if $|A_k| = n$, $1 \leq k \leq n$ and $|A_{k_1} \cap A_{k_2}| \leq 1$, then one can color the elements of $\bigcup_{k=1}^{n} A_k$ with n colours so that every set gets all the n colors. It is surprising that this simple statement should present so much difficulty. I gladly offer 100 dollars for a proof or disproof. Greenwell and Lovász proved the statement if the number of sets is at most

$[\frac{n+1}{2}]$ and it is easy to see that the result does not have to be true if the number of sets is $n + 1$.

Many modifications are possible .Let $|A_k| = n$, $|A_{k_1} \cap A_{k_2}| \leq 1$. It is true that one can color the elements by n colours so that every A_k gets all the colors, as long as the number of pairs $A_{k_1} \cap A_{k_2} \neq \phi$ is $\leq f(n)$. Determine $f(n)$ at least for the small values of n. Also what is the smallest $g(n,r)$ so that if $|A_{k_1} \cap A_{k_2}| \leq 1$, $1 \leq k_1 < k_2 \leq g(n,r)$, then one can color the elements with r colours so that every A_k gets all the r colours? The condition $|A_{k_1} \cap A_{k_2}| \leq 1$ could be replaced by $|A_{k_1} \cap A_{k_2}| \leq \ell$ etc. The interest of these generalizations depends of course whether one can prove or at least conjecture non-trivial and interesting answers.

Let $|S| = 2n$, $A_k \subset S$, $|A_k| = n$, $1 \leq k \leq t_n$. Assume that the number of pairs $A_{k_1} \cap A_k = \phi$ is $\geq 2^{2n}$. Is it true that then $t_n > (1-\varepsilon)2^{n+1}$? The 2^{n+1} subsets of $(1,n)$ and $(n+1, 2n)$ show that ε can not be 0.

A very recent question of Rado and myself is: Determine or estimate the largest $g(n)$ for which one can give $g(n)$ sets $|A_k| = n$, $1 \leq k \leq g(n)$ so that for every three of them the union of some two contains the third. I observed $g(n)^{1/n} \to 1$ and J. Larson that $g(2n) \geq (n+1)^2$. Perhaps for every k, $g(n)/n^k \to \infty$ as $n \to \infty$. **Frankl just wrote me that he proved** $g(n) < e^{c\sqrt{n}}$.

This problem arose as a modification of our old problem on Δ-systems. A family of sets $\{A_k\}$ is called a strong Δ system if the intersection of any two of them is identical. Denote by $g(n)$ the smallest integer for which, if $\{A_k\}$, $1 \leq k \leq g(n)$, are $g(n)$ sets of size n, then there are always three of them which form a strong Δ-system. Is it true that $g(n) < C^n$ for a certain absolute constant C? I offer 500 dollars for a proof or disproof. **Abbott showed** $g(3) = 21$. **Milner, Rado and I investigated** weak Δ systems. A system of sets $\{A_k\}$ is a weak Δ-system if the intersection of any two of them has the same size. Our principal unsolved problem states:

Let $f(n)$ be the smallest integer so that if $\{A_k\}$, $1 \le k \le f(n)$, is any family of sets of size n, then there always are three of them which form a weak Δ system. Is it true that $f(n) < c^n$?

(P. Erdös and R. Rado, Intersection theorems for systems of sets I and II, Journal London Math. Soc. 35 (1960), 85-90 and 44(1969), 467-479.

P. Erdös, E. Milner and R. Rado, Intersection theorems for finite sets III; Journal Australian Math. Soc. 18 (1974), 22-40.)

III

Finally I state some problems in combinatorial number theory and also some questions which can only be attacked by some preliminary computation.

Let $a_1, \ldots, a_n$ be a sequence of n numbers. Consider the numbers $a_i + a_j$, $a_i a_j$. I conjectured about 18 months ago that there are more than $n^{2-\varepsilon}$ distinct numbers amongst them. The conjecture presumably holds whether the a's are integers or real or complex numbers, but I have not succeeded in proving the much weaker inequality $n^{1+\varepsilon}$, even when the a's are integers.

Many various and interesting questions can be asked about the iteration of number-theoretic functions - unfortunately they are almost always hopeless. Sivasankaranarayana Pillai first investigated the iterates of Euler's ϕ function. Denote by $g(n) = k$ the smallest integer k for which $\phi_k(n) = 1$, $\phi_k(n) = \phi(\phi_{k-1}(n))$. Pillai proved that $g(n)$ is between $\frac{\log n}{\log 3}$ and $\frac{\log n}{\log 2}$. H. S. Shapiro, who took up this question independently, proved that $g(n)$ is essentially additive and stated many further questions. For more than 40 years I have wondered whether $g(n)/\log n$ has a distribution function and if so, what its nature might be. Unfortunately I have no results about this. (H.S. Shapiro, An Arithmetic Function Arising from the ϕ function, Amer. Math. Monthly 50 (1943), 18-30.)

For the distribution of primes, the following modification of $\phi(n)$

may be more interesting. Put $Q(n) = \prod_{p|n} p$ and $\phi'(n) = \phi(Q(n))$ and $\phi_k'(n) = \phi'(\phi_{k-1}'(n))$. Let $G(n)$ be the smallest integer k for which $\phi_k'(n) = 1$. Then $G(2^n) = 1$, but the maximal possible values of $G(n)$ seem to me to be of some interest. The first step would be to prove $G(n) = \sigma(\log n)$; perhaps this is not quite hopeless.

The most famous conjecture about iterates of number-theoretic functions is the old conjecture of Catalan: Put $s_1(n) = \sigma(n) - n$ and $s_k(n) = s_1(s_{k-1}(n))$; then $s_k(n)$ is never unbounded. The Lehmers, Selfridge and Guy now believe that this conjecture is probably false. In any case I doubt if "We" will know the truth in the next thousand years or so.

It is generally believed that there is no n for which $s_{k+1}(n) > s_k(n)$ for every k. In this case I certainly agree with the majority.

It seems very likely that $\lim_{k\to\infty}(\sigma_k(n))^{1/k} = \infty$ for every n and perhaps even $\sigma_{k+1}(n)/\sigma_k(n) \to \infty$ for every n.

It has been conjectured that the sequences $\{\sigma_k(n)\}$ and $\{\sigma_k(m)\}$ are never disjoint. This would of course imply that they can differ in only a finite number of terms, for if $\sigma_k(n) = \sigma_\ell(m)$ then $\sigma_{k+t}(n) = \sigma_{\ell+t}(m)$ for every $t > 0$. Selfridge is convinced that the conjecture is false.

I thought the difficulty is that $\sigma(n)$ increases too fast. I tried to replace $\sigma(n)$ by a function which is always greater than n but which increases much more slowly. Put $h(k) = n + \nu(n)$, where $\nu(n)$ denotes the number of distinct prime factors of n. I tried to prove that no two sequences $\{h_k(n_1)\}_{k\geq 1}$ and $\{h_k(n_2)\}_{k\geq 1}$ can be disjoint. I am afraid I did not succeed, though perhaps I overlooked an obvious argument. But I did come across the following problem which seems of some interest. Let $f(n) > 0$ be a number-theoretic function; the integer m is called a barrier if $m \geq n + f(n)$ for every $n < m$. Is it true that $\nu(n)$ has infinitely many barriers? If the answer is affirmative then of course no two of the sets $\{h_k(n)\}$ can be disjoint, but unfortunately our ("our" in this case stands for the human race) strength is not quite sufficient to prove this - to be convinced observe that if m is a barrier for $\nu(n)$ then $m-1$ must be a power of a prime and $m-k$ can have at most k distinct prime factors. As far as I know the sieve methods are not yet strong enough to attack such a problem. I could not even prove that there is an $\varepsilon > 0$ such that

$$(1) \qquad n + \varepsilon\nu(n) < m$$

for infinitely many m and every $n < m$.

Denote by $\nu^*(m)$ the number of prime factors of m, with multiple factors counted accordingly. It seems very likely that $\nu^*(m)$ also has infinitely many barriers.

Selfridge and I thought that even $d(n)$, the number of divisors of n, might have a barrier in the following modified form.

There is a fixed k such that, for infinitely many n,

$$(2) \qquad n + d(n) < m \qquad \text{if} \quad n < m - k.$$

This is certainly hopeless at the present state of science - perhaps it can be disproved but we could not do this even for $k = 2$.

On the other hand it is not too difficult to prove the following theorem:

If $d^+(n)$ is the number of divisors of n which are squares, then $d^+(n)$ has infinitely many barriers.

Finally I will state three recent questions on iteration of number-theoretic functions. Mr. Finucane considered the following problem: Put $\psi_1(n) = \phi(n) + 1$ and consider the sequence $\{\psi_k(n)\}_{k\ge1}$. Clearly $\psi_1(n) < n$ except if $n = p$. Thus to every n there is a prime $p^{(n)}$ for which $\psi_k(n) = p^{(n)}$. On the average, in how many steps do you reach $p^{(n)}$? Are there primes p for which $p^{(n)} = p$ for infinitely many n? Probably the answer is negative.

I could not answer any of Finucane's questions. I can only show that, for almost all primes p,

$$p^{(n)} = p \text{ only if } n = p \text{ or } n = 2p .$$

One can modify Finucane's question as follows. Let $f_1(n) = \sigma(n)-1$, then $f_1(n) > n$ except if $n = p$. It seems likely that $f_k(n)$ will never tend to infinity but will end at a prime after relatively few steps. I have only crude and unconvincing probability arguments for these conjectures, so perhaps it would be worthwhile to try it for $n < 1000$. These conjectures have been checked for $n \leq 200000$ by Mohan-Lal and S. Weintraub e.g., $f_{29}(133200) = 14591399 = p$ and every other number needs fewer steps.

Put $F_1(n) = n + \phi(n)$. Selfridge and I observed that $F_{k+2}(n) = 2F_k(n)$ for $n = 10$ and 94. We also found an integer m so that $F_{k+9}(m) = 9F_k(m)$, but of course we could not prove any general result. It seems that $\{f_k(210)\}$ tends to infinity without any sign of periodicity.

Straus and I conjectured that for every sufficiently large prime p_k there are indices i for which

$$p_k^2 < p_{k+i} \cdot p_{k-i} . \tag{3}$$

(3) if true is certainly very deep. It would be interesting to find large (say primes $> 10^{10}$) primes p_k for which (3) does not hold i.e. for which for every $1 \leq i < k$

$$p_k^2 > p_{k+i} \cdot p_{k-i} . \tag{4}$$

I think the following sharper conjecture is true:

For every r there is a $k_0 = k_0(r)$ so that for every $k > k_0$ there are indices $1 \leq i_j < \dots < i_{2r} < k$ so that

$$(-1)^j \; (p_k^2 - p_{k-i_j} \cdot p_{k+i_j}) > 0 \quad \text{for } j = 1,2,\dots,2r . \tag{5}$$

I can not even prove that (5) or (3) holds for a set of k's of density 1.