SOME EXTREMAL PROBLEMS ON r-GRAPHS

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1. Introduction.

By an r-graph we mean a fixed set of vertices together with a class of unordered subsets of this fixed set, each subset containing exactly r elements and called an r-tuple. In the language of Berge [2] this is a simple uniform hypergraph of rank r. The concept becomes interesting only for $r > 1$: for $r = 2$ we obtain ordinary (i.e., Michigan) graphs. We shall represent an r-graph by a capital Latin letter followed by a superscripted (r), as $H^{(r)}$. If in some context this symbol is followed by (n) or (n,m), as $H^{(r)}(n)$ or $H^{(r)}(n,m)$, this will mean that $H^{(r)}$ has exactly n vertices and, in the second case, at least m r-tuples; thus n will always be an integer, but m need not be. Any of the foregoing notations, when applied to the symbol for a family of r-graphs, will be intended to apply to all members of the family: for example, if $\mathcal{H}^{(r)}$ is a family of r-graphs, and we write $\mathcal{H}^{(r)}(n)$, we shall be saying that every member of $H^{(r)}$ has exactly n vertices. The letter G will be reserved for a <u>general</u> r-graph: in the sense that we may write "any $G^{(r)}(n;m)$" when we mean "any r-graph having n vertices and at least m r-tuples"; it will not be used as the name of a specific r-graph. As another example, if we use the symbol $H^{(r)}(n)$ defined above, we are saying that every member of the family $H^{(r)}$ is a $G^{(r)}(n)$. The superscripted (r) will sometimes be omitted from the symbol for a family of r-graphs, but will normally be included in the symbol for a specific r-graph -- except possibly when r = 2.

For any fixed family H of r-graphs and any positive integer n, the <u>extremal number</u> ex(n; H) is the largest integer t for which there exists a $G^{(r)}(n,t)$ containing no member of H as a sub-r-graph. More precisely, H is a family of isomorphism classes of r-graphs, none of which contains an r-graph, which may be extended, possibly by adjoining new vertices and/or new r-tuples, to yield this $G^{(r)}(n,t)$. In the case of graphs, r = 2, much work has been done on extremal numbers . As usual, K_t denotes the complete graph with t vertices. Turán [11] generalized a result of Mantel-Wythoff [9] by evaluating $ex(n; \{K_t^{(2)}(t,\binom{t}{2})\})$ for every t. Numerous other results, both exact and asymptotic, have since been discovered for graphs: indeed, for every family $\mathcal{H}$ of graphs, $n^{-2}ex(n; \mathcal{H})$ approaches a known limit [8] as n approaches infinity, the value of the limit depending only on the minimum of the chromatic numbers of members of $\mathcal{H}$; descriptions of general extremal results for graphs may be found in [5] etc. In [4] we investigated several problems for $r = 3$. For example, we studied the asymptotic behavior of $ex(n; \mathcal{T}^{(3)})$ where $\mathcal{T}^{(3)}$ is the class of all triangulations of the sphere -- thereby generalizing the trivial statement that any graph $G^{(2)}(n;n)$ contains a polygon.

Let $\mathcal{G}^{(r)}(n,m)$ denote the class of <u>all</u> $G^{(r)}(n,m)$. In this paper we shall denote $ex(n;\mathcal{G}^{(r)}(k,h))$ by $f^{(r)}(n;k,s) - 1$. Thus $f^{(r)}(n;k,s)$ denotes the smallest t for which every $G^{(r)}(n,t)$ contains at least one $G^{(r)}(k,s)$. This problem was studied for graphs in [5] and for $r = 3$ in [4]. In the former case, many exact values are known, as well as asymptotic information; in the latter case, there are many gaps -- even for small values of k. We shall give examples of some known results for $r = 2$ and

$r = 3$. The main result of this paper consists of the determination of a lower bound for $f^{(r)}(n;k,s)$. The method of proof is that called by one of us [6] "probabilistic" -- we employ a counting argument to prove the existence of r-graphs containing no $G^{(r)}(k,s)$ and having the desired number of r-tuples, but we make no attempt to exhibit the r-graphs explicitly. The bound we obtain is not always best possible, but does in some cases improve on our earlier results for $r = 3$.

The letter c, possibly subscripted, will be reserved for positive constants which appear in inequalities for extremal numbers. We shall not be concerned with best possible values for such constants.

2. Some known values of $f^{(2)}(n;k,s)$

We shall not attempt an exhaustive discussion here, but refer the reader to [5] remarking, however, that some of the results there stated have been improved upon by various authors. We discuss below the behavior when $s \leq k$.

First, for the range $s < k$,

$$f^{(2)}(n;k,s) = \begin{cases} s & s \leq k/2 \\ 1 + [n(2s-k)/2s-k+1] & k/2 < s < k. \end{cases}$$

When s = k an exact result [9,11] is available only for k = 3, where

$$f^{(2)}(n;3,3) = [n^2/4] + 1.$$

Some information is available on the asymptotic behavior of $f^{(2)}(n;k,k)$. When k = 4, it follows from a previous result [3,7] that

$$\lim_{n\to\infty} n^{-3/2} f^{(2)}(n;4,4) = 1/2.$$

Erdös has proved the existence of positive constants ε_k, a_k, b_k such that the inequality

$$a_k n^{1+\varepsilon_k} < f^{(2)}(n;k,k) < b_k n^{1+1/[k/2]}$$

holds for all k: indeed, with $\varepsilon_k = 1/[k/2]$ for $k \leq 5$ at least. This stronger lower bound is easily seen to be valid for k = 6,7 and for k = 10,11 using graphs derived from families constructed by Benson [1] and Singleton [10].

3. Some known values of $f^{(3)}(n;k,s)$.

Much of [4] was devoted to a discussion of other structural problems. But we did determine asymptotic bounds for $f^{(3)}(n;k,s)$ for $k \leq 6$. We reproduce here the list of inequalities proved in Theorem 4 of that paper:

$$\lim_{n\to\infty} n^{-2} f^{(3)}(n;4,2) = 1/6$$

$$c_3 n^3 < f^{(3)}(n;4,3) < f^{(3)}(n;4,4)$$

$$f^{(3)}(n;5,2) = [n/3] + 1$$

$$c_4 n^2 < f^{(3)}(n;5,3) < c_5 n^2$$

$$c_6 n^{5/2} < f^{(3)}(n;5,4) < c_7 n^{5/2}$$

$$c_8 n^3 < f^{(3)}(n;5,5) < \dots < f^{(3)}(n;5,10)$$

$$f^{(3)}(n;6,2) = 2$$

$$c_9 n^{3/2} < f^{(3)}(n;6,3)$$

$$c_{10} n^2 < f^{(3)}(n;6,4) < n^2/4$$

$$f^{(3)}(n;6,6) < c_{11} n^{5/2}$$

$$f^{(3)}(n;6,8) < c_{12} n^{11/4}$$

$$c_{13} n^3 < f^{(3)}(n;6,9) < \dots < f^{(3)}(n;6,20)$$

Perhaps the most interesting question we were unable to answer is whether $f^{(3)}(n;6,3) = o(n^2)$. Our main result will provide improved lower bounds for $f^{(3)}(n;6,5)$ and $f^{(3)}(n;6,6)$; also we shall be able to generalize the pair of inequalities for $f^{(3)}(n;6,4)$ to $f^{(3)}(n;k,k-2)$.

4. A lower bound for $f^{(r)}(n;k,s)$.

Our main result is now stated.

Theorem. For integers $k > r$ and $s > 1$ there exists a positive constant $c_{k,s}$ such that

$$f^{(r)}(n;k,s) > c_{k,s}n^{(rs-k)/(s-1)}$$

Before proceeding with the proof, which uses the so-called "probabilistic" methods of [6], we remark that the exponent of n in the above inequality is not always best possible. It can, however, be shown to be best possible when $s - 1$ divides $rs - k$. For example, when $k = 5$ and $s = 4$ we know that

$f^{(3)}(n;5,4) = O(n^{5/2})$, but here we obtain only $f^{(3)}(n;5,4) > cn^{7/3}$.

Proof of the theorem. Let r,k,s be fixed integers ($k > r$, $s > 1$). Let n be any integer "sufficiently large", and m an integer to be further specified in inequality (1) below. Let V be a fixed set of cardinality n. The set of r-graphs having vertex set V and exactly m r-tuples will be denoted by M; it has exactly $\binom{\binom{n}{r}}{m}$ members. For any r-graph $H^{(r)}$ in

M, a subset K of V of cardinality k is called "$H^{(r)}$-bad" if at least s r-tuples of $H^{(r)}$ are contained in K. Denoting by $b(H^{(r)})$ the number of $H^{(r)}$-bad subsets of V, we shall choose m so that the following inequality will hold:

$$(1) \qquad \Sigma \ b(H^{(r)}) \div \binom{\binom{n}{r}}{m} \leq m \div 2 \binom{k}{r},$$

where the sum is taken over all $H^{(r)} \in M$, both in (1) and below. Since the left member of this inequality is just the average number of $H^{(r)}$-bad subsets in graphs in family M, (1) ensures that there exists an r-graph $H_o^{(r)}$ in M such that $b(H_o^{(r)}) < m/\binom{k}{r}$. If we omit from $H_o^{(r)}$ every r-tuple which occurs in an $H_o^{(r)}$-bad k-tuple we may construct a $G^{(r)}(n,m)$ containing no $H_o^{(r)}$-bad k-tuple: hence it will follow that

$$f^{(r)}(n;k,s) \geq m.$$

It remains to determine for given n the largest m for which (1) holds. The total number of $H^{(r)}$-bad k-tuples can be counted in the following way: first we fix a k-tuple K, then select s r-tuples consisting entirely of vertices of K, and m − s r-tuples from among the remaining $\binom{n}{r} - s$. Thus

$$\Sigma \quad b(H^{(r)}) \leq \binom{n}{k} \binom{\binom{k}{r}}{s} \binom{\binom{n}{r} - s}{m - s}$$

and hence

$$(2) \quad \Sigma \quad b(H^{(r)}) \div \binom{\binom{n}{r}}{m} \leq c_1 n^k \quad \frac{\binom{\binom{n}{r} - s}{m - s}}{\binom{\binom{n}{r}}{m}}$$

$$\leq c_1 n^k \left(\frac{m}{\binom{n}{r}}\right)^s$$

where c_1 is a constant depending on k, r, and s. The last inequality is a special case of the following:

If $B \leq C \leq A$ then $(C-i)/(A-i)$ is a decreasing function of i and so

$$\left(\frac{\binom{A - B}{C - B}}{\binom{A}{C}}\right) = \frac{C(C-1)\ldots(C-B+1)}{A(A-1)\ldots(A-B+1)} \leq (C/A)^B.$$

Inequality (2) will imply (1), provided

$$c_1 n^k \left(\frac{m}{\binom{n}{r}}\right)^s < c_2 m$$

i.e., $m^{s-1} < c_3 n^{rs-k}$; thus we may fix $m = c_4 \, n^{(rs-k)/(s-1)}$ thereby proving the theorem.

5. The order of magnitude of $f^{(3)}(n;k,k-2)$.

By our theorem, $f^{(3)}(n;k,k-2) > cn^2$. We prove that (constant) x n^2 triples suffice to ensure the existence of a $G^{(3)}(k,k-2)$. Let

$$G^{(3)} = G^{(3)} \left(n, \frac{1}{3} \left(n \left[\frac{k-2}{k-1} \cdot (n-1)\right] + 1 \right)\right).$$

Then some vertex x has the property that the pairs of vertices which together with x constitute triples of the 3-graph form a

$$G^{(2)} \left(n-1, \left[\frac{k-2}{k-1}(n-1)\right] + 1\right)$$

on the vertex set of our $G^{(3)}$ with x omitted. By a result quoted in Section 2, such a graph must contain a $G^{(2)}(k-1,k-2)$, hence $G^{(3)}$ contains a $G^{(3)}(k,k-2)$.

We conjecture that $\lim_{n\to\infty} n^{-2} f^{(3)}(n;k,k-2)$ exists, but have succeeded [4] in proving this only for $k = 4$.

Many interesting new problems arise if we also consider the structure of the graphs $G^{(r)}(k;s)$ and we have some very preliminary results; but many unsolved problems remain even for $r = 2$.

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