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OSCILLATIONS OF BASES FOR THE NATURAL NUMBERS

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ABSTRACT. Let A be a set of positive integers. Then A is a basis if every sufficiently large integer n can be written in the form $n = a_i + a_j$ with $a_i, a_j \in A$. Otherwise, A is a nonbasis. In this paper we construct sets which oscillate from basis to nonbasis to basis or from nonbasis to basis to nonbasis under finite perturbations of the sets.

1. Introduction. A number is a positive integer. A set is a set of numbers. If $A = \{a_i\}_{i=1}^{\infty}$ is a set such that every sufficiently large number n can be written in the form $n = a_i + a_j$, then A is called an asymptotic basis of order 2, or simply, a *basis*. If A is not a basis, then infinitely many numbers are not of the form $a_i + a_j$, and A is called an asymptotic nonbasis of order 2, or, simply, a *nonbasis*.

A basis $A = \{a_i\}_{i=1}^{\infty}$ is *minimal* if no proper subset of A is a basis; that is, $A \backslash \{a_j\}$ is a nonbasis for every $a_j \in A$. Minimal bases were introduced by Stöhr [4], and Härtter [2] and Nathanson [3] constructed examples of minimal bases, and also of bases no subset of which is minimal.

A nonbasis $A = \{a_i\}_{i=1}^{\infty}$ is *maximal* if no proper superset of A is a nonbasis; that is, $A \cup \{b\}$ is a basis for every number $b \notin A$. Maximal nonbases were introduced by Nathanson [3], and examples were constructed by Erdös and Nathanson [1]. It is still not known if every nonbasis is contained in a maximal nonbasis.

Minimal bases and maximal nonbases are examples of sets which oscillate under small perturbations from bases to nonbases and from nonbases to bases. Such oscillations are the theme of this paper.

Notation. Numbers will be denoted by lower case Latin letters, and sets by upper case Latin letters. The set of all numbers is denoted N. We denote by $|A|$ the cardinality of the set A, and by $A \backslash B$ the complement of B in A. The set of numbers between a and b is denoted $[a, b]$. If $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_j\}_{j=1}^{\infty}$, then the sum of A and B is $A + B = \{a_i + b_j | a_i \in A, b_j \in B\}$. The sum $A + A$ is written $2A$. Finally, the number of elements of A which do not exceed n is denoted $A(n)$, and the set A has density δ if $\lim_{n\to\infty} A(n)/n = \delta$.

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Lemma. *Let $Q = \{2q_k + 1\}_{k=1}^{\infty}$ be a set of odd numbers such that $q_k \geq 5q_{k-1} + 3$. Let*

$$A^Q = \bigcup_{k=2}^{\infty} \{[2q_{k-1} + 2, q_k - q_{k-1}] \cup [q_k + 1, q_k + q_{k-1}]\}.$$

Then $2A^Q \subset N \setminus Q$, and $2A^Q$ contains all but finitely many of the numbers in $N \setminus Q$. Moreover, if F and G are any finite sets, then $2(A^Q \cup G)$ and $2(A^Q \setminus F)$ differ from $2A^Q$ by only finitely many numbers; that is, $|2(A^Q \cup G) \setminus 2(A^Q \setminus F)| < \infty$.

Proof. Let $I_k = [2q_{k-1} + 2, q_k - q_{k-1}]$ and $J_k = [q_k + 1, q_k + q_{k-1}]$. If $2q_k + 1 = a + b$, then one of the summands, say a, must be greater than q_k. If $a \in A^Q$, then $a \in J_k$. But then $b = 2q_k + 1 - a \notin A^Q$. Therefore, $2q_k + 1 \notin 2A^Q$, and so $2A^Q \subset N \setminus Q$. Moreover,

$$(1) \qquad I_k + J_k = [2q_{k-1} + q_k + 3, 2q_k] \subset 2A^Q,$$

$$(2) \qquad 2I_k = [4q_{k-1} + 4, 2q_k - 2q_{k-1}] \subset 2A^Q,$$

$$(3) \qquad 2J_{k-1} = [2q_{k-1} + 2, 2q_{k-1} + 2q_{k-2}] \subset 2A^Q.$$

Let $a \in A^Q$. Then

$$(4) \qquad I_k + \{a\} = [2q_{k-1} + 2 + a, q_k - q_{k-1} + a] \subset 2A^Q,$$

If $2q_{k-2} > a$, then the intervals (1)–(4) completely cover $[2q_{k-1} + 2, 2q_k]$, and so all sufficiently large numbers in $N \setminus Q$ belong to $2A^Q$. If a finite set F is removed from A^Q, then A^Q still contains the intervals I_k and J_k for large enough k, and the argument above shows that $2(A^Q \setminus F)$ consists of all but finitely many numbers in $N \setminus Q$. If a finite set G is added to A^Q, then $2(A^Q \cup G)$ will contain only finitely many elements of Q.

This Lemma will be applied frequently; in particular, $A^Q = \bigcup_{k=1}^{\infty} \{I_k \cup J_k\}$ will always denote the set constructed from a set $Q = \{2q_k + 1\}_{k=1}^{\infty}$ satisfying $q_k \geq 5q_{k-1} + 3$.

2. **Minimal bases.** A basis A is *r-minimal* if $A \setminus F$ is a basis whenever $|A \cap F| < r$, but $A \setminus F$ is a nonbasis if $|A \cap F| \geq r$. The minimal bases discussed in the introduction are precisely the 1-minimal bases. We shall construct for each r a class of r-minimal bases. The basis A is called $\aleph_0$-minimal if $A \setminus F$ is a basis for every finite subset F of A, but for no infinite subset F of A. We shall construct a class of $\aleph_0$-minimal bases. This answers a question posed in [1]. Finally, we prove that there does not exist a basis $A = \{a_i\}_{i=1}^{\infty}$ such that $A \setminus \{a_u\}_{u \in U}$ is a basis if and only if U has density zero.

Theorem 1. *Every A^Q is contained in an r-minimal basis.*

Proof. Let $\{U_k\}_{k=r+2}^{\infty}$ be a sequence of sets each containing exactly r numbers, such that $u \le k$ for $u \in U_k$, and such that every set of r numbers occurs infinitely often in the sequence. We shall construct an r-minimal basis $A = \bigcup_{k=2}^{\infty} A_k$ such that $I_k \cup J_k \subset A_k \subset [2q_{k-1} + 2, 2q_k]$. Let $A_k = I_k \cup J_k$ for $k \le r+1$. Suppose that A_i has been determined for all $i < k$. Write the elements of the finite set $\bigcup_{i<k} A_i$ in increasing order $a_1 < a_2 < a_3 < \cdots$. Let $A_k = I_k \cup J_k \cup \{2q_k + 1 - a_u\}_{u \in U_k}$. Then $A^Q \subset A = \bigcup_{k=2}^{\infty} A_k$, and every sufficiently large element of Q has exactly r representations of the form $2q_k + 1 = a_i + a_j$, where $a_i, a_j \in A$ and $a_i < a_j$. Moreover, every r-element subset of A is required for the representation of infinitely many elements of Q. It follows from the Lemma that $A \setminus F$ is still a basis if $|A \cap F| < r$, but that $A \setminus F$ is a nonbasis if $|A \cap F| \ge r$.

Theorem 2. *Every A^Q is contained in an $\aleph_0$-minimal basis.*

Proof. We construct an $\aleph_0$-minimal basis $A = \bigcup_{k=2}^{\infty} A_k$ such that $I_k \cup J_k \subset A_k \subset [2q_{k-1} + 2, 2q_k]$. Let $A_2 = I_2 \cup J_2 \cup \{(2q_2 + 1) - (2q_1 + 2)\}$. If A_i has been determined for $i < k$, and if $\bigcup_{i<k} A_i$ consists of the elements $a_1 < a_2 < a_3 < \cdots$, then let $A_k = I_k \cup J_k \cup \{2q_k + 1 - a_{k-1}\}$. Then $A^Q \subset A = \bigcup_{k=2}^{\infty} A_k$, and A is a basis. Moreover, each element a_i of A is required for the representation of exactly one element of Q. It follows that $A \setminus F$ is a basis if F is any finite subset of A, but that $A \setminus F$ is a nonbasis if F is an infinite subset of A.

Theorem 3. *There does not exist a basis $A = \{a_i\}_{i=1}^{\infty}$ such that $A \setminus \{a_u\}_{u \in U}$ is a basis if U has zero density and a nonbasis if U has positive density.*

Proof. Let $A = \{a_i\}_{i=1}^{\infty}$ be a basis such that $A \setminus \{a_u\}_{u \in U}$ is a nonbasis if U has positive density. We shall construct a set U_0 of zero density such that $A \setminus \{a_u\}_{u \in U_0}$ is a nonbasis. Let $U_2 = \{u_i^{(2)}\}_{i=1}^{\infty}$ be a strictly increasing sequence of numbers with density ½. Then $A \setminus \{a_u\}_{u \in U_2}$ is not a basis, so there exists a number s_2 such that $2(A \setminus \{a_{u_j^{(2)}}\}_{j=1}^{s_2}) \ne 2A$. If $A \setminus \{a_{u_j^{(2)}}\}_{j=1}^{s_2}$ is a nonbasis, we are done, since the finite set $\{u_j^{(2)}\}_{j=1}^{s_2}$ has density zero. Suppose that $A \setminus \{a_{u_j^{(2)}}\}_{j=1}^{s_2}$ is a basis. Let $U_3 = \{u_j^{(3)}\}_{j=1}^{\infty}$ be a subsequence of U_2 of density 1/3 such that $u_j^{(3)} = u_j^{(2)}$ for $j \le s_2$. Then $A \setminus \{a_u\}_{u \in U_3}$ is not a basis, and so there is a number $s_3 > s_2$ such that $2(A \setminus \{a_{u_j^{(3)}}\}_{j=1}^{s_3}) \ne 2(A \setminus \{a_{u_j^{(2)}}\}_{j=1}^{s_2})$. Continuing inductively, we obtain either a finite set U_0 such that $A \setminus \{a_u\}_{u \in U_0}$ is a nonbasis, or a sequence of integers $s_2 < s_3 < s_4 < \ldots$, and a sequence of sets $U_2 \supset U_3 \supset U_4 \supset \cdots$ such that

(i) U_n has density $1/n$;

(ii) If $U_n = \{u_j^{(n)}\}_{j=1}^{\infty}$, then $u_j^{(n)} = u_j^{(n+1)}$ for $j \le s_n$;

(iii) $2(A \setminus \{a_{u_j^{(n+1)}}\}_{j=1}^{s_{n+1}}) \ne 2(A \setminus \{a_{u_j^{(n)}}\}_{j=1}^{s_n})$.

Let $U_0 = \bigcap_{n=2}^{\infty} U_n$. Then $A \setminus \{a_u\}_{u \in U_0}$ is a nonbasis, and U_0 has density zero.

By modifying slightly the proof of Theorem 3, one can show that if ϕ is any function tending monotonically to infinity, then there cannot exist a basis $A = \{a_i\}_{i=1}^{\infty}$ such that $A \setminus \{a_u\}_{u \in U}$ is a basis if and only if $\limsup_{n \to \infty} A(n)/\phi(n) = 0$.

3. Maximal nonbases. A nonbasis A is *s-maximal* if $A \cup G$ is a nonbasis whenever $|G \setminus A| < s$, but $B \cup G$ is a basis if $|G \setminus A| \geq s$. The maximal nonbases discussed in the introduction are precisely the 1-maximal nonbases. We shall construct a class of s-maximal nonbases for each s. A nonbasis A could be called $\aleph_0$-maximal if $A \cup G$ is a nonbasis if and only if $|G \setminus A| < \infty$. But we prove that $\aleph_0$-maximal nonbases do not exist.

Theorem 4. *Every A^Q is contained in an s-maximal nonbasis.*

Proof. Let $\{V_k\}_{k=s+2}^{\infty}$ be a sequence of sets each containing exactly $s-1$ numbers, such that $v \leq k$ for $v \in V_k$, and such that every set of $s-1$ numbers occurs infinitely often in the sequence. We shall construct an s-maximal nonbasis $A = \bigcup_{k=2}^{\infty} A_k$ such that $I_k \cup J_k \subset A_k \subset [2q_{k-1}+2, 2q_k]$. Let $A_k = I_k \cup J_k$ for $k \leq s+1$. Suppose that A_i has been determined for all $i < k$. Let $B_k = [1, 2q_{k-1}] \setminus \bigcup_{i<k} A_i$. Write the elements of the finite set B_k in increasing order $b_1 < b_2 < b_3 < \cdots$. Let

$$A_k = I_k \cup J_k \cup \{2q_k + 1 - b \mid b \in B_k \text{ and } b \neq b_v \text{ for } v \in V_k\}.$$

Then $A^Q \subset A = \bigcup_{k=2}^{\infty} A_k$, and A is a nonbasis. If $G \cap A = \emptyset$ and $|G| = s-1$, then $G = \{b_{v_1}, b_{v_2}, \ldots, b_{v_{s-1}}\}$ for some set $V = \{v_1, v_2, \ldots, v_{s-1}\}$. But $V = V_k$ for infinitely many k, and, for each of these k, the number $2q_k + 1 \notin 2(A \cup G)$. Therefore, $A \cup G$ is a nonbasis. But if $|G| \geq s$, then $A \cup G$ is a basis.

Theorem 5. *There does not exist an $\aleph_0$-maximal nonbasis.*

Proof. If A is a nonbasis such that $A \cup G$ is a nonbasis for every finite set G, then there must exist an infinite set G such that $A \cup G$ is a nonbasis. Indeed, we have proved that there exists an infinite set G such that $A \cup G$ is a maximal nonbasis [1].

4. Double oscillations. Now we consider sets which oscillate from basis to nonbasis to basis, or from nonbasis to basis to nonbasis. Let A be a set, and let F and G be finite sets such that $F \subset A$ and $G \cap A = \emptyset$. Then A is an *(r, s)-basis* if (i) $A \setminus F$ is a basis if and only if $|F| < r$, and (ii) if $|F| = r$ (and so $A \setminus F$ is a nonbasis), then $(A \setminus F) \cup G$ is a nonbasis if and only if $|G| < s$. We shall construct a class of (r, s)-bases. The set A

is an (s, r)-*nonbasis* if (i) $A \cup G$ is a nonbasis if and only if $|G| < s$, and (ii) if $|G| = s$ (and so $A \cup G$ is a basis), then $(A \cup G)\backslash F$ is a basis if and only if $|F| < r$. We also construct a class of (s, r)-nonbases.

Theorem 6. *Every A^Q is contained in an (r, s)-basis.*

Proof. Let $\{U_k\}_{k=r+s}^{\infty}$ and $\{V_k\}_{k=r+s}^{\infty}$ be sequences of sets containing r elements and $s-1$ elements, respectively, such that $u \le k$ and $v \le k$ for $u \in U_k$ and $v \in V_k$, and such that every set with r elements occurs infinitely often in the sequence $\{U_k\}_{k=r+s}^{\infty}$ and every set with $s-1$ elements occurs infinitely often in the sequence $\{V_k\}_{k=r+s}^{\infty}$. If $|U| = r$, let $K^U = \{k | U_k = U\}$. Then K^U is an infinite set, and we may write $K^U = \{k_j^U\}_{j=1}^{\infty}$, where $k_1^U < k_2^U < \cdots$.

We shall construct an (r, s)-basis $A = \bigcup_{k=2}^{\infty} A_k$ such that $I_k \cup J_k \subset A_k \subset [2q_{k-1}+2, 2q_k]$. Let $A_k = I_k \cup J_k$ for $k \le r+s$. Suppose that A_i has been determined for $i < k$. Write the elements of the finite set $\bigcup_{i<k} A_i$ in increasing order $a_1 < a_2 < a_3 < \cdots$. Let $A_k' = I_k \cup J_k \cup \{2q_k + 1 - a_u\}_{u \in U_k}$. If $U = U_k$, then $k \in K^U$, and so $k = k_j^U$ for some $j \le k$. Let $B_k = [1, 2q_{k-1}]\backslash\bigcup_{i<k} A_i$. Write the elements of B_k in increasing order $b_1 < b_2 < b_3 < \cdots$. Let

$$A_k = A_k' \cup \{2q_k + 1 - b \,|\, b \in B_k \text{ and } b \ne b_v \text{ for } v \in V_j\}.$$

Then $A = \bigcup_{k=2}^{\infty} A_k$ has all of the desired properties.

Theorem 7. *Every A^Q is contained in an (s, r)-nonbasis.*

Proof. Let $\{U_k\}_{k=r+s}^{\infty}$ and $\{V_k\}_{k=r+s}^{\infty}$ be sequences of sets exactly as in the proof of Theorem 6. We shall construct an (s, r)-nonbasis $A = \bigcup_{k=2}^{\infty} A_k$ such that $I_k \cup J_k \subset A_k \subset [2q_{k-1}+2, 2q_k]$. Let $A_k = I_k \cup J_k$ for $k \le r+s$. Suppose that A_i has been determined for $i < k$. If $k = 2m$ is even, write the elements of the finite set $\bigcup_{i<k} A_i$ in increasing order $a_1 < a_2 < a_3 < \ldots$, and let $A_k = I_k \cup J_k \cup \{2q_k + 1 - a_u\}_{u \in U_m}$. If $k = 2m+1$ is odd, write the elements of the finite set $B_k = [1, 2q_{k-1}]\backslash\bigcup_{i<k} A_i$ in increasing order $b_1 < b_2 < b_3 < \ldots$, and let

$$A_k = I_k \cup J_k \cup \{2q_k + 1 - b \,|\, b \in B_k \text{ and } b \ne b_v \text{ for } v \in V_m\}.$$

Then $A = \bigcup_{k=2}^{\infty} A_k$ has all of the desired properties.

We are not able to construct a set that oscillates infinitely often from basis to nonbasis to basis to nonbasis . . . . The existence of such infinitely oscillating sets can be proved by a probabilistic method. This will be discussed in a later paper.

Note added in proof. We have constructed a set which oscillates infinitely often from basis to nonbasis to basis ... as random elements are successively deleted from and added to the set. This is a special case of a theorem proved in the paper *Partitions of the natural numbers into infinitely oscillating bases and nonbases* (to appear).

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