PROBLEMS AND RESULTS ON DIOPHANTINE APPROXIMATIONS (II)

P. ERDÖS

In a previous paper (referred to as (I)) I discussed several problems. First of all I will report on any progress made on these questions and at the end of the paper I will state a few new questions. I will try to make the references as complete as possible, but I have not done much work on this subject recently and I hope the reader (and writer) will forgive me if I omitted any reference.

P. Erdös, "Problems and results on diophantine approximations", Compositio Math. 16, 52-65 (1964).

1. Let $x_1, x_2, \ldots$ be an infinite sequence of real numbers in $(0,1)$. Put

$$A_k = \lim_{n=\infty} \left| \sum_{j=1}^{n} e^{2\pi i k x_j} \right| .$$

I conjectured that $\limsup_{k=\infty} A_k = \infty$ and expected the proof to be simple. Indeed, not much later I obtained a simple proof. Clunie proved a much stronger result, he showed that for infinitely many k $|A_k| > c\sqrt{k}$. Perhaps $|A_k| > ck$ holds for infinitely many values of k. Clunie gives an example for which $|A_k| \leq k$ for all k.

J. Clunie, "On a problem of Erdös", J. London Math. Soc. 42, 133-136 (1967).

2. Following van der Corput, define the discrepancy $D(x_1,\ldots,x_n)$ of $x_1,\ldots,x_n$ (the x_i are all in $(0,1)$) as follows:

$$D(x_1,\ldots,x_n) = \sup_{0 \leq a < b \leq 1} |N_n(a,b) - (b-a)n|$$

where $N_n(a,b)$ is the number of x_i in the interval (a,b).

Sharpening previous results of Mrs. Aardenne-Ehrenfest and K. F. Roth, W. Schmidt proved

$$\max_{1 \leq m \leq n} D(x_1, \ldots, x_m) > 10^{-2} \log n$$

which is best possible apart from the value of 10^{-2} .

I asked: Is there an infinite sequence $x_1, \ldots$ so that for every (a,b) , $0 \leq a < b \leq 1$

$$\limsup_{n \to \infty} N_n(a,b) < \infty \ ? \tag{1}$$

Schmidt proved that the answer is negative, in fact he showed that for fixed a there are only denumerably many values of b for which (1) holds. Schmidt in fact proved that there is an α for which

$$\limsup_{n \to \infty} \frac{N_n(0,\alpha)}{\log n} > \frac{1}{2000} \tag{2}$$

and that for almost all α

$$\limsup N_n(0,\alpha)/\log \log n > 0 \ . \tag{3}$$

Schmidt of course observes the gap between (2) and (3) and raises the question wether $\log \log n$ in (3) can be improved and perhaps even replaced by $\log n$.

Let β be an irrational number, $x_n = n\beta - [n\beta]$. Hecke and Ostrowski proved that

$$|N_n(a,b) - n(b-a)| \tag{4}$$

is bounded for this sequence if $|a-b| = n\beta - [n\beta]$. P. Szüsz and I conjectured that the converse also holds, i.e., if (4) is bounded, then $|a-b| = n\beta - [n\beta]$. This conjecture was proved by Kesten.

I raised several problems in (1) about discrepancies of sets of points in higher dimensions or in the plane or surface of the sphere if the sets in which we want the discrepancy to be small are spherical caps or circles. All these questions asked in (1) have been resolved and extended in various deep papers of W. Schmidt. As I conjectured, the discrepancies increase like a power of n, but the best exponent is not yet determined. Several interesting results have recently been obtained by W. Philipp on the discrepancy of sequences of the form $n_k\alpha - [n_k\alpha]$, where $n_{k+1}/n_k > c > 1$. It is not yet clear wether these results can be extended if we only assume $n_{k+1} > n_k$. For the older results in this direction of Cassels, Koksma and myself see I.

T. von Aardenne-Ehrenfest, "On the impossibility of a just distribution", Indig. Math. 11, 264-269 (1949).

K. F. Roth, "On irregularities of distribution", Mathematika 1, 73-79 (1955).

W. M. Schmidt, "On irregularities of distribution I to IX", Quaterly J. Math. 19, 181-191 (1968); Trans. Amer. Math. Soc. 136, 347-360 (1969); Pacific J. Math. 29, 225-234 (1969); Compositio Math. 24, 63-74 (1972); Acta Arith. 21, 45-50 (1972); VIII will appear in Trans. Amer. Math. Soc. and IX in Acta Arith. See also Schmidt's lecture notes, published by the Tata Institute.

H. Kesten, "On a conjecture of Erdös and Szüsz related to uniform distribution mod 1 ", Acta Arith. 12,193-212 (1968).

For another conjecture of Szüsz, Turan and the author mentioned in I, see

H. Kesten and V.T. Sos , "On two problems of Erdös, Szüsz and Turan concerning diophantine approximations", Acta Arith. 12, 183-192 (1968).

W. Philipp, Limit theorems for lacunary series and uniform distribution mod 1 , will appear in Acta Arithmetica.

3. Let E be a measurable subset of $(0,1)$. Put

$$f_n(E,\alpha) = \sum_{1 \le k \le n} 1$$

where the summation is extended over those k for which $(k\alpha) = k\alpha - [k\alpha]$ is in E.

Khintchine conjectured that for almost all α and for every E

$$(1) \qquad \lim_{n=\infty} f_n(E,\alpha)/n = m(E) \; .$$

It was a great surprise to me when Marstrand disproved (1). In fact he proved that there is a closed set E for which

$$\limsup_{n=\infty} f_n(E,\alpha)/n = 1 \quad , \quad \liminf_{n=\infty} f_n(E,\alpha)/n = 0$$

holds for almost all α . In fact Marstrand proves a more general theorem which answers many of the relevant problems here in the negative, but some interesting unsolved problems remain (see I).

A sequence $x_1, x_2, \ldots$ in $(0,1)$ is said to be well distributed if to every $\varepsilon > 0$ there exists a $k_0 = k_0(\varepsilon)$ so that for every $k > k_0$, $n > 0$ and $0 \le a < b \le 1$

$$\left| \frac{1}{k} N_{n,n+k}(a,b) - (b-a) \right| < \varepsilon$$

where $N_{n,n+k}(a,b)$ denotes the number of x_m's , $n < m \le n+k$ in (a,b) . As far as I know, well distributed sequences were introduced by Hlawka and Petersen. I claimed in I that there is an irrational α for which $(p_n\alpha)$ is not well distributed where p_n , $n = 1, 2, \ldots$ is the sequence of primes. There is no doubt that this result is true but I was not able to reconstruct my proof (and thus perhaps it never existed). I am sure that in fact for every α $(p_n\alpha)$ is not well distributed.

J. M. Marstrand, "On Khintchine's conjecture about strong uniform distribution", Proc. London Math. Soc. 21, 540-556 (1970).

4. Let $n_1 < n_2 < \ldots$ be an infinite sequence of integers. I proved that the necessary and sufficient condition that for almost all α

$$\left| \alpha - \frac{a}{n_k} \right| < \frac{\varepsilon}{n_k^2} \quad , \quad (a, n_k) = 1$$

should have infinitely many solutions is that

$$(2) \qquad \sum_{q=1}^{\infty} \frac{\varepsilon_q \, \phi(q)}{q} = \infty \ .$$

This old conjecture, of course, contains my theorem (in fact I was led to it by this conjecture). I hope and believe that my method will lead to a proof of (2) but the technical difficulties seem too great and I was not able to overcome them.

In connection with problems on diophantine approximations Cassels introduced a property of sequences which seemed to me to have independent interest. Let $n_1 < n_2 < \dots$ be an infinite sequence of integers. Denote by $\phi(n_1,\dots,n_{k-1};n_k)$ the number of integers $1 \le a \le n_k$ for which $\frac{a}{n_k} \ne \frac{b}{n_j}$ for every $1 \le j < k$. Clearly $\phi(n_1,\dots,n_{k-1};n_k) \ge \phi(n_k)$.
Cassels calls the sequence $\{n_k\}$ a Σ sequence if

$$(3) \qquad \liminf_{k = \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{\phi(n_1,\dots,n_{i-1};n_i)}{n_i} > 0 \ .$$

Cassels shows that there are sequences which are not Σ -sequences. I then asked : Is there a sequence $n_1 < n_2 < \dots$ for which

$$\lim_{k = \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{\phi(n_1,\dots,n_{k-1};n_k)}{n_i} = 0 \ .$$

I thought that such a sequence does not exist and was very surprised to learn that Haight constructed such a sequence in his thesis (his construction is not yet published). As far as I know it has not yet been investigated how fast the sum $\frac{1}{k} \sum_{i=1}^{k} \frac{\phi(n_1,\dots,n_{i-1};n_i)}{n_i}$ can tend to 0.

I proved in I that $\liminf_{k \to \infty} \frac{\phi(n_1,\dots,n_{k-1};n_k)}{n_k} = 0$ implies

$$\limsup_{k \to \infty} \frac{\phi(n_1,\dots,n_{k-1};n_k)}{k} = 1 \ .$$

P. Erdös, "On the distribution of the convergents of almost all real numbers", Journal of Number Theory 2, 425-441 (1970).

See also, W. Philipp, Mixing sequences of random variables and probabilistic number theory, Memoirs Amer. Math. Soc. 114 (1971), see chapters 2 and 3.

5. The following problem is due to Leveque : Let $A = \{a_1 < a_2 \ldots\ldots\}$ be an infinite sequence tending to infinity satisfying

$$a_{n+1}/a_n \to 1 \quad . \text{ Let } \quad a_1 \leq x_n \leq a_{i+1} \quad . \text{ Put } \quad Y_n = \frac{x_n - a_i}{a_{i+1} - a_i} \quad , \quad 0 \leq Y_n \leq 1 \; .$$

Leveque calls the sequence x_n , $1 \leq n < \infty$ uniformly distributed mod A if Y_n , $1 \leq n < \infty$ is uniformly distributed. Is it true that for almost all α the sequence $n\alpha$, $1 \leq n < \infty$ is uniformly distributed mod A ? Leveque proved this in some special cases, and Davenport and I proved it if $a_n > n^{\frac{1}{2}+\varepsilon}$. We conjectured that this always holds but this was disproved by W. Schmidt, and it does not seem to be easy to give a necessary and sufficient condition in terms of A of the uniform distribution of $n\alpha$ for almost all α mod A .

Schmidt posed the following problem : Does there exist a set S of real numbers having infinite measure so that the quotient of two elements of S is never an integer ? Haight and Szemeredi independently of each other constructed such a set. One can ask : Denote by $S(x)$ the intersection of S with the interval $(0,x)$ and assume that the quotient of two elements of S is never an integer. How fast can $m(S(x))$ tend to infinity ? ($m(S(x))$ is the measure of $S(x)$.). As far as I know this question has not yet been answered.

The following very interesting question is due to Haight : Let S be an arbitrary set of positive measure. The set $S(\infty)$ is defined as follows : $Y \in S(\infty)$ if there is an $x \in S$ and an integer n with $nx = Y$.
Is it true that to almost all z there is an $n(z)$ so that for every $m > n(z)$ $m.z \in S(\infty)$? Another result of Haight states : Let S be a set (of positive numbers) so that for every $x > 1$ there is at most one multiple of x in S . Then $m(S) < 9$. I wonder if perhaps $m(S) \leq \frac{3}{2}$. $\frac{3}{2}$ if true is best possible (S is the interval $(\frac{3}{2},3)$).

W. Schmidt, "Disproof of some conjectures on diophantine approximations", Studia Sci. Math. Hungary 4, 137-144 (1969).

P. Erdös and H. Davenport, "A theorem on uniform distribution", Publ. Math. Inst. Hung. Acad. Sci. 8, 3-11 (1963).

A. Haight, "A Linear set of infinite measure with no two points having integral ratio", Mathematatika 17, 133-138 (1970).

E. Szemerédi, "On a problem of W. Schmidt", Studia Sci. Math. Hung. 6, 287-288 (1971).

See also C. G. Lekkerkerker, "Lattice points in unbounded point sets", Indig. Math. 20, 197-205 (1958).

6. Let there be given n points $x_1, \ldots, x_n$ in the unit circle. Denote by $A(x_{i_1}, \ldots, x_{i_k})$ the area of the least convex polygon containing $x_{i_1}, \ldots, x_{i_k}$. Put

$$f_k(n) = \max_{x_1, \ldots, x_n} \min A(x_{i_1}, \ldots, x_{i_k}) .$$

Heilbronn raised the problem of estimating $f_3(n)$ from above and below. $f_3(n) < \frac{c}{n}$ is obvious, $f_3(n) > \frac{c_1}{n^2}$ is easy. The first non trivial result is due to K. F. Roth who proved

$$f_3(n) < \frac{c}{n(\log \log n)^{1/2}} .$$

This was improved by Schmidt to

$$f_3(n) < \frac{c}{n(\log n)^{1/2}}$$

Finally K. F. Roth proved

$$f_3(n) < \frac{c}{n^{1+\alpha}} \quad , \quad \alpha = \frac{1}{8}\left(9 - 65^{1/2}\right) .$$

It would be very interesting to decide wether $f_3(n) < c/n^2$ is true.

The first results on $f_k(n)$, for $k > 3$ are due to Schmidt.

He proved : $f_4(n) > \frac{c}{n^{3/2}}$ and conjectures $f_k(n) = \sigma(\frac{1}{n})$.

In conversation with K. F. Roth, we observed the following fact which might be interesting. There are two trivial ways to obtain $f_3(n) < \frac{c}{n}$.

One can find the triangle with the smallest angle or smallest diameter, both methods give $f_3(n) < \frac{c}{n}$. Now let there be given n points in the unit sphere, denote

$$f_3^{(3)}(n) = \max_{x_1,\dots,x_n} \min A(x_1,x_j,x_r) \quad .$$

The smallest angle method here gives $f_3^{(3)}(n) < \frac{c}{n^{1/2}}$ but the smallest diameter gives $f_3^{(3)}(n) < \frac{c}{n^{2/3}}$.

Is $f_3^{(3)}(n) = \sigma(\frac{1}{n^{2/3}})$ true ? It is not immediately clear if any of the methods so far developed apply here.

K. F. Roth, " On a problem of Heilbronn II and III", Proc. London Math. Soc. 25, 193-212 and 543-549 (1972).

W. Schmidt, " On a problem of Heilbronn", J. London Math. Soc. (2) 4, 545-550 (1971-1972).

Finally I state a few disconnected problems. Let $n_1 < n_2 < \dots$ be an infinite sequence of integers satisfying $n_{k+1}/n_k > c > 1$. Is it true that there always is an irrational α for which the sequence $(n_k\alpha)$ is not everywhere dense ? Taylor and I proved that the set of α's for which $(n_k\alpha)$ is not uniformly distributed has Hausdorff dimension one.

Let $\alpha > 1$ and β be real numbers. We call the sequence $[t\alpha + \beta]$, t integer a generalized arithmetic progression. Let $\{n_k\}$ tend to infinity sufficiently fast. Is it true that the complement of $\{n_k\}$ contains an infinite generalized arithmetic progression ?

Let $\phi(n)$ be Euler's ϕ function. Denote by $f(x)$ the density of integers n for which $\frac{\phi(n)}{n} < x$. I. Schoenberg proved that the density exists for every x and I proved that $f(x)$ is purely singular. Thus for almost all x the derivative of $f(x)$ is 0 and it is easy to see that for some x the derivate of $f(x)$ from the right is infinite. Can the derivate of $f(x)$ exist and be different from 0 ? Or more generally, can its value be any given number α ?

As far as I know no progress has been made with the following problem : Let $|z_n| = 1$, $n = 1,2,\ldots$ be an infinite sequence of points on the unit circle. Put

$$A_n = \max_{|z|=1} \prod_{i=1}^{n} |z - z_i| .$$

Is it true that A_n is unbounded ? A simple example of Hayman shows that $A_n \leq n$ is possible for all n .

More generally the following question can be asked : Let D be a closed set without interior points. Put

$$E_n(D) = \min_{f_n(z)} \max_{z \in D} |f_n(z)|$$

where the minimum is extended over all monic polynomials whose roots are all in D . Let $z_1, z_2, \ldots$ be an infinite sequence of points in D . Put

$$A_n(D) = \max_{z \in D} \prod_{i=1}^{n} |z - z_i| .$$

Is it true that $\limsup_{n = \infty} A_n(D)/E_n(D) = \infty$?

If D is the unit circle and its interior the result clearly fails ($z_n = 0$ for all n), thus the condition that z_n has no interior points can not be entirely dropped, but it may remain true for some such domains.

Let finally $1 < a_1 < \ldots$ be a sequence of integers no one of which divides any other. I proved

$$(1) \qquad \sum_{k=1}^{\infty} \frac{1}{a_k \log a_k} < c$$

and Behrend proved

(2) $$\sum_{a_k<x} \frac{1}{a_k \log a_k} < c_1 \log x(\log\log x)^{-\frac{1}{2}}$$

(1) and (2) was sharpened and generalized by Sárkozy, Szemerédi and myself. As far as I know the following problem of diophantine character has not yet been much investigated : Let $1 < a_1 < \ldots$ be a sequence of real numbers. Assume that for every integer i, j and ℓ

(3) $$| a_j - \ell a_i | \geq 1$$

Does (3) imply (1) and (2) ? If the a's are integers then (3) states that no $\underline{a}$ divides any other. I could not even prove that (3) implies

$(A(x) = \sum_{a_k<x} 1)$

$$\liminf_{x = \infty} A(x)/_x = 0 \ .$$

Haight proved that if the a's are rationally independent and (3) holds then

(4) $$\lim_{x = \infty} A(x)/_x = 0$$

A well known result of Besicovitch states that (3) does not imply (4) if the a's are integers. The real difficulty seems to be if the a_i are rational.

REFERENCES

[1] ERDÖS, P., TAYLOR, J.: On the set of points of convergence of a lacunary trigonometric series and the equidistribution properties of related sequences. Proc. London Math. Soc. 7, 598-615 (1957).

[2] ERDÖS, P.: On the smoothness of the asymptotic distribution of additive arithmetical functions. Amer. J. Math. 61, 722-725 (1939).

[3] ERDÖS, P., SARKOZY, A. and SZEMEREDI, E.: On divisibility properties of sequences of integers. Number Theory Colloquium, Janus Bolyai Math. Soc. North Holland Amsterdam, 36-49 (1968).

[4] SCHOENBERG, I. J.: Über die asymptotische Verteilung reeller Zahlen mod 1 . Math. Zeitschrift 28, 171-190 (1928).

Mathematical Institute
The Hungarian Academy of Sciences
Budapest V., Realtanado u. 13-15
Hungary