MATHEMATICS OF COMPUTATION, VOLUME 28, NUMBER 126, APRIL 1974, PAGES 647–649

A New Function Associated with the Prime Factors of $\binom{n}{k}$

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Abstract. Let $g(k)$ denote the least integer $> k + 1$ so that all the prime factors of $\binom{g(k)}{k}$ are greater than k. The irregular behavior of $g(k)$ is studied, obtaining the following bounds:

$$k^{1+c} < g(k) < \exp(k(1 + o(1))).$$

Numerical values obtained for $g(k)$ with $k \leqq 52$ are listed.

The prime factors of $\binom{n}{k}$ have been studied a great deal. In a recent paper, Erdös [2] stated several results and unsolved problems on this subject. In this paper, we discuss one of the problems stated there: Denote by $g(k)$ the least integer $> k + 1$* so that all prime factors of $\binom{g(k)}{k}$ are greater than k. Determine or estimate $g(k)$.

The behavior of $g(k)$ is surprisingly irregular. We searched for values of $g(k) \leqq 2500000$ for $2 \leqq k \leqq 100$; the results of this search are reported in Table 1. In reviewing Table 1, we noticed the surprising example $g(28) = 284$. This motivated a second search for other such examples with $g(k) \leqq 100000$ and $101 \leqq k \leqq 500$; none were found.

TABLE 1. *Values of* $g(k) \leqq 2500000$ *for* $2 \leqq k \leqq 100$

k	$g(k)$	k	$g(k)$	k	$g(k)$	k	$g(k)$	k	$g(k)$
		11	47	21	14871	31	341087	41	*B*
2	6	12	174	22	19574	32	371942	42	96622
3	7	13	2239	23	35423	33	6459	43/	*B*
4	7	14	239	24	193049	34	69614	45	
5	23	15	719	25	2105	35	37619	46	692222
6	62	16	241	26	36287	36	152188	47/	*B*
7	143	17	5849	27	1119	37	152189	51	
8	44	18	2098	28	284	38	487343	52	366847
9	159	19	2099	29	240479	39	767919	53/	*B*
10	46	20	43196	30	58782	40	85741	100	

B: $g(k)$ *exceeds the search bound of* 2500000

The following conjectures on $g(k)$ all seem certainly true, and perhaps some of them will not be difficult to prove. First, we conjecture

$$\limsup_{k\to\infty} g(k+1)/g(k) = \infty \quad \text{and} \tag{1}$$

$$\liminf_{k\to\infty} g(k+1)/g(k) = 0. \tag{2}$$

Received May 7, 1973.

AMS (MOS) subject classifications (1970). Primary 10H15.

* The condition $g(k) > k + 1$ was inserted to avoid the special case $k + 1 = p$, a prime.

Also, it seems that $g(k)$ is not of polynomial growth—in other words, for every n and $k > k_0(n)$,

$$g(k) > k^n. \tag{3}$$

On the other hand,

$$\lim_{k\to\infty} g(k)^{1/k} = 1 \tag{4}$$

certainly seems to hold, and we expect that

$$g(k) < \exp(c_1\pi(k)) \tag{5}$$

is true.

We now give lower and upper bounds for $g(k)$. For a lower bound, we show there is an absolute constant $c > 0$ such that

$$g(k) > k^{1+c}. \tag{6}$$

We first show that $g(k) > 2k$ (for $k > 4$) always holds. By definition, $g(k) > k + 1$, and $g(k) \neq 2k$ since $\binom{2k}{k}$ is always even. Suppose $g(k) = k + t$ with $1 < t < k$. We have $\binom{k+t}{k} = \binom{k+t}{t}$. Ecklund [1] showed that $\binom{k+t}{t}$ has a prime factor not exceeding $(k + t)/2 < k$, the only exception being $\binom{7}{3}$ which corresponds to the case $k = 4$, $t = 3$. Erdös and Selfridge [2, p. 406] proved that if $m \geqq 2k$, then $\binom{m}{k}$ always has a prime factor $< m/k^c$, for some absolute constant $c > 0$. This immediately implies (6).

Next, we give a very crude upper bound on $g(k)$. Denote by L_k the least common multiple of the integers $1, 2, \cdots, k$ and put $P_l = \prod_{p\leq l} p$. Let $N(k, l) = L_kP_l$. If $n + 1$ is any multiple of $N(k, l)$, then

$$\binom{n}{k} = \left(\frac{mN(k, l)}{1} - 1\right)\left(\frac{mN(k, l)}{2} - 1\right) \cdots \left(\frac{mN(k, l)}{k} - 1\right)$$

has no prime factors less than l. Thus,

$$g(k) < N(k, k) = \prod_{p\leq k} p^{\alpha_p+1}, \tag{7}$$

where $\alpha_p = [\log_p k]$. For $k > k_0$, this upper bound can be improved a bit. We show

$$g(k) < k^2L_kP_l \quad \text{with } l = [6k/\log k]. \tag{8}$$

To prove (8), consider the integers $tL_kP_l - 1$ for $1 \leqq t \leqq k^2$. We show that, for at least one of these values of t,

$$p \nmid \binom{tL_kP_l - 1}{k} \quad \text{for every } p \leqq k. \tag{9}$$

For $p \leqq l$, (9) holds as before. If $l < p \leqq k$,

$$p \,\Big|\, \binom{tL_kP_l - 1}{k}$$

can only hold if there is a j, $1 \leqq j \leqq k$, for which

$$tL_kP_l \equiv j \pmod{p^{\alpha_p+1}}. \tag{10}$$

The number of integers t with $1 \leqq t \leqq k^2$, for which (10) holds, is at most

(11) $$k([k^2/p^2] + 1), \quad \text{since } \alpha_p = 1 \text{ for } p > l.$$

Thus, by (10) and (11), the number of integers t, $1 \leqq t \leqq k$, for which (10) holds for some prime p, $l < p \leqq k$, is at most

(12) $$\sum_{l<p\leqq k} k([k^2/p^2] + 1) < k^3 \sum_{p>l} 1/p^2 + k\pi(k).$$

It easily follows from the prime number theorem that, for $k > k_0$,

(13) $$\sum_{p>l} 1/p^2 < \frac{2}{l \log l} < \frac{1}{2k}.$$

From (12) and (13), for $k > k_0$, the number of integers t, $1 \leqq t \leqq k$, for which (10) holds, is less than $k^2/2 + k\pi(k) < k^2$. Thus, there is a $t \leqq k^2$ with (9) holding for every $p \leqq k$. Thus, $g(k) < k^2 L_k P_l$ as stated. The value 6 could be replaced by a smaller constant, but we cannot prove $g(k) < L_k$, which seems to hold for all k.

It is well known that $L_k < \exp(k(1 + o(1)))$ and $k^2 P_l < \exp(o(k))$. Thus, $g(k) < \exp(k(1 + o(1)))$. So $g(k) < L_k$ should be achievable.

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2. P. Erdös, "Some problems in number theory," in *Computers in Number Theory*, Academic Press, London, 1971, pp. 405–414.