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ON THE APPLICATION OF COMBINATORIAL ANALYSIS TO MUMBER THEORY GEOMETRY AND ANALYSIS

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In this lecture I will discuss the application of some well known and less well known theorems in combinatorial analysis to various other branches of mathematics. In other words I will not mention combinatorial types of reasoning the use of which is of course very wide-spread (e.g. the classical proof of Carleson on the almost everywhere convergence of Fourier series of functions in L_2 is full of combinatorial reasoning), but will restrict myself to cases where definite quotable theorems are used. My paper in no ways claim to give a complete survey of all the applications of combinatorial theorems and is certainly heavily biased towards my own work. Though combinatorics has been successfully applied to many branches of mathematics these can not be compared neither in importance nor in depth to the applications of analysis in number theory or algebra to topology, but I hope that time and the ingenuity of the younger generation will change this.

First we discuss some applications of Ramsey's theorem. The classical theorem of Ramsey states as follows : Let S be an infinite set. Split the k-tuples of S into r classes. Then there is an infinite subset S_1 of S all whose k-tuples are in the same class. The finite form of Ramsey's theorem states that to every k and $u_1, \ldots, u_r$ there is a smallest integer $R_k^{(r)}(u_1, \ldots, u_r)$ so that if we split the k-tuples of a set $|\mathfrak{S}| = R_k(u_1, \ldots, u_r)$ into r classes then for at least one i there is an $S_i \subset S$, $|S_i| \geqslant u_i$ all whose k-tuples are in the i-th class. The exact determination, or even good estimation of $R_k^{(r)}(u_1, \ldots, u_r)$ is a difficult problem which is very far from being solved and we do not discuss it here.

Ramsey's theorem was often rediscovered. Szekeres [1] rediscovered it in connection with the problem of Miss Klein. Miss Klein observed that if there are 5 points in the plane no three of them on a straight line then there are always 4 of them which determine a convex quadrilateral. She then asked : Is there a smallest $f(n)$ so that if there are $f(n)$ points in the plane no three on a line then there are always n of them which determine the vertices of a convex n-gon. Szekeres observed that

$$(1) \qquad f(n) \leqslant R_4^{(2)}(5, n)$$

since an n-gon all whose quadrilaterals are convex is itself convex. Thus Ramseys theorem immediately gives a positive answer to Miss Klein's question.

(1) gives a very poor upper bound for $f(n)$. Szekeres in fact conjectured $f(n) = 2^{n-2} + 1$. This is Miss Klein's result for $n = 4$ and for $n = 5$ it was

proved by Turán and E. Makai by methods of elementary geometry. $n > 5$ is not settled so far. Szekeres and I proved $f(n) \geqslant 2^{n-2} + 1$ [2] (there are some minor inaccuracies in our proof which were corrected by Kalbfleisch), and I [1] proved $f(n) \leqslant \binom{2n-4}{n-2}$.

Ramsey originally discovered his theorem for the purpose of some logical applications. Hajnal, Rado and I in our partition calculus [3] systematically studied the generalisations of Ramseys theorem to nigher cardinal numbers, our results have applications to logic and model theory, also Hajnal and Juhász applied our results to set theoretic topology, but I do not discuss these transfinite applications here. Also Ramsey's theorem has many generalisations and extensions but I can not discuss them here. (Erdós-Rado London Journal 1950, Nash-Williams. . . Cambridge Phil. Soc.).

It is obvious that if there are $n + 2$ points in n dimensional space then not all the distances can be equal. Schoenberg and Seidel in fact determined the minimum of the ratio of the maximal distance divided by the minimal distance. Several years ago Coxeter asked me to determine or estimate the smallest integer $f(n)$ so that if there are $f(n)$ points in n dimensional space then they determine at least three different distances. It immediately follows from Ramseys theorem that

$$f(n) \leqslant R_2\,(n + 2, n + 2) \tag{2}$$

(2) in fact is a very poor estimate, probably $f(n) < c_1 n^{c_2}$ and perhaps $f(n) = \left(\frac{1}{2} + o\,(1)\right) n^2$. By fairly complicated arguments I can prove $f(n) < \exp\,(n^{1-\epsilon})$.

Let $A\,(k\,,n)$ be the smallest integer so that if there are given any $A\,(k\,,n)$ points in k-dimensional space one can always find n of them so that all their distances are distinct. It seems quite difficult to determine $A\,(k\,,n)$ even for $k = 1$, only crude upper bounds are known for the general case. $A\,(2\,,3) = 7$ and Croft proved $A\,(3\,,3) = 9$ [4].

Ramsey's theorem easily implies that to every $\epsilon > 0$ and n there is a $B\,(\epsilon\,,n)$, so that if there are $B\,(\epsilon\,,n)$ points in the plane then there are always n of them $x_1, \ldots, x_n$ which determine a convex polygon for which the angle $(\chi_i\,,\chi_j\,,\chi_r)$ is greater than $\pi - \epsilon$ (for every $1 \leqslant i < j < r \leqslant n$).

A well known theorem of Schur states that if we split the integers not exceeding $e\,n\,!$ into n classes then the equation $x + y = z$ is satisfied in at least one of the classes. V.T. Sós communicated to me the following simple proof of Schur's theorem : Consider the partition of pairs $(i\,,j)$ so that the pair $(i\,,j)$, $i < j$, belongs to the r-th class if $j - i$ belongs to the r-th class. By Ramsey's theorem at least one of the new classes contains a triangle $(i\,,j\,,l)$, but then

$$(j - i) + (l - j) = l - i,$$

or $x + y = z$ is solvable in the original class. It is known that

$$R_3^{(n)}\,(3, \ldots, 3) \leqslant [\,e\,n\,!\,]$$

which completes the proof of Schur's theorem.

The determination of the exact bound in Schur's theorem is a very difficult problem, probably $e\,n\,!$ can be replaced by c^n. The value of $f_2^{(r)}(3, \ldots, 3)$ is not known for $r > 3$. [5].

It seems that the following theorem of J. Sanders can not be proved so simply : To every r and n there is an $f_r(n)$ so that if we split the integers from 1 to $f_r(n)$ into r classes there are n distinct integers $a_1 < \ldots < a_n$ so that all the $2^n - 1$ sums

$$\sum_{i=1}^{n} \epsilon_i a_i, \qquad \epsilon_i = 0 \text{ or } 1, \text{ not all } \epsilon_i = 0$$

are in the same class. Rados results [10] imply the theorem of Sanders.

Graham and Rotschild [6] have a very general theorem from which this follows as a special case. They have the following very interesting problem : Split the integers into two (or more generally into r) classes. Is it true that there is an infinite sequences $a_1 < \ldots$ so that all the sums

$$\sum_{i} \epsilon_i a_i \qquad \epsilon_i = 0 \text{ or } 1, \text{ not all } \epsilon_i = 0 \tag{3}$$

are in the same class ? (in (3) of course only a finite number of ϵ's are 1). It is not even known if there is an infinite sequence $a_1 < \ldots$ for which $a_1 < a_2 < \ldots$ and $a_i + a_j$, $1 \leqslant i < j < \infty$ all belong to the same class.

On the other hand it immediately follows from Ramsey's theorem that for every l there is an infinite subsequence $a_1^{(l)} < \ldots$ so that all distinct sums taken l at a time belong to the same class. I do not know if there is an infinite subsequence $a_{l_1} < \ldots$ so that for every t, $(t = 1, 2, \ldots)$, all distinct sums taken t at a time belong to the same class, the class may depend on t.

It is easy to see that every infinite sequence of integers contains an infinite subsequence so that either no two members of the subsequence divide each other or each term of the subsequence divides the subsequent one. This follows immediately from Ramsey's theorem but perhaps is not a good example of its use since the direct proof is easier.

Császár [7] proved the following theorem which arose in his joint work with Czipszer : Let $g_1(x) \ldots, g_n(x)$ be n bounded real functions and $f(x)$ another real function. Assume that there are two real numbers $\epsilon > 0$, $\delta > 0$ so that whenever $f(x) - f(y) > \epsilon$ there is an i, $1 \leqslant i \leqslant n$ so that $g_i(x) - g_i(y) > \delta$. Then $f(x)$ is also bounded. Császár gave a direct proof of this theorem and V.T. Sós observed that it immediately follows from Ramsey's theorem.

A theorem of Van der Waerden states that if we split the integers into two classes at least one of them contains an arbitrarily long arithmetic progression. The finite form of Van der Waerden's theorem states that there is a smallest $f(n)$ so that if we split the integers from 1 to $f(n)$ into two classes at least one of them contains an arithmetic progression of n terms. No satisfactory upper bound is know for $f(n)$, the best lower bound is due to Berlekamp [8].

Van der Waerden's theorem also has many applications e.g. A Brauer [9] proved that if $p > p_0(k)$ is a sufficiently large prime then there are k consecutive quadratic residues and non-residues mod p. Rado [10] gives many interesting generalisations and applications to new number theoretic and combinatorial problems. The theorem of Graham and Rotschild [6] can be considered as a generalisation of Van der Waerdens theorem. Finally I would like to drow your attention to a beautiful conjecture of Rota [6] which seems very deep. Added in proof : Rotas conjecture has been proved by Graham, Leeb and Rotschild.

Turán and I [11] raised the following problem in combinatorial number theory : Denote by $r_k(n)$ the maximum value of l for which there exists a sequence of integers $a_1 < \ldots < a_l \leqslant n$ which do not contain an arithmetic progression of k terms. Determine or estimate $r_k(n)$. If we could prove that for every k there is an $n_0(k)$ so that for $n > n_0(k)$ $r_k(n) < n/2$, then Van der Warden's theorem would immediately follow. Unfortunately this has never been proved.

It is known that

$$n^{1-c_1/\sqrt{\log n}} < r_3(n) < c_2 n/\log\log n$$

The lower bound is due to Behrend [12] and the upper to Roth [13]. Szemerédi [14] proved $r_4(n) = o(n)$.

I would like to mention one more old conjecture of mine from combinatorial number theory : Let $g(n) = \pm 1$ be an arbitrary function. Then to every c there is a d and m so that

$$\left| \sum_{k=1}^{m} g(kd) \right| > c$$

Now we give some applications of combinatoral inequalities and extremal problems. Let $a_1 < \ldots < a_k \leqslant n$ be a sequence of integers no a divides any other. Then it is easy to see that max $k = [(n+1)/2]$. On the other hand if we assume that no a divides the product of two others then [15]

$$\pi(n) + c_1 n^{2/3} < \max k < \pi(n) + c_2 n^{2/3}/(\log n)^2 \tag{4}$$

The proof of both the upper and the lower bound in (4) uses combinatorial results. The lower bound uses Steiner triplets and the upper bound the trivial result that a graph of n vertices and n edges contains a circuit. Assume now that all the products $a_i a_j$ are distinct. Then [16]

$$\pi(n) + c_3 n^{3/4}/(\log n)^{3/2} < \max k < \pi(n) + c_4 n^{3/4}/(\log n)^{3/2} \tag{5}$$

Here the lower bound uses the existence of finite geometries for $n = p^2 + p + 1$ and the upper bound uses the following result : Let $\mathcal{G}$ be a graph of t_1 vertices which contains no rectangle, further assume that there are t_2 vertices so that every edge of our graph is incident to one of these vertices. Then the number $R(\mathcal{G})$ of edges of $\mathcal{G}$ is less than

$$t_1 + t_1\left[\frac{t_2}{t_1^{1/2}}\right] + t_2^2\left(1 + \left[\frac{t_2}{t_1^{1/2}}\right]\right)^{-1}$$

Thus, in particular, if $\mathcal{G}$ has n vertices and contains no rectangle then $R(\mathcal{G}) < cn^{3/2}$. W. Brown and Rényi, V.T. Sós and I proved that [17] if $\mathcal{G}$ contains no rectangle then

$$\max c(\mathcal{G}) = \left(\frac{1}{2} + \sigma(1)\right) n^{3/2}.$$

Assume now that the number of solutions of $a_i a_j = m$ is bounded. Then we have the following result : Let $a_1 < \ldots < a_k \leqslant n$, $n > n_0(\epsilon , l)$. Assume

$$(6) \qquad k > (1 + \epsilon)\, n\, (\log\log n)^{l-1}/(l-1)\,!\, \log n$$

Then for some m the number of solutions of $m = a_i a_j$ is $\geqslant 2^l$. (6) is best possible in the sense that it fails if $1 + \epsilon$ is replaced by $1 - \epsilon$ [18].

(6) implies that if $a_1 < \ldots$ is an infinite sequence of integers so that every large integer can be written in the form $a_i a_j$ then the number of solutions of $n = a_i a_j$ is unbounded. Now I state an old conjecture of Turán and myself which is an additive analogue of this result (and which in fact lead me to this result) : Let $a_1 < \ldots$ be an infinite sequence of integers, denote by $f(n)$ the number of solutions of $n = a_i + a_j$. Assume that $f(n) > 0$ for all sufficiently large n. Then $\limsup f(n) = \infty$. This conjecture if true is probably very deep.

The combinatorial theorem needed for the proof of (6) states as follows. Let r and t be given, $\epsilon = \epsilon(r , t)$ small and $n > n_0(\epsilon , r , t)$. Let

$$|\mathcal{S}| = n \qquad \text{and} \qquad A_1, \ldots, A_u, u > n^{r-\epsilon}$$

are r-tuples contained in $\mathcal{S}$. Then there are rt distinct elements $\chi_i^{(j)}$, $i = 1, \ldots, t$; $j = 1, \ldots, r$ of $\mathcal{G}$ so that all the t^r sets $(\chi_{i_1}^{(1)}, \chi_{i_2}^{(2)}, \ldots \chi_{i_r}^{(r)})$ occur among the A's. For $r = 2$ this is a theorem of Kóvári and the Turáns [19].

A well known theorem of Turán [20] states that in a graph of n vertices which has more than

$$(7) \quad f(n , r) = \frac{r-2}{2(r-1)}(n^2 - h^2) + \binom{h}{2}, n \equiv h (\operatorname{mod}\ r-1), 0 \leqslant h < r-1$$

edges there is a complete r-gon. This theorem and its extensions have many applications in geometry and potential theory. A. Meir, the Turáns and I are publishing a series of joint papers on this subject. Here I state only one such application. Let $\mathcal{S}$ be a set of diameter 1 and $\chi_1, \ldots, \chi_n$, n points in $\mathcal{S}$. Let the packing constants of $\mathcal{S}$ be $p_2 = 1 \geqslant p_3 \geqslant \ldots$. Then at most $f(n , r)$ of the distances $d(x_1 , x_j)$ are greater than p_r. Let C_r be the r-th covering constant of $\mathcal{S}$, using a theorem of Moser and myself (Australien J. Math. XI (1970), 92-97), V.T. Sós obtained a lower bound for the number of distances $d(x_1 , x_j) > C_r$. This will also appear in the quadruple paper.

Another application of (7) is due to Katona [21]. Let $f_1(x), \ldots, f_n(x)$ be n functions which satisfy $\int f_i(X)^2\, dx \geqslant 1$. Then there are at most $[n^2/4]$ pairs (i , j), $i < j$ for which $\int (f_i(X) + f_j(X))^2\, dx < 1$.

I investigated the following question : Let $\chi_1, \ldots, \chi_n$ be n distinct points in k-dimensional Euclidean space. For how many pairs i , j $(i < j)$ can we have

$d(x_i, x_j) = 1$? Denote this maximum by $f(k, n)$. For $k = 2$ and $k = 3$ I have no good estimations for $f(k, n)$, e.g. for $k = 2$ I only know that

$$n^{1+c/\log\log n} < f(2, n) < c' n^{3/2} \tag{8}$$

It seems that in (8) the lower bound is close to being best possible.

For $k \geqslant 4$ one knows very much more. Lenz and I proved [23]

$$\lim_{n=\infty} f(k, n)/n^2 = \frac{1}{2}\left(1 - \frac{1}{[k/2]}\right)$$

and if $k = 4$, $n \equiv 0 \pmod 8$ I proved [23]

$$f(k, n) = \frac{n^2}{4} + n. \tag{9}$$

(9) follows from the following result of Simonovits and myself wich will soon appear in Acta Hungarica : Denote by $\mathcal{G}(n; l)$ a graph of n vertices and l edges. $k(u_1, \ldots, u_r)$ denotes the complete r-chromatic graph where there are u_i vertices of the i-th colour and every two vertices of different colour are joined. We proved that every $\mathcal{G}\left(n; \left[\frac{n^2}{4}\right] + n + 1\right)$ contains $a\, k(1, 3, 3)$. This result is best possible, there is a $\mathcal{G}\left(n \left[\frac{n^2}{4}\right] + n\right)$ which does not contain a $k(1, 3, 3)$.

A well known theorem of Sperner [24] states that if $|\mathcal{S}| = n, A_i \subset \mathcal{S}, 1 \leqslant i \leqslant k$ is a family of subsets, no one of which contains any other, then

$$\max k = \binom{n}{[n/2]}, \tag{10}$$

This result and its generalisations and extensions has many applications. Using (10) Behrend [25] proved that if $a_1 < \ldots < a_k \leqslant n$ is a primitive sequence (a sequence of integers is called primitive if no a divides any other) then

$$\sum_{i=1}^{k} \frac{1}{a_i} < c \log n/(\log\log n)^{1/2}. \tag{11}$$

and Pillai proved that (11) is best possible.

Using a more complicated refinement of (10) Sárközi, Szemerédi and I [26] proved that if $a_1 < \ldots$ is an infinite primitive sequence then

$$\lim \sum_{a_i < X} \frac{1}{a_i} \left(\frac{(\log\log x)^{1/2}}{\log x}\right)^{-1} = 0.$$

We also proved that

$$\max \sum_{a_i < X} \frac{1}{a_i} = (1 + \sigma(1)) \frac{\log x}{(2\pi \log\log x)^{1/2}}$$

where the maximum is taken over all primitive sequences [27].

I made strong use of Sperners theorem in my papers on the distribution function of additive arithmetic functions [28].

Let $|\mathcal{G}| = n$, $A_i \subset \mathcal{G}$ $1 \leqslant i \leqslant k$. Assume that the union of two A's never equals a third. I conjectured that then $k < c\binom{n}{[n/2]}$. Kleitman [29] proved this conjecture as well as several other related conjectures, all of which have number theoretic applications [30].

Sharpening a result of Littlewood and Offord [31] I immediately deduced from (10) that if $\chi_i \geqslant 1$, $i = 1, \ldots, n$ then the number of sums

$$\sum_{i=1}^{n} \epsilon_i \chi_i, \epsilon_i = \pm 1$$

which fall inside an interval of length 2 is at most $\binom{n}{[n/2]}$. I conjectured that the same holds if the x_1 are vectors in a Banach space (the interval of length 2 has to be replaced by a sphere of radius 1). This was first proved for the plane independently by Katona and Kleitman [32] using an ingenious extension of (10). Very recently Kleitman proved my general conjecture in a surprisingly simple way without using Sperners theorem. Kleitman's proof is not yet published.

Rado and I proved the following theorem. Let $a \geqslant 2$ and $b > 1$ be integers. Then there is a smallest integer $f(a, b)$ so that if we have $f(a, b) + 1$ sets each having at most b elements there are always $a + 1$ of them which have pairwise the same intersection [33]. We proved

$$f(a, b) \leqslant b!\, a^{b=1} \left(1 - \frac{1}{2!a} - \frac{2}{3!a^2} - \ldots - \frac{b-1}{b!\, a^{b-1}}\right) \tag{12}$$

(12) is far from being best possible and very likely

$$f(a, b) < c^{b+1} a^{b+1} \tag{13}$$

We are very far from being able to prove (13), even for $a = 2$ (12) has not even been proved with $o(b!)$.

Sauer determined $f(a, 2)$ for every a. For $b > 2$ there are only relatively crude upper and lower bounds for $f(a, b)$.

(12) has many applications which could be significantly strengthened if (13) would be proved.

Denote by $f_t(n)$ the smallest integer so that if

$$1 \leqslant a_1 < \ldots < a_l \leqslant n, l = f_t(n)$$

is an arbitrary sequence of integers, one can always find t a's which have pairwise the same greatest common divisor. First I proved by number theoretical methods that

$$f_t(n) < \frac{n}{\exp((\log n)^{1/2-\epsilon})}$$

Later I observed that (12) implies that for every t and $\epsilon > 0$ there is an n_0 so that for all $n > n_0(t, \epsilon)$

$$\exp(c_t \log n/\log\log n) < f_t(n) < n^{3/4+\epsilon}$$

(13) would imply that the lower bound gives the right order of magnitude,

Using (12) I proved [35] that for every k there are squarefree integers satisfying ($V(n)$ denotes the number of distinct prime factors of n)

$$(a_i, a_j) = 1, \varphi(a_i) = \varphi(a_j), \quad V(a_i) = V(a_j), \quad 1 \leqslant i < j \leqslant k$$

If (13) would hold we could add $\sigma(a_i) = \sigma(a_j)$.

(12) has been improved by Abbot and others but as far as I know nobody came close to (13).

Dodson [36] investigated the following problem : Denote by $\Gamma^*(k, p^n)$ the smallest value of s for which for every choice of the integers $a_1, \ldots, a_s$ $\sum_{i=1}^{s} a_i x_i^k \equiv 0 \pmod{p^n}$ has a non trivial solution in integers x_i, $i = 1, \ldots, s$) (i.e not all the x_i are multiples of p). In one of the cases (12) was needed.

(12) has also many applications to combinatorial problems and set theory (see Engelking and others).

Before completing the paper I want to state a few miscellaneous combinatorial results which have applications in various branches of mathematics.

Let $|\mathfrak{S}| = n$, $A_i \subset \mathfrak{S}$, $1 \leqslant i \leqslant r$, $r \to \infty$ as $n \to \infty$, $|A_i| > cn$, $0 < c < 1$ for

$$1 \leqslant i \leqslant r, r \to \infty \quad \text{as} \quad n \to \infty$$

Then there are two indices i and j for which $|A_i \cap A_j| > (c^2 + o(1))n$. This statement can be proved easily by using the characteristic functions of the sets A_i and it is easy to state various generalisations for the intersection of more than two sets, one can also reformulate the result for measurable sets [37]. This theorem has many applications to combinatorial analysis, number theory and analysis.

A theorem of Szekeres and myself states that if there are given 2^n points in the plane they always determine an angle $> \pi\left(1 - \frac{1}{n}\right)$. This result is best possible since Szekeres showed previously that to every $\epsilon > 0$ one can give 2^n points in the plane so that all the angles are $< \pi\left(1 - \frac{1}{n}\right) + \epsilon$; see [2].

The fact that 2^n points determine an angle $\geqslant \pi\left(1 - \frac{1}{n}\right)$ follows from the fact that the complete graph of $2^n + 1$ points is not the union of n bipartite graphs. The sharper result that one of the angles is $> \pi\left(1 - \frac{1}{n}\right)$ follows from a more careful study of the decompositions of the complete graph of 2^n vertices into n bipartite graphs.

Probability methods have often been applied successfully to solve combinatorial problems which seemed intractable by more direct methods and conversely combinatorial results often imply unexpectedly beautiful results in probability. e.g. the arc sine law of Andersen [38], see also my paper with Hunt [39]. Finally I want to mention that Davies and Rogers [40] rediscovered ans used a little known theorem of Hajnal and myself on chromatic graphs in the study of Hausdorff dimension of sets.

REFERENCES

[1] Erdös P and Szekeres G. — A combinatorial problem in geometry, *Compositio Math.* 2, 1935, p. 463-470.

[2] Erdös P. and Szekeres G. — On some extremum problems in elementary geometry, *Annales Univ. Sci. Budapest,* 3-4, 1960, p. 53-62.

[3] Erdös P., Hajnal A. and Rado R. — *Partition relations for cardinal numbers,* 16, 1965, p. 93-196.

[4] Croft H.T. — 9-point and 7 point configurations in three space, *Proc. London Math. Sci.* XII, 1962, p. 400-424.

[5] Greenwood R.E. and Gleason A.M. — Combinatoxial theorems and chromatic graphs, *Canad. J. Math.* 7, 1955, p. 1-7.

[6] Graham R.L. and Rotschild B.L. — Ramsey's theorem for *n-parameter* sets, *Colloquium, Comb. theory and applications* Balatonfüred, Hungary 1969.

[7] Csaszar A. — Sur une critère d'approximation uniforme, *Publ. Mat. Inst. Hung. Acad.* 8, 1963, p. 413-416.

[8] Berlekamp E.R. — A construction for partitions which avoid long arithmetic progressions. *Canad. Bull. Math.* 11, 1968, p. 409-414.

[9] Brauer A. — Über Sequenzen von Potenzresten I und II, *Sitrungsber. Preuss. Akad. Wiss. Phys. Math. Klass* 19, 1931, p. 329-341.

[10] Rado R. — Studien zur Kombinatorik, *Math. Zeitschrift* 1933, p. 424-480.

[11] Erdös P. and Turan P. — On some sequences of integers. *J. London Math. Soc.,* 11, 1936, p. 261-264.

[12] Behrend F.A. — On sets of integers which contain no three terms in arithmetical progression, *Proc. Mat. Acad. Sci., U.S.A.,* 32, 1946, p. 331-332.

[13] Roth K.F. — On certain sets of integers, *J. London Math. Soc.,* 28, 1953, p. 104-109.

[14] Szemerédi E. — On sets of integers containing no four elements in arithmetic progressions, *Acta Math. Acad. Sci. Hung.,* 20, 1969, p. 89-104.

[15] Erdös P. — On sequences of integers no one of which divides the product of two others and on some related problems, *Tomsk. Gos. Univ. Vcen Zap.,* 2, 1938, p. 74-82.

[16] Erdös P. — On some applications of graph theory to number theoretic problems, *Publ. Ramanujan Inst.* No. 1, p. 132-136.

[17] Brown W.G. — On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.,* 9, 1966, p. 281-285, Erdos P., Rényi A. and Sos V.T., On a problem of graph theory, *Studia Sci. Math. Hung,* 1, 1966, p. 215-235.

[18] Erdös P. — On the multiplicative representation of integers, *Israel J. Math.,* 2, 1964, p. 251-261.

[19] ERDÖS P. — On extremal problems of graphs and generalised graphs, *Israel J. Math.*, 2, 1964, p. 183-190. KOVARI T., SOS V.T. and TURAN P. — On a problem of K. Zarankievicz, *Colloquium Math.*, 3, 1950, p. 50-57.
[20] TURAN P. — Eine Extremalaufgabe aus der Graphentheorie (in Hungarian), *Mat. és Fiz. Lapok*, 48, 1941, p. 436-462.
[21] KATONA Gy. — Graphs, vectors and inequalities in probability theory (in Hungarian), *Mat. Lapok*, 20, 1969, p. 123-127.
[22] ERDÖS P. — On sets of distances of n points, *Amer. Math. Monthly*, 53, 1946 p. 258-260.
[23] ERDÖS P. — On sets of distances of n points in an Euclidean space, *Publ. Math. Inst. Hung. Acad.*, 5, 1960, p. 165-169; ERDOS P., On some applications of graph theory to geometry, *Canadian J. Math.*, 19, 1967, p. 968-971.
[24] SPERNER E. — Ein Satz über Untermengen einer endlichen Menge, *Math. Zeitschrift*, 27, 1928, p. 544-548.
[25] BEHREND F. — On sequences of numbers not divisible one by another, *J. London Math. Soc.*, 10, 1935, p. 42-49, see also chapter V of the book *Sequences* of H. Halberstam and K.F. Roth.
[26] ERDÖS P., SARKOZI A. and SZEMERÉDI E. — On a theorem of Behrend, *J. Australian Math. Soc.* 7, 1967, p. 9-16.
[27] ERDÖS P., SARKOZI A. and SZEMERÉDI E. — On an extremal problem concerning primitive sequences, *J. London Math. Soc.*, 42, 1967, p. 484-488.
[28] ERDÖS P. — On the density of some sequences of numbers I, II and III. *J. London Math. Soc.*, 11, 1935, p. 120-125; 13, 1937, p. 7-11; 14, 1938, p. 185-192.
[29] KLEITMAN D. — On a combinatorial problem of Erdos, *Proc. Amer. Math. Soc.*, 17, 1966, p. 139-141.
[30] ERDÖS P., SARKOZI A. and SZEMERÉDI E. — On the solvability of the equations $[a_i, a_j] = a_r$ and $(a_i . a_j) = a_r$ in sequences of positive density, *J. Math. Anal. and Applications*, 15, 1966, p. 60-64.
[31] ERDÖS P. — On a Lemma of Littlewood and Offord, *Bull. Amer. Math. Soc.*, 31, 1945, p. 898-902.
[32] KATONA G. — On a conjoncture of Erdos and a stronger form of spencer's theorem, *Studia Sci. Math. Hungar.*, 1, 1966, p. 59-63, KLEITMAN D. — On a lemma of Littlewood and Offord on the distribution of certain sums, *Math. Z.*, 90, 1965, p. 251-259.
[33] ERDÖS P. and RADO R. — Intersection theorems for systems of sets, *J. London Math. Soc.*, 35, 1960, p. 85-90.
[34] ERDÖS P. — On a problem in elementary number theory and a combinatorial problem, *Math. of Computation*, 18, 1964, p. 644-646.
[35] ERDÖS P. — Some remarks on the functions φ and σ, *Bull. Acad. Polon. Sci.*, X, 10, 1962, p. 617-619.
[36] DODSON M. — Homogeneous additive congruences, *Philos. Trans. Roy. Soc. London*, Ser. A, 261, 1967.
[37] ERDÖS P., NEVEU J. and RÉNYI A. — An elementary inequality between the probabilities of events, *Math. Scand.*, 13, 1963, p. 99-104. See also papers by GILLIS, LORENTZ, SUCHESTON, SHAPIRO H.S.
[38] ANDERSEN E.S. — On the number of positive sums of random variables, *Skandinavisk Aktuarietidsskrift*, 32, 1949, p. 27-36.
[39] ERDÖS P. and HUNT G.A. — Changes of sign of sums of random variables, *Pacific J. Math.*, 3, 1953, p. 673-687.

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