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ON SOME APPLICATIONS OF GRAPH THEORY III

BY

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In memory of Leo Moser, a friend and colleague

1. In the first and second parts of this sequence we dealt with applications of graph theory to distance distribution in certain sets in euclidean spaces, to potential theory, to estimations of the transfinite diameter [1] and to value distribution of "triangle functionals" (e.g. perimeter, area of triangles) [2]. The basic tool is provided in all these applications by the result formulated as Lemma 2. This, an essentially pure logical result, proves to be a very flexible and versatile instrument in applications.

Here the same method is used in an abstract setting. First we deduce certain results for the density of a given family of subsets of an abstract set S in another family of subsets of the same S. Then we apply the results obtained to distance distribution in certain (e.g. totally bounded or compact) sets in metric spaces, in particular in a normed linear function space. Applications of this method to functionals on Hilbert spaces were given by Katona [3].

2. Let S denote an infinite set and F an infinite family of its finite subsets satisfying

(2.1) F contains arbitrary large subsets

(2.2) $f \in F$ and $f_1 \subset f$ imply $f_1 \in F$.

Let G be a given family of finite subsets of S. We shall be interested in the relative density of G-subsets in F-subsets.

For fixed $f \in F$ and fixed integer k we denote by $L_k(f, G)$ the number of sets $g \in G$ with $|g| = k$ such that $g \subset f$. Then for fixed n, $n \geq k$ we define

$$l_{n,k}(F, G) = \frac{1}{\binom{n}{k}} \inf_{\substack{f \in F \\ |f| = n}} L_k(f, G). \tag{2.3}$$

The quantities $l_{n,k}(F, G)$ are lower bounds for the density of G-subsets of cardinality k in F-subsets of cardinality n. As we shall prove later, the following result holds:

LEMMA 1. *For* $n \geq k$

$$l_{n+1,k}(F, G) \geq l_{n,k}(F, G). \tag{2.4}$$

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It follows thus from (2.4) that

$$\lambda_k(F, G) \overset{\text{def}}{=} \lim_{n\to\infty} l_{n,k}(F, G) \tag{2.5}$$

exists for every fixed k. By (2.3) clearly $0 \le \lambda_k(F, G) \le 1$.

As an immediate consequence of the definitions and (2.4) we obtain:

COROLLARY. *If every subset $f \in F$ with $|f| = N$ contains a subset $g \in G$ (with $|g| = k$), then for $n \ge N$*

$$l_{n,k}(F, G) \ge \binom{N}{k}^{-1}. \tag{2.6}$$

In particular, $\lambda_k(F, G) \ge \binom{N}{k}^{-1}$.

Although (2.6), in general, is a weak lower bound for $l_{n,k}$, in some cases it yields nontrivial conclusions (see [2]).

3. In case $k = 2$ we can improve (2.6) substantially on using the graph theoretic Lemma 2. In fact we shall show that in certain cases we can determine the best possible lower bounds for $l_{n,2}$ and the exact values of $\lambda_2(F, G)$.

Our main result for $k = 2$ is the following:

THEOREM 1. *Suppose every subset f in F with $|f| = N+1$ contains a pair $g \in G$. Then for $n \ge N+1$,*

$$l_{n,2}(F, G) \ge \frac{1}{N} - \frac{1}{n}. \tag{3.1}$$

In particular, $\lambda_2(F, G) \ge \frac{1}{N}$.

4. Applications. (i) Let S be a set, ϕ and ψ functionals defined on all k-tuplets in S. Let F consist of all finite subsets $\{x_1, \ldots, x_n\}$ of S for which

$$\max_{1 \le i_j \le n} \phi(x_{i_1}, \ldots, x_{i_k}) \le 1.$$

Let G consist of all those finite subsets $\{y_1, \ldots, y_m\}$ for which

$$\min_{1 \le i_j \le m} \psi(y_{i_1}, \ldots, y_{i_k}) \le \theta$$

with some fixed $0 < \theta < 1$.

If among any set of N points in S there is a k-tuplet such that the corresponding value of ψ is $\le \theta$, then among any set of n $(> N)$ points there are at least

$$\binom{n}{k} \Big/ \binom{N}{k}$$

k-tuplets whose corresponding ψ-value is $\le \theta$. Specializing this result we get Theorem 1 of [2].

(ii) Let K be a compact (and therefore totally bounded) set in a complete metric space (X, ρ). Let the sequence of positive numbers $d_i = d_i(K)$ (the "packing constants") be defined by

$$d_i = \sup_{\substack{\kappa \subset K \\ |\kappa| = i}} \inf_{\substack{x, y \in \kappa \\ x \neq y}} \rho(x, y), \quad i = 2, 3, \ldots \tag{4.1}$$

and the "critical indices" i_j $(j = 2, 3, \ldots)$ by

$$d_2 = \cdots d_{i_2} > d_{i_2+1} = \cdots = d_{i_3} > \cdots. \tag{4.2}$$

Observe that from the definition it follows that $d_{i+1} \leq d_i$ for all i and $d_{i_{j+1}} = d_{i_j+1}$. Because of the total boundedness of K, $\lim_{i \to \infty} d_i = 0$.

In order to apply our previous result, we choose all finite subsets of K as F and all pairs of points $\{P_r, P_s\}$ in K for which $\rho(P_r, P_s) \leq \theta$ $(0 < \theta \leq d_2)$ as G. Then we have

THEOREM 2. *For any finite subset f of K and arbitrary fixed θ $(0 < \theta \leq d_2)$ let $L(f, \theta)$ denote the number of pairs $\{P_r, P_s\}$ in f satisfying $\rho(P_r, P_s) \leq \theta$. Let*

$$l_n(K, \theta) = \frac{1}{\binom{n}{2}} \inf_{\substack{f \subset K \\ |f| = n}} L(f, \theta) \tag{4.3}$$

and let i_j be the integer so that

$$d_{i_j+1} \leq \theta < d_{i_j}. \tag{4.4}$$

Then for $n > i_j$

$$l_n(K, \theta) > \frac{1}{i_j} - \frac{1}{n}. \tag{4.5}$$

Moreover, inequality (4.5), in general, cannot be replaced by $l_n(K, \theta) \geq (i_j)^{-1}$. If we define the "lower distance distribution" of K for $0 < \theta \leq d_2$ by

$$\lambda(K, \theta) \overset{\text{def}}{=} \lim_{n \to \infty} l_n(K, \theta), \tag{4.6}$$

we have with the above used notation

THEOREM 3. *If K is a perfect, compact set in a complete metric space (X, ρ), then for $j = 2, 3, \ldots$*

$$\lambda(K, \theta) = \frac{1}{i_j}, \qquad d_{i_j+1} \leq \theta < d_{i_j}. \tag{4.7}$$

Thus, $\lambda(K, \theta)$ is a right-continuous step function with jumps at $\theta = d_{i_j}$, $j = 3, 4, \ldots$

A particular case of Theorem 2 deserves special attention.

THEOREM 4. *Let X denote the set of functions $\{x(t)\}$ such that $x(t) \in C[0, 1]$, $x(0) = 0$ and $|x(t_1) - x(t_2)| \leq |t_1 - t_2|$ whenever $0 \leq t_1 < t_2 \leq 1$. Let the distance of any two functions x, y in X be defined by the usual maximum norm $\|x - y\|$.*

Then for $\nu=1, 2, \ldots$ *if* $n>2^\nu$ *and* $x_1, \ldots, x_n$ *are any functions in* X, *the number of distances* $\|x_i-x_j\|$ *which are* $\leq 2/\nu$ *is at least*

$$\frac{n^2}{2^\nu}-\frac{n}{2}.$$

This estimate, in general, is best possible.

An illuminating interpretation of Theorem 4 is that the probability that randomly chosen $x, y \in X$ satisfy $\|x-y\| \leq 2/\nu$ is at least $1/(2^{\nu-1})$ $(\nu=1, 2, \ldots)$.

5. **Proofs.** We need the following:

LEMMA 2 [6]: *Let* Γ *be a graph (with simple edges and no loops) having* n *vertices and* e *edges. Let* $n=N\cdot m+\nu$, $0\leq \nu<N$ *and suppose that*

$$e > \frac{N-1}{2N}(n^2-\nu^2)+\binom{\nu}{2}.$$

Then Γ *contains a complete subgraph of order* $N+1$.

In order to prove our Theorem 1, let f be any fixed set in F with $|f|=n$, $n\geq N+1$. Denote the elements of f by $x_1, \ldots, x_n$. Corresponding to f we define a graph on the vertices $P_1, \ldots, P_n$ as follows: The pair (P_i, P_j) should be an edge in Γ if and only if the pair (x_i, x_j) is *not* in G. Then, by the assumption of Theorem 1, Γ cannot contain a complete subgraph of order $N+1$. Thus, by Lemma 2 the number of edges e in Γ satisfies

$$e \leq \frac{N-1}{2N}(n^2-\nu^2)+\binom{\nu}{2} \tag{5.1}$$

where $n=N\cdot m+\nu$, $0\leq \nu<N$. Returning to f, inequality (5.1) implies that at least

$$\binom{n}{2}-\frac{N-1}{2N}(n^2-\nu^2)-\binom{\nu}{2}$$

pairs $\{x_i, x_j\}$ *are* members of G. In other words

$$L_2(f, G) \geq \binom{n}{2}-\frac{N-1}{2N}(n^2-\nu^2)-\binom{\nu}{2}. \tag{5.2}$$

Since as one easily calculates, the right-hand expression is

$$\geq \frac{n^2}{2}\left(\frac{1}{N}-\frac{1}{n}\right)$$

and f is an arbitrary set in F, (3.1) follows from (5.2).

Inequality (4.5) is a consequence of Theorem 1. Namely, if θ satisfies (4.4), then by the definition (4.1), among any set of i_j+1 points in K there is a pair (P_r, P_s) with distance $\rho(P_r, P_s)\leq\theta$. Hence the conditions of Theorem 1 are satisfied with $N=i_j$.

Now let K be a perfect set and $n=i_j\cdot m$ (m an integer). From the definition of the packing constants it follows that there exist points, say $Q_1, \ldots, Q_{i_j}$ in K such that $\rho(Q_r, Q_s) > \theta$ for $1 \le r < s \le i_j$. Since K is perfect, for each r, $1 \le r \le i_j$ there exist m points in K, say $Q_{r,1}, Q_{r,2}, \ldots, Q_{r,m}$, "near" Q_r so that $\rho(Q_{r,p}, Q_{s,q}) > \theta$ whenever $1 \le r < s \le i_j$, for all values of p, q. Hence in the set of n elements $\{Q_{r,p}\}$, $(1 \le r \le i_j, 1 \le p \le m)$, the number of distances $\le \theta$ is not greater than $i_j\binom{m}{2}$. This implies that

$$l_n(K, \theta) \le \frac{m-1}{n-1} < \frac{1}{i_j}.$$

This completes the proof of Theorem 2.

The proof of Theorem 3 follows from the above. Namely, since $l_n(K, \theta)$ is an increasing sequence, $l_n(K, \theta) < 1/i_j$ holds for all $n > i_j$. Hence (4.7) follows from (4.5) and (4.6).

Theorem 4 follows from (4.5) and a result of Newman and Raymon [5] (see also [4]). Namely, in our notation, it was shown in [5] that for the set X of Theorem 4,

$$d_{2^\nu+1} = d_{2^\nu+2} = \cdots = d_{2^{\nu+1}} = \frac{2}{\nu+1}, \nu = 0, 1, \ldots$$

and thus $i_\nu = 2^{\nu-1}$, $v = 1, 2, \ldots$.

Finally, we prove Lemma 1. Suppose

$$l_{n+1,k}(F, G) = \binom{n+1}{k}^{-1} L_k(f^*, G).$$

Let $f_1, f_2, \ldots, f_{n+1}$ denote all subsets of f^* with cardinality n. By (2.2), $f_i \in F$ for $1 \le i \le n+1$. Now, if for some $g \in G$ we have $g \subset f^*$, then $g \subset f_i$ will hold for $n-k+1$ of the f_i's. Hence

$$l_{n+1,k}(F, G) = \binom{n+1}{k}^{-1} \frac{1}{n-k+1} \sum_{i=1}^{n+1} L_k(f_i, G)$$

which by the definition (2.3) of the $L_{n,k}$'s is

$$\ge \binom{n+1}{k}^{-1} \frac{1}{n-k+1} \cdot \binom{n}{k} (n+1) l_{n,k}(F, G) = l_{n,k}(F, G).$$

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