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Generalized Ramsey Theory for Multiple Colors

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In this paper, we study the generalized Ramsey number $r(G_1, ..., G_k)$ where the graphs $G_1, ..., G_k$ consist of complete graphs, complete bipartite graphs, paths, and cycles. Our main theorem gives the Ramsey number for the case where $G_2, ..., G_k$ are fixed and $G_1 \simeq C_n$ or P_n with n sufficiently large. If among $G_2, ..., G_k$ there are both complete graphs and odd cycles, the main theorem requires an additional hypothesis concerning the size of the odd cycles relative to their number. If among $G_2, ..., G_k$ there are odd cycles but no complete graphs, then no additional hypothesis is necessary and complete results can be expressed in terms of a new type of Ramsey number which is introduced in this paper. For $k = 3$ and $k = 4$ we determine all necessary values of the new Ramsey number and so obtain, in particular, explicit and complete results for the cycle Ramsey numbers $r(C_n, C_l, C_k)$ and $r(C_n, C_l, C_k, C_m)$ when n is large.

1. INTRODUCTION

The *Ramsey number* $r(G_1, G_2, ..., G_k)$ is the least positive integer p such that if $(E_1, E_2, ..., E_k)$ is an arbitrary partition of the edges of the complete graph K_p, then, for some i, the edge-induced subgraph $\langle E_i \rangle$ contains a graph isomorphic to G_i. The classes $E_1, E_2, ..., E_k$ are usually visualized as color classes and the partition $(E_1, E_2, ..., E_k)$ is thought of as an edge coloring using k colors. In the case $G_1 \simeq G_2 \simeq \cdots \simeq G_k \cong G$ the Ramsey number is denoted $r(G; k)$.

Although the classical problem of determining $r(K_m, K_n)$ remains relatively untouched, several interesting results in generalized Ramsey theory are known for the case $k = 2$. Progress in determining generalized Ramsey numbers for $k > 2$ has been comparatively slow, but some interesting results have been obtained recently. The path Ramsey number for three colors, $r(P_l, P_m, P_n)$, is studied in [6]. Although exact values are, in general, unknown, there are several papers which give upper and lower bounds for Ramsey numbers of the type $r(G; k)$. A useful survey of recent results in generalized Ramsey theory is given in [3].

In this paper, we study the Ramsey number $r(G_1, G_2, ..., G_k)$ where k is arbitrary and where the graphs $G_1, G_2, ..., G_k$ consist of complete graphs, complete bipartite graphs, paths, and cycles. This otherwise inaccessible problem is made tractable by taking $G_1 \simeq C_n$, a cycle of length n, where n is sufficiently large, and by imposing certain other conditions concerning the size of odd cycles. Although our general result must be expressed in terms of some unknown Ramsey numbers, by appropriate specializations of the main result we obtain some explicitly stated, nontrivial results. In particular, we obtain all of the cycle Ramsey numbers $r(C_n, C_l, C_k)$ and $r(C_n, C_l, C_k, C_m)$ for the case where n is sufficiently large.

For the most part, our notation will conform to that used in [1] or [8]. One exception will be in our description of certain m-partite structures. Let $V_1, V_2, ..., V_m$ denote disjoint sets of vertices. The collection $(V_1, V_2, ..., V_m)$ will be called an *m-partite vertex set*. If $|V_1| = |V_2| = \cdots = |V_m| = a$, we shall write $(V_1, V_2, ..., V_m) \simeq K_a^m$. Given $(V_1, V_2, ..., V_m)$, we shall say that $(X_1, X_2, ..., X_m)$ is a *subpartite vertex set* if $X_i \subseteq V_i$ for $i = 1, ..., m$. We define

$$E(V_i) = \{\{u, v\} \mid u, v \in V_i, u \neq v\}$$

and

$$E(V_1, ..., V_m) = \{\{u, v\} \mid u \in V_i, v \in V_j, i \neq j\}.$$

Let $(E_1, ..., E_k)$ denote a partition of $E(V_1, ..., V_m)$ into k color classes. The coloring is called *canonical* if, for every pair $i \neq j$, $E(V_i, V_j)$ meets only one color class.

The following notation is introduced for the purpose of stating our main result in a more concise way. Let $[K]$, $[B]$, and $[C]$ denote the following sequences of complete graphs, complete bipartite graphs, and odd cycles, respectively:

$$[K] = (K_{a_1}, ..., K_{a_r}),$$

$$[B] = (K(b_1, c_1), ..., K(b_s, c_s)), \qquad b_i \leqslant c_i, \quad i = 1, ..., s,$$

$$[C] = (C_{2d_1+1}, ..., C_{2d_t+1}).$$

Now, using an obvious notation, we note that the Ramsey number under consideration in this paper is $r(C_n, [K], [B], [C])$. Note that the number of graphs, and hence color classes, is $k = r + s + t + 1$. It will be advantageous to introduce a second, more convenient, indexing scheme for the edge classes. To this end, we define

$$\begin{aligned} EK_i &= E_{1+i}, & i &= 1,..., r, \\ EB_i &= E_{r+1+i}, & i &= 1,..., s, \\ EC_i &= E_{r+s+1+i}, & i &= 1,..., t. \end{aligned}$$

In other words, EK_i is the edge class which is associated with the ith complete graph, etc. We shall adhere to this notation even in the case where one of the sequences $[K]$, $[B]$, or $[C]$ is missing. Finally, we define

$$\begin{aligned} l &= \sum_{i=1}^{s} (b_i - 1), & s &> 0, \\ &= 0, & s &= 0. \end{aligned}$$

2. A Critical Coloring

A coloring of the complete graph of order $r(G_1, ..., G_k) - 1$ in which no $\langle E_i \rangle$ contains a graph isomorphic to G_i is called a *critical coloring*. Fortunately, in generalized Ramsey theory it is often the case that there is a critical coloring which is of the canonical type. We shall now describe some special operations for producing canonical edge partitions. These operations will be used to give a critical coloring of the complete graph of order $r(C_n, [K], [B], [C]) - 1$.

Let $K_p\{E; k\}$ denote the complete graph of order p, together with a certain edge partition $E = (E_1, ..., E_k)$. The *bi-expansion* of $K_p\{E; k\}$, denoted $b(K_p\{E; k\})$, is the complete graph of order $2p$ with its edges partitioned into $k + 1$ classes that is obtained by taking two isomorphic copies of $K_p\{E; k\}$ with all of the edges from one copy to the other in the additional edge class E_{k+1}. Note that the ith iterate, $b^i(K_p\{E; k\})$, is a complete graph of order $2^i p$ with its edges partitioned into $k + i$ classes and with no odd cycle in any of the edge-induced subgraphs $\langle E_{k+1} \rangle, ..., \langle E_{k+i} \rangle$.

Let $K_p\{E; k\}$ and $K_q\{E^*; j\}$ be given. The *expand*, denoted $e(K_p\{E; k\}, K_q\{E^*; j\})$, is the complete graph of order pq with edges partitioned into $k + j$ classes that is obtained by replacing each vertex of $K_p\{E; k\}$ with an isomorphic copy of $K_q\{E^*; j\}$ and by making all edges between two copies in the same class as the edge which joined the two vertices which the copies replace.

We now describe a special $K_p\{E; k\}$, called the *canonical ladder*. With $p_1 + p_2 + \cdots + p_k = p$, let $(V_1, \ldots, V_k)$ be a k-partite vertex set such that $|V_i| = p_i$ for $i = 1, \ldots, k$. Letting I denote the finite sequence $(p_1, \ldots, p_k)$, the canonical ladder, denoted CL[I], is the complete graph of order p with edge partition $(E_1, \ldots, E_k)$ defined as follows:

$$E(V_i) \subseteq E_i, \qquad \text{for } i = 1, \ldots, k,$$
$$E(V_i, V_j) \subseteq E_i \qquad \text{for all } j < i.$$

Using the concept of canonical ladder, bi-expansion, and expand we can describe the desired critical coloring. Letting $I = (n-1, b_1 - 1, \ldots, b_s - 1)$, form the canonical ladder CL[I] and note that this yields a partition of the edges of the complete graph of order $n + l - 1$ into $s + 1$ color classes in which $\langle E_1 \rangle$ does not contain C_n and $\langle EB_i \rangle$ does not contain $K(b_i, c_i)$ for $i = 1, \ldots, s$. In fact, $\langle E_1 \rangle$ contains no P_n and $\langle EB_i \rangle$ contains no P_{2b_i} for $i = 1, \ldots, s$. Now form $b^t(\text{CL}[I])$ and note that this yields a complete graph of order $2^t(n + l - 1)$ with edges partitioned into $s + t + 1$ color classes in which $\langle E_1 \rangle$ does not contain C_n, $\langle EB_i \rangle$ does not contain $K(b_i, c_i)$ for $i = 1, \ldots, s$, and $\langle EC_i \rangle$ does not contain C_{2d_i+1} for $i = 1, \ldots, t$.

Let r^* denote the Ramsey number $r([K])$. We know that there exists a complete graph of order $r^* - 1$ with edges partitioned into r color classes such that $\langle EK_i \rangle$ does not contain K_{a_i} for $i = 1, \ldots, r$. Let us denote this example as $K_{r^*-1}\{E^*; r\}$. Finally, form the expand,

$$e(K_{r^*-1}\{E^*; r\}, b^t(\text{CL}[I])).$$

We thus obtain a complete graph of order $2^t(r^* - 1)(n + l - 1)$ with edges partitioned into $k = r + s + t + 1$ color classes such that $\langle E_1 \rangle$ does not contain C_n, $\langle EK_i \rangle$ does not contain K_{a_i} for $i = 1, \ldots, r$, $\langle EB_i \rangle$ does not contain $K(b_i, c_i)$ for $i = 1, \ldots, s$ and $\langle EC_i \rangle$ does not contain C_{2d_i+1} for $i = 1, \ldots, t$. Hence, we know that $r(C_n, [K], [B], [C]) > 2^t(r^* - 1)(n + l - 1)$. In our main theorem, we shall prove that if n is sufficiently large and if certain conditions on $[C]$ are satisfied, then the example given is critical, i.e.,

$$r(C_n, [K], [B], [C]) = 2^t(r^* - 1)(n + l - 1) + 1.$$

3. Preliminary Results

It is well known that given arbitrary positive integers k and q, there exists a least positive integer $f(k, q)$ with the property that if $p \geqslant f(k, q)$ and if the edges of $K(p, p)$ are colored using k colors, then there will be a monochromatic $K(q, q)$. The function f has not been determined precisely [5, Chap. 12]. However, for our purposes it suffices to note the existence of f. The existence of f is the bipartite case of the first of two lemmas dealing with related ideas.

LEMMA 1. *Let $(E_1, \ldots, E_k)$ be a partition of $E(V_1, \ldots, V_m)$, where $(V_1, \ldots, V_m) \simeq K_p{}^m$. Let q be an arbitrary positive integer. Then, if p is sufficiently large, there exists a subpartite set $(X_1, \ldots, X_m) \simeq K_q{}^m$ for which the induced coloring of $E(X_1, \ldots, X_m)$ is canonical.*

Proof. The proof is by induction on m. The case of $m = 2$ is the bipartite result which is discussed above. Let r be an arbitrary positive integer. Then, by the induction hypothesis, there exists a subpartite set $(W_1, \ldots, W_{m-1}) \simeq K_r^{m-1}$ such that the induced coloring of $E(W_1, \ldots, W_{m-1})$ is canonical. By means of the bipartite result, there exist $Y_1 \subseteq V_m$ and $X_1 \subseteq W_1$ with $|X_1| = q$, $|Y_1|$ as large as desired, and such that $E(X_1, Y_1)$ meets only one color class. Similarly, one obtains sets $X_2, \ldots, X_{m-1}$ and $Y_2, \ldots, Y_{m-1}$ where each X_i is a q-subset of the corresponding W_i, the Y's are nested $Y_{m-1} \subseteq Y_{m-2} \subseteq \cdots \subseteq Y_1$, $|Y_{m-1}| = q$, and, for all $i \leqslant j$, $E(X_i, Y_j)$ meets only one color class. Hence, setting $X_m = y_{m-1}$, we find that the induced coloring of $E(X_1, \ldots, X_m)$ is canonical as claimed. ▮

The second lemma is similar, except that a set of vertices whose size is fixed is involved.

LEMMA 2. *Let $(E_1, \ldots, E_k)$ be a partition of $E(V_1, \ldots, V_m, X)$, where $(V_1, \ldots, V_m) \simeq K_p{}^m$ and $X = \{x_1, \ldots, x_b\}$. Let q be an arbitrary positive integer. Then, if p is sufficiently large, there exists $(Y_1, \ldots, Y_m) \simeq K_q{}^m$, a subpartite set of $(V_1, \ldots, V_m)$, such that the induced coloring of $E(Y_1, \ldots, Y_m, \{x_1\}, \ldots, \{x_b\})$ is canonical.*

Proof. Let r be an ariitrary positive integer. By Lemma 1, there exists a subpartite set $(U_1, \ldots, U_m) \simeq K_r{}^m$ such that the induced coloring of $E(U_1, \ldots, U_m)$ is canonical. Consider vertex x_1 in X. Since the r edges from x_1 to any given U_i are divided into k classes, it is clear that there exists $(W_1, \ldots, W_m) \simeq K_s{}^m$, a subpartite set of $(U_1, \ldots, U_m)$, such that the induced coloring of $E(W_1, \ldots, W_m, \{x_1\})$ is canonical. By repeating this process, we obtain the nested sets $Y_i \subseteq \cdots \subseteq W_i \subseteq U_i$, $i = 1, \ldots, m$, where the induced coloring of $E(Y_1, \ldots, Y_m, \{x_1\}, \ldots, \{x_b\})$ is canonical as claimed. ▮

The basic results contained in the next three lemmas can be expressed as follows. For r fixed and n sufficiently large, let (E_1, E_2) be a partition of the edges of a complete graph of order $p = m(n-1) + 1$ for which $\langle E_1 \rangle$ contains no C_n and $\langle E_2 \rangle$ contains no K_r. Then, the following lemmas give the existence of m vertex-disjoint cycles in $\langle E_1 \rangle$ which are sufficiently large and, in a sense to be defined, maximal. The large cycles are ordered in such a way that each vertex in a subsequent cycle or not on any cycle is adjacent in $\langle E_1 \rangle$ to at most $r - 2$ vertices of a given cycle.

LEMMA 3. *For $p \geqslant n(r-1)$, let (E_1, E_2) be a partition of $E(K_p)$ for which $\langle E_2 \rangle$ contains no K_r. Then $\langle E_1 \rangle$ contains a cycle of length at least n.*

Proof. By Turan's theorem [1, Chap. 17],

$$| E_2 | \leqslant \frac{(p^2 - j^2)(r-2)}{2(r-1)} + \binom{j}{2},$$

where $p \equiv j \pmod{(r-1)}$, $0 \leqslant j < r - 1$. It follows that

$$| E_1 | \geqslant \frac{1}{2} \left\{ p \left[\frac{p}{r-1} - 1 \right] + j \left[1 - \frac{j}{r-1} \right] \right\} \geqslant \frac{p}{2} \left[\frac{p}{r-1} - 1 \right].$$

Hence, by the Erdös–Gallai theorem [4], $\langle E_1 \rangle$ contains a cycle of length at least

$$(p^2 - 2(r-1))/((r-1)(p-1)).$$

Finally, if $p \geqslant n(r-1)$, then $\langle E_1 \rangle$ contains a cycle of length at least n. ∎

LEMMA 4. *For $r \geqslant 3$, let (E_1, E_2) be a partition of $E(K_p)$ such that $\langle E_2 \rangle$ contains no K_r. Then*

(a) *if $\langle E_1 \rangle$ contains a C_l and no C_{l+1}, each vertex $V(K_p) - V(C_l)$ is adjacent in $\langle E_1 \rangle$ to at most $r - 2$ vertices of C_l, and*

(b) *if $\langle E_1 \rangle$ contains a C_l for $l \geqslant 2r$, then $\langle E_1 \rangle$ contains a smaller cycle, of length j where $l + 3 - 2r \leqslant j < l$.*

Proof. (a) Let $C_l = (x_1, x_2, \ldots, x_l, x_1)$ and let $x \in V(K_p) - V(C_l)$. Let x be adjacent in $\langle E_1 \rangle$ to vertices $x_{i_1}, \ldots, x_{i_s}$ of the cycle, where $i_1 < \cdots < i_s$. Observe that no two of these vertices can be consecutive vertices of the cycle, for otherwise $\langle E_1 \rangle$ would contain a cycle of length $l + 1$. Consider the set of vertices

$$A = \{x_{i_1 - 1}, \ldots, x_{i_s - 1}\}.$$

By the previous argument, no vertex in A is adjacent in $\langle E_1 \rangle$ to x. Also, no two vertices of A are adjacent in $\langle E_1 \rangle$, for otherwise $\langle E_1 \rangle$ would contain a cycle of length $l + 1$. Therefore, A together with x are vertices of a complete graph in $\langle E_2 \rangle$. Since $\langle E_2 \rangle$ contains no K_r, we have $s + 1 < r$, or equivalently $s \leqslant r - 2$.

(b) Let $C_l = (x_1, x_2, ..., x_l, x_1)$ and consider the set of vertices

$$B = \{x_1, x_3, ..., x_{2r-1}\}.$$

Since $\langle E_2 \rangle$ contains no K_r, there are two vertices of B which are adjacent in $\langle E_1 \rangle$. Hence, there is a cycle of length at least $l - (2r - 3) = l + 3 - 2r$ in $\langle E_1 \rangle$. ∎

LEMMA 5. *For $p = m(n - 1)$ and $r \geqslant 3$, let (E_1, E_2) be a partition of $E(K_p)$ such that $\langle E_1 \rangle$ contains no C_n and $\langle E_2 \rangle$ contains no K_r. Let $q \geqslant 2r$ be fixed. Then, if n is sufficiently large, $\langle E_1 \rangle$ contains m disjoint cycles $C_{l_1}, ..., C_{l_m}$, where $q \leqslant l_i < n$ for $i = 1, ..., m$. Each cycle C_{l_i} is of maximal length, subject to the bound $l_i < n$, in the subgraph of $\langle E_1 \rangle$ induced by the set of vertices not contained in any cycle C_{l_j} for $j < i$.*

Proof. Without loss of generality, take $n > q(r - 1)$ and note that, by Lemma 3, $\langle E_1 \rangle$ contains a cycle C_l with $l \geqslant q$. If $l > n$, then by applying Lemma 4(b), several times if necessary, we obtain a cycle of length l where $q \leqslant l < n$. Take l_1 to be the length of a maximal such cycle.

If the vertices of C_{l_1} are deleted, we have a complete graph of order at least $(m - 1)(n - 1)$ and with edge partition (E_1, E_2) satisfying the same conditions as before. Thus, for this graph we obtain a C_{l_2} in $\langle E_1 \rangle$, where l_2 is maximal subject to the bounds $q \leqslant l_2 < n$. Since, originally, $p = m(n - 1)$, repetition of the basic argument yields the m cycles $C_{l_1}, ..., C_{l_m}$ as claimed. ∎

Applications of the previous lemmas can lead to the following situation. There is an m-partite set $(V_1, ..., V_m)$ and an edge partition (E_1, E_2) of $E(V_1, ..., V_m)$ such that each vertex in V_i is adjacent in $\langle E_1 \rangle$ to a limited number of vertices in V_j for $j < i$. The final two lemmas then deal with that situation.

LEMMA 6. *Let (E_1, E_2) be a partition of $E(V_1, V_2)$, where $|V_1| = p$ and $|V_2| = q$. If each vertex in V_2 is adjacent in $\langle E_1 \rangle$ to at most r vertices in V_1 and if $p > qr$, then there is an $X_1 \subseteq V_1$ where $|X_1| = p - qr$ such that $E(X_1, V_2) \subseteq E_2$.*

Proof. This is obviously so, since the total number of vertices in V_1 which are adjacent in $\langle E_1 \rangle$ to some vertex of V_2 is at most qr. ∎

LEMMA 7. *Let (E_1, E_2) be a partition of $E(V_1, ..., V_m)$ where $(V_1, ..., V_m) \simeq K_p{}^m$ and assume that each vertex of V_i is adjacent in $\langle E_1 \rangle$*

to at most r vertices of V_j for $j < i$. Let q be an arbitrary positive integer. Then, if p is sufficiently large, there exists $(X_1, ..., X_m) \simeq K_q{}^m$, a subpartite set of $(V_1, ..., V_m)$, such that $E(X_1, ..., X_m) \subseteq E_2$.

Proof. The proof is by induction on m. For $m = 2$, set $p = (r + 1)q$ and apply Lemma 6. Applying the induction hypothesis, let $(W_1, ..., W_{m-1}) \simeq K_{q'}^{m-1}$ be a subpartite set of $(V_1, ..., V_{m-1})$ such that $E(W_1, ..., W_{m-1}) \subseteq E_2$. Let X_m be an arbitrary q-subset of V_m. Note that each vertex of X_m is adjacent in $\langle E_1 \rangle$ to at most r vertices of W_j. Setting $q' = (r + 1)\, q$ and applying Lemma 6 to (W_j, X_m) for $j = 1, ..., m - 1$, we obtain the stated result. ∎

4. The Main Theorem

THEOREM. *Let [K], [B], and [C] denote the following fixed sequences of graphs:*

$$[K] = (K_{a_1}, ..., K_{a_r}),$$
$$[B] = (K(b_1, c_1), ..., K(b_s, c_s)), \qquad b_i \leqslant c_i,$$
$$[C] = (C_{2d_1+1}, ..., C_{2d_t+1}).$$

Further, let

$$\begin{aligned} r^* &= r([K]), \qquad & r > 0 \\ &= 2 & r = 0 \end{aligned}$$

and define

$$l = \sum_{i=1}^{s} (b_i - 1).$$

Require that $d_i \geqslant 2^{t-i}(r^* - 1)$ for $i = 1, ..., t$. Then, if n is sufficiently large,

$$r(C_n, [K], [B], [C]) = 2^t(r^* - 1)(n + l - 1) + 1.$$

Proof. The example given in Section 2 shows that

$$r(C_n, [K], [B], [C]) > 2^t(r^* - 1)(n + l - 1).$$

For $p = 2^t(r^* - 1)(n + l - 1) + 1$ and $k = r + s + t + 1$, let $(E_1, ..., E_k)$ be a partition of the edges of K_p. We shall steadfastly assume that $\langle E_1 \rangle$ contains no C_n, $\langle EK_i \rangle$ contains no K_{a_i} for $i = 1, ..., r$, $\langle EB_i \rangle$ contains no $K(b_i, c_i)$ for $i = 1, ..., s$, and that $\langle EC_i \rangle$ contains no C_{2d_i+1} for $i = 1, ..., t$. These assumptions lead ultimately to a contradiction. Let

$\bar{E} = E - E_1$ and $\bar{r} = r([K], [B], [C])$. Let $q^* \geqslant 2\bar{r}$ be fixed. Since $\langle \bar{E} \rangle$ contains no $K_{\bar{r}}$, by Lemma 5 we find that if n is sufficiently large then $\langle E_1 \rangle$ contains m disjoint cycles $C_{l_1}, ..., C_{l_m}$, where $m = 2^t(r^* - 1)$ and $l_i \geqslant q^*$, $i = 1, ..., m$. Let

$$V_i = V(C_{l_i}),\ i = 1, ..., m.$$

Since each C_{l_i} is maximal in the sense given in Lemma 5, by Lemma 4(a) we find that each vertex in V_i is adjacent in $\langle E_1 \rangle$ to at most $\bar{r} - 2$ vertices of V_j for $j < i$.

Consider the partition of $E(V_1, ..., V_m)$ induced by $(E_1, \bar{E})$. Let q be an arbitrary positive integer. By Lemma 7, there exists $(W_1, ..., W_m) \simeq K_q{}^m$, a subpartite set of $(V_1, ..., V_m)$, such that $E(W_1, ..., W_m) \subseteq \bar{E}$. Note that since each of the $m = 2^t(r^* - 1)$ cycles is of order at most $n - 1$ and since the order of the graph is $p = 2^t(r^* - 1)(n + l - 1) + 1$, there are at least $ml + 1$ vertices of the graph which are not contained in any of the large cycles. Setting $b = ml + 1$, let $X = \{x_1, ..., x_b\}$ denote such a set of vertices. Note that Lemma 6 implies that there exist sets $X_1, ..., X_m$ with $X_i \subseteq W_i$ and $|X_i| = q - (\bar{r} - 2)(ml + 1)$ for $i = 1, ..., m$ such that $E(X_1, ..., X_m, X) \subseteq \bar{E}$. Let $\bar{q}$ be an arbitrary positive integer. By Lemma 2 we obtain $(Y_1, ..., Y_m) \simeq K_{\bar{q}}{}^m$, a subpartite vertex set of $(X_1, ..., X_m)$, such that the induced coloring of $E(Y_1, ..., Y_m, \{x_1\}, ..., \{x_b\})$ is canonical.

Since $\bar{q}$ is arbitrary, we can surely assume that $\bar{q} \geqslant \max\{c_1, ..., c_s\}$. Since $\langle EB_i \rangle$ contains no $K(b_i, c_i)$ for $i = 1, ..., s$, it must be true that $E(Y_1, ..., Y_m) \cap EB_i = \varnothing$. Also, for the same reason, the number of vertices of X with which the vertices of a given Y_j are adjacent in $\langle EB_i \rangle$ is at most $b_i - 1$. Hence, the number of vertices of X which are adjacent in some $\langle EB_i \rangle$ to the vertices of some Y_j is at most ml. This leaves at least one vertex of X which is adjacent to vertices of $(Y_1, ..., Y_m)$ only in $\langle EK_i \rangle$, $i = 1, ..., r$, or $\langle EC_i \rangle$, $i = 1, ..., t$. Designate this vertex as $\bar{x}$.

Consider the $m + 1$-partite graph $\langle E(Y_1, ..., Y_m, \{\bar{x}\}) \cap EC_i \rangle$. Let C_{2a+1} denote the smallest odd cycle, if any, contained by this graph. Since $|Y_j| = \bar{q}$, which is arbitrarily large, it is clear that the graph contains all odd cycles C_{2b+1} for all $b \geqslant a$ up to some arbitrarily large limit, providing that a exists. Since, by assumption, the graph contains no C_{2d_i+1}, we must have $a > d_i$.

For $j = 1, ..., m$, select $y_j \in Y_j$ and consider the graph with vertex set $V = \{y_1, ..., y_m, \bar{x}\}$ and with $E(V)$ partitioned according to the canonical coloring of $\langle E(Y_1, ..., Y_m, \{\bar{x}\}) \rangle$. Consider first the graph $\langle E(V) \cap EC_1 \rangle$. This is a graph with $m + 1 = 2^t(r^* - 1) + 1$ vertices and, by the previous argument, containing no odd cycle of length less than or equal to $2d_1 + 1 \geqslant 2^t(r^* - 1) + 1$. Hence, the graph contains no odd cycle and is therefore bipartite. Splitting the vertex set as evenly as possible, we see that there exists $U_1 \subseteq V$ with $|U_1| = 2^{t-1}(r^* - 1) + 1$ such that $E(U_1) \cap EC_1 = \varnothing$. Now consider the graph $\langle E(U_1) \cap EC_2 \rangle$. Again, this graph is bipartite since it is a graph with $2^{t-1}(r^* - 1) + 1$ vertices which contains no odd cycle of length less than or equal to $2d_2 + 1 \geqslant 2^{t-1}(r^* - 1) + 1$. It follows that there exists $U_2 \subseteq U_1$ with $|U_2| = 2^{t-2}(r^* - 1) + 1$ such that $E(U_2) \cap EC_i = \varnothing$ for $i = 1, 2$. Continuing in this manner, we finally obtain a set U_t with $|U_t| = r^*$ such that $E(U_t) \cap EC_i = \varnothing$ for $i = 1,..., t$. We have already established that for this same set $E(U_t) \cap EB_i = \varnothing$ for $i = 1,..., s$. If $r = 0$, then there is at this point a contradiction. If $r > 0$, the fact that $E(U_t)$ meets only the edge classes EK_i, $i = 1,..., r$ and $|U_t| = r^*$ implies that $\langle EK_i \rangle$ contains a K_{a_i} for some i, and again we have a contradiction. ∎

5. Additional Results

In our discussion of the critical coloring, it was noted that with $I = (n - 1, b_1 - 1,..., b_s - 1)$ the canonical ladder CL[I] is such that $\langle E_1 \rangle$ contains no P_n and $\langle EB_i \rangle$ contains no P_{2b_i} for $i = 1,..., s$. Hence, $\langle E_1 \rangle$ contains no C_n and $\langle EB_i \rangle$ contains neither C_{2b_i} nor P_{2b_i+1} nor $K(b_i, c_i)$. On the other hand, if $\langle E_1 \rangle$ contains a C_n, it necessarily contains a P_n and if $\langle EB_i \rangle$ contains a $K(b_i, c_i)$, it necessarily contains P_{2b_i}, P_{2b_i+1}, and C_{2b_i}. Let CP_n denote either C_n or P_n and let [BPC] denote a list of s graphs, the ith one of which is a connected bipartite graph which has parts of size b_i and c_i' ($b_i \leqslant c_i' \leqslant c_i$). Of particular interest is the case where the ith graph in [BPC] is either $K(b_i, c_i)$, P_{2b_i+1} (if $b_i < c_i$), P_{2b_i}, or C_{2b_i}. By the observations just made, the following result is an immediate corollary of the main theorem.

COROLLARY 1. *If $d_i \geqslant 2^{t-i}(r^* - 1)$ for $i = 1,..., t$ and if n is sufficiently large, then*

$$r(CP_n, [K], [BPC], [C]) = 2^t(r^* - 1)(n + l - 1) + 1.$$

Except for a small number of cases, $r^* = r([K])$ is unknown. Therefore, in order to obtain explicit results, the list of complete graphs [K] needs to be restricted in some way. If [K] consists of only one complete graph, K_m, then $r^* = m$ and we have, for all sufficiently large n and if $d_i \geqslant 2^{t-i}(m - 1)$ for $i = 1,..., t$,

$$r(C_n, K_m, [B], [C]) = 2^t(m - 1)(n + l - 1) + 1.$$

In particular, if n is sufficiently large, then

$$r(C_n, K_m) = (m-1)(n-1) + 1,$$

a result given originally by Bondy and Erdös in [2].

By eliminating $[K]$ altogether, we obtain the following basic result.

COROLLARY 2. *If $d_i \geqslant 2^{t-i}$ for $i = 1,..., t$ and if n is sufficiently large, then*

$$r(CP_n, [BPC], [C]) = 2^t(n + l - 1) + 1.$$

If, in addition, we eliminate $[C]$ altogether, the following result is obtained.

COROLLARY 3. *If n is sufficiently large, then*

$$r(CP_n, [BPC]) = n + l.$$

We note that for the special case in which all of the graphs in the list CP_n, $[BPC]$ are paths, the result of Corollary 3 answers a question posed in [6].

Except for the case in which $[C]$ is eliminated altogether, all of the results stated thus far have involved a condition imposed on the size of the odd cycles relative to their number. By introducing a new type of Ramsey number, we are able to state results which no longer involve this type of condition. We define $r(\leqslant [C])$ to be the least integer p such that if $(EC_1,..., EC_t)$ is an arbitrary partition of the edges of K_p, then, for some i, $\langle EC_i \rangle$ contains an odd cycle of length less than or equal to $2d_i + 1$.

Now we can prove that if $d_i \geqslant 2^{t-i}$ for $i = 1,..., t$, then $r(\leqslant [C]) = 2^t + 1$. Let $K_2\{E; 1\}$ be the complete graph of order two with its one edge in class EC_1. Then, the iterated bi-expansion, $b^{t-1}(K_2\{E; 1\})$, gives the desired critical coloring since it is a complete graph of order 2^t with edge partiton $(EC_1,..., EC_t)$ such that for $i = 1,..., t$, $\langle EC_i \rangle$ contains no odd cycle. Now we proceed as in the proof of the main theorem. Consider the complete graph of order $2^t + 1$ with edge partition $(EC_1,..., EC_t)$ and assume that for $i = 1,..., t$, $\langle EC_i \rangle$ contains no odd cycle of length less than or equal to $2d_i + 1$. Since $d_1 \geqslant 2^{t-1}$, we have $2d_1 + 1 \geqslant 2^t + 1$ and so $\langle EC_1 \rangle$ is bipartite. Thus, there is a vertex set U_1 with $|U_1| \geqslant 2^{t-1} + 1$ such that $E(U_1) \cap EC_1 = \varnothing$. Continuing in this manner, we ultimately obtain a set U_{t-1} with $|U_{t-1}| \geqslant 3$ and such that $E(U_{t-1}) \subseteq EC_t$. Since $d_t \geqslant 1$, this is an obvious contradiction.

In the cases where these conditions on the d_i are not satisfied, a knowledge of $r(\leqslant [C])$ allows us to restate our basic result for the case where $[K]$ is absent in a way which involves no conditions other than that n is large.

The following result is a corollary to the proof of the main theorem.

COROLLARY 4. *If n is sufficiently large, then*

$$r(CP_n, [BPC], [C]) = (r(\leqslant [C]) - 1)(n + l - 1) + 1.$$

Proof. With $q = r(\leqslant [C]) - 1$, let $K_q\{E; t\}$ denote a complete graph of order $r(\leqslant [C]) - 1$ together with the edge partition $(EC_1, ..., EC_t)$ such that $\langle EC_i \rangle$ contains no odd cycle of length less than or equal to $2d_i + 1$ for $i = 1, ..., t$. Then the expand,

$$e(K_q\{E; t\}, \mathrm{CL}[I])$$

is a complete graph of order $(r(\leqslant [C]) - 1)(n + l - 1)$ such that $\langle E_1 \rangle$ contains no P_n, $\langle EB_i \rangle$ contains no P_{2b_i} for $i = 1, ..., s$ and $\langle EC_i \rangle$ contains no C_{2d_i+1} for $i = 1, ..., t$. Hence, the desired Ramsey number is at least $(r(\leqslant [C]) - 1)(n + l - 1) + 1$. Now we let $p = (r(\leqslant [C]) - 1) \times (n + l - 1) + 1$ and follow the proof of the main theorem. With $m = r(\leqslant [C]) - 1$, we easily arrive at the point of having established that there exist $Y_1, ..., Y_m$ and $\bar{x}$ with $|Y_j| = \bar{q}$ for $j = 1, ..., m$ and such that the coloring of $E(Y_1, ..., Y_m, \{\bar{x}\})$ is canonical. Again, $\langle E(Y_1, ..., Y_m, \{\bar{x}\}) \cap EC_i \rangle$ must contain no odd cycle of length less than or equal to $2d_i + 1$. Select $y_j \in Y_j$ for $j = 1, ..., m$ and consider the graph with vertex set $V = \{y_1, ..., y_m, \bar{x}\}$ and $E(V)$ partitioned according to the canonical coloring of $E(Y_1, ..., Y_m, \{\bar{x}\})$. By assumption $\langle E(V) \cap EC_i \rangle$ contains no odd cycle of length less than or equal to $2d_i + 1$. But this leads to an immediate contradiction, since $|V| = m + 1 = r(\leqslant [C])$. ∎

In the remainder of this section, we shall confine ourselves to determining $r(\leqslant [C])$ for the cases $t = 2$ and $t = 3$. Using these results, we can completely determine the cycle Ramsey numbers $r(C_n, C_l, C_k)$ and $r(C_n, C_l, C_k, C_m)$ if n is sufficiently large.

The case of $t = 2$ is easily completed. If $l \geqslant 2$ and $k \geqslant 1$, we know that

$$r(\leqslant (C_{2l+1}, C_{2k+1})) = 2^2 + 1 = 5.$$

The only other case is for $l = m = 1$, for which we have the familiar result $r(C_3, C_3) = 6$.

Now consider the case of $t = 3$. First of all, we have the result

$$r(\leqslant (C_{2l+1}, C_{2k+1}, C_{2m+1})) = 2^3 + 1 = 9$$

if $l \geqslant 4$ and $k \geqslant 2$. If $k = m = 1$ and $l \geqslant 5$, the following argument shows that

$$r\,(\leqslant (C_{2l+1}\,, C_3\,, C_3)) = 11.$$

Let $K_5\{E; 2\}$ be the complete graph on five vertices with edge partition $(EC_2\,, EC_3)$ such that neither $\langle EC_2\rangle$ nor $\langle EC_3\rangle$ contains a C_3. Then the bi-expansion $b(K_5\{E; 2\})$ is a complete graph of order 10 with edge partition $(EC_1\,, EC_2\,, EC_3)$ such that $\langle EC_1\rangle$ contains no odd cycle, and neither $\langle EC_2\rangle$ nor $\langle EC_3\rangle$ contain a C_3. Hence, $r\,(\leqslant (C_{2l+1}\,, C_3\,, C_3)) \geqslant 11$. Consider a complete graph on 11 vertices with edge partition $(EC_1\,, EC_2\,, EC_3)$ such that $\langle EC_1\rangle$ contains no odd cycle of length less than or equal to $2l + 1$. If $l \geqslant 5$, then $\langle EC_1\rangle$ contains no odd cycle and is therefore bipartite. Now one of the two parts must contain at least six vertices and since $r(C_3\,, C_3) = 6$ it follows that either $\langle EC_2\rangle$ or $\langle EC_3\rangle$ contains a C_3.

Note that the two results already obtained cover all but 11 special cases of $r\,(\leqslant [C])$ for $t = 3$. Of these remaining cases, perhaps $r\,(\leqslant (C_3\,, C_3\,, C_3))$ requires the most sophisticated argument, but, fortunately, the result $r(C_3\,, C_3\,, C_3) = 17$ is already known [7].

Now it so happens that three examples provide all of the critical colorings necessary to establish the remaining Ramsey numbers, Example 1 is $b^2(K_2\{E; 1\})$. This is a complete graph of order 8 with edge partition $(EC_1\,, EC_2\,, EC_3)$ such that for $i = 1, 2, 3$, $\langle EC_i\rangle$ contains no odd cycle. Example 2 is the complete graph with vertex set $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and having the edge partition $(EC_1\,, EC_2\,, EC_3)$ defined as follows:

$$\begin{aligned} EC_1 &= \{\{i, j\} \mid i - j \equiv 4 \text{ or } 7 \pmod{11}\},\\ EC_2 &= \{\{i, j\} \mid i - j \equiv 1, 3, 8, \text{ or } 10 \pmod{11}\},\\ EC_3 &= \{\{i, j\} \mid i - j \equiv 2, 5, 6, \text{ or } 9 \pmod{11}\}. \end{aligned}$$

It is easy to verify that $\langle EC_1\rangle$ contains no odd cycle of length less than or equal to 9, $\langle EC_2\rangle$ contains no C_3, and $\langle EC_3\rangle$ contains no C_3. Example 3 is the complete graph with vertex set $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and having edge partition $(EC_1\,, EC_2\,, EC_3)$ defined as follows:

$$\begin{aligned} EC_1 &= \{\{i, j\} \mid j > 1, j - i = 4 \text{ or } 5\},\\ EC_2 &= \{\{i, j\} \mid j > i, j - i = 1, 6, \text{ or } 8\},\\ EC_3 &= \{\{i, j\} \mid j > i, j - i = 2, 3, \text{ or } 7\}. \end{aligned}$$

It is readily verified that $\langle EC_1\rangle$ contains no odd cycle of length less than or equal to 7, $\langle EC_2\rangle$ contains no odd cycle of length less than or equal to 5, and that $\langle EC_3 \rangle$ contains no C_3. Specific proofs verify that these three examples indeed provide critical colorings for the remaining Ramsey numbers, leading to the following conclusion.

THEOREM. *Take $l \geqslant k \geqslant m$. Then*

$$\begin{aligned} r(\leqslant (C_{2l+1}, C_{2k+1}, C_{2m+1})) &= 9, \qquad l \geqslant 4, k \geqslant 2, m \geqslant 1 \\ &= 11, \qquad l \geqslant 5, k = 1, m = 1. \end{aligned}$$

Moreover,

$$\begin{aligned} r(\leqslant (C_7, C_7, C_7)) &= r(\leqslant (C_7, C_7, C_5)) = r(\leqslant (C_7, C_7, C_3)) \\ &= r(\leqslant (C_7, C_5, C_5)) = r(\leqslant (C_5, C_5, C_5)) = 9, \\ r(\leqslant (C_7, C_5, C_3)) &= r(\leqslant (C_5, C_5, C_3)) = 10, \\ r(\leqslant (C_9, C_3, C_3)) &= r(\leqslant (C_7, C_3, C_3)) = r(\leqslant (C_5, C_3, C_3)) = 12, \end{aligned}$$

and

$$r(\leqslant (C_3, C_3, C_3)) = 17.$$

By simply substituting our results for $r(\leqslant [C])$ into the general statement of Corollary 4, we obtain all Ramsey numbers $r(C_n, C_l, C_k)$ and $r(C_n, C_l, C_k, C_m)$ for n sufficiently large. Thus, we have, if n is large,

$$\begin{aligned} r(C_n, C_{2l}, C_{2k}) &= n + k + l - 2 \\ r(C_n, C_{2l}, C_{2k+1}) &= 2(n + l) - 3, \end{aligned}$$

and

$$\begin{aligned} r(C_n, C_{2l+1}, C_{2k+1}) &= 4n - 3 \qquad l \geqslant 2, k \geqslant 1, \\ &= 5n - 4 \qquad l = 1, k = 1. \end{aligned}$$

Similarly, for n sufficiently large, we have

$$\begin{aligned} r(C_n, C_{2l}, C_{2k}, C_{2m}) &= n + l + k + m - 3, \\ r(C_n, C_{2l}, C_{2k}, C_{2m+1}) &= 2(n + l + k) - 3, \\ r(C_n, C_{2l}, C_{2k+1}, C_{2m+1}) &= 4(n + l) - 3, \qquad k \geqslant 2, m \geqslant 1, \\ &= 5(n + l) - 4, \qquad k = 1, m = 1, \\ r(C_n, C_{2l+1}, C_{2k+1}, C_{2m+1}) &= 8n - 7, \qquad l \geqslant 4, k \geqslant 2, m \geqslant 1, \\ &= 10n - 9, \qquad l \geqslant 5, k = 1, m = 1, \end{aligned}$$

plus 11 special cases which are readily written down.

6. Questions

Among other things, the results of this paper have pointed to the value of studying the class of Ramsey numbers typified ty $r(\leqslant [C])$. More generally, we have the following concept. Let $\{G\}_1, \ldots, \{G\}_k$ denote specified sets of graphs. We define the Ramsey number $r(\{G\}_1, \ldots, \{G\}_k)$ to be the smallest integer p such that if $(E_1, \ldots, E_k)$ is an arbitrary partition of the edges of K_p, then for some i, $\langle E_i \rangle$ contains at least one of the graphs from the set $\{G\}_i$. Investigations of this type of Ramsey number may shed light on several questions concerning the more standard Ramsey numbers, $r(G_1, \ldots, G_k)$.

A general determination of $r(\leqslant [C])$ for $t > 3$ would be of great interest. More specifically, it would be helpful to know exactly for what cases the result $r(\leqslant [C]) = 2^t + 1$ holds.

All of our results have been for the case where there is exactly one large cycle. Let $[X]$ denote some specified list of graphs. It would be quite enlichtening to know $r(C_n, C_m, [X])$, where n and m are comparably large.

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