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NON COMPLETE SUMS OF MULTIPLICATIVE FUNCTIONS

by

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1. It is well known that $\sum_{d|n}\mu(d) = 0$ for all $n > 1$. We are interested concerning the upper estimate of

$$M(n) = \max_z M(n,z) = \max_z \left| \sum_{\substack{d|n \\ d\leq z}} \mu(d) \right| .$$

Previously it was proved that

$$M(n) \leq \binom{\omega(n)}{\left[\frac{\omega(n)}{2}\right]} < c\,\frac{2^{\omega(n)}}{\sqrt{\omega(n)}}, \tag{1.1}$$

where $\omega(n)$ denotes the number of different prime factors of n (See [1], [2]).

One of us asked in a recent paper [3] whether $M(n)$ has a better upper estimate for almost all n. Explicitly it was asked whether

$$M(n) \leq 2^{\alpha\omega(n)} \tag{1.2}$$

holds for almost all integers n with a suitable constant $\alpha < 1$. Now we prove a more general theorem, whence (1.2) will immediately follow.

2. THEOREM. *Let $f(n)$ be a multiplicative function satisfying the conditions: a) $|f(n)| < 1$; b) Let $\mathscr{P}$ denote the set of primes p for which $f(p) = -1$, let $\sum_{p\in\mathscr{P}} \frac{1}{p} = \infty$.*
Then

$$\max_{1\leq z\leq n} \left| \sum_{\substack{d|n \\ d\leq z}} f(d) \right| < 2^{\alpha\omega(n)}$$

for almost all n, where α is an arbitrary constant $> 1/2$.

To prove this we need two lemmas.

Let $x_1 = \log x$, $x_2 = \log x_1$, $y_1 = \log y$, $y_2 = \log y_1$, $\Omega(n)$ be the number of all prime divisors of n counted each of them by their multiplicity. Let ε be an arbitrarily small positive constant, $R = (1 + \varepsilon)x_2$. The symbol Σ' denotes a sum extended over those n for which $\Omega(n) \leq R$. Since, by the well known theorem of HARDY and RAMANUJAN $|\Omega(n)-x_2|<\varepsilon x_2$ holds for all $n\leq x$ except at most $o(x)$ of them, therefore the Σ' is extended over almost all n. Let for an arbitrary $A>1$ $\tau(n',z,A)=\sum_{d|n}1,\ \frac{z}{A}<d\leq z$. $d(n)$ denotes the number of divisors of n.

LEMMA 1. *We have*

$$\left(\Sigma \overset{\text{def}}{=}\right)\sum_n{}' \sum_{A^k\leq x}\tau^2(n;A^{k+1},A)\leq cx\,.\,2^R x_2 (\log A).$$

PROOF. For $d|n$, $\delta|n$ let $(d,\delta)=a$, $d=au$, $\delta=av$, $(u,v)=1$. If $A_k\leq d\leq\delta\leq A^{k+1}$, then $u\leq n\leq Au$. Hence

$$\Sigma=\Sigma'\sum_{A^k\leq x}\tau^2(n;\ A^{k+1},A)\leq\sum_{n=auvl}{}'\ 1\overset{\text{def}}{=}\Sigma_1,$$

where the last sum is extended over those a, u, v, l for which $n\leq x$, $\Omega(n)\leq$ $\leq R$, $u\leq v\leq Au$. Therefore $\Sigma_1<\sum_{uv\leq x}\Sigma_{u,v}$,

$$\Sigma_{u,v}=\sum_{\substack{m\leq x/uv\\ \Omega(m)\leq R-\Omega(uv)}}d(m).$$

Since $d(m)<2^{\Omega(m)}$, hence

$$\Sigma_{u,v}\leq\frac{x}{uv}\cdot 2^{R-\Omega(uv)}$$

and consequently

$$\Sigma_1\leq x\,2^R\sum_{uv\leq x}\frac{2^{-\Omega(uv)}}{uv}\leq x\,2^R\sum_{u\leq x}\frac{2^{-\Omega(u)}}{u}\Sigma_u, \tag{2.1}$$

where

$$\Sigma_u=\sum_{u\leq v\leq Au}\frac{2^{-\Omega(v)}}{v}. \tag{2.2}$$

To estimate Σ_u we use the following theorem due to HARDY and RAMANUJAN: *if $\pi_r(y)$ is the number of $n\leq y$ with $\Omega(n)=r$, then*

$$\pi_r(y)<\frac{y(y_2+c)^{r-1}}{y_1(r-1)!}.$$

Hence

$$\gamma(y)\overset{\text{def}}{=}\sum_{v\leq y}2^{-\Omega(v)}<\frac{y}{y_1}\sum_{r=1}^{\infty}\frac{(y_2+c)^{r-1}2^{-r+1}}{(r-1)!}= \tag{2.3}$$

$$=\frac{y}{y_1}\exp\left(\frac{1}{2}y_2+\frac{c}{2}\right)<c\frac{y}{\sqrt{y_1}}.$$

Hence

$$\Sigma_u\leq\sum_{2^t\leq A}\frac{1}{u\,2^t}\gamma(u\,2^{t+1})<c\sum_{2^t\leq A}\frac{1}{\sqrt{\log u2^t}}\leq c\frac{\log A}{\sqrt{\log u}}.$$

Taking this estimate into (2.1), we have

$$\Sigma_1 \leq cx\, 2^R \log A \sum_{u \leq x} \frac{2^{-\Omega(u)}}{u\sqrt{\log u}}. \tag{2.4}$$

Furthermore, by (2.3)

$$\sum_{2 \leq u \leq x} \frac{2^{-\Omega(u)}}{u\sqrt{\log u}} \leq c \sum_{2^t \leq x} \frac{1}{2^t \sqrt{t}} \gamma(2^t) \leq c \sum_{2^t \leq x} \frac{1}{t} \leq c\, x_2.$$

Consequently, from (2.4)

$$\Sigma_1 \leq cx\; 2^R (\log A) x_2,$$

which proves the lemma.

Let $p(n)$ denote the smallest prime divisor p of n for which $p \in \mathcal{P}$, $p^2 \nmid n$. We take $p(n) = \infty$ if such p does not exist.

Lemma 2. *We have,*

$$\frac{1}{x} \sum_{\substack{n \leq x \\ p(n) > A_x}} 1 \to 0,$$

if $A_x \to \infty$ arbitrarily slowly.

This can be proved, by using the Eratosthenes' sieve; therefore we omit its proof.

3. Now we prove the theorem. We have

$$\sum_{\substack{d|n \\ d \leq z}} f(d) = \sum_{\substack{d|n' \\ d \leq z}} f(d) + f(p(n)) \sum_{\substack{d|n' \\ d \leq z/p(n)}} f(d), \qquad n' = \frac{n}{p(n)}.$$

Since $f(p(n)) = -1$, therefore

$$\Big| \sum_{\substack{d|n \\ d \leq z}} f(d) \Big| \leq \Big| \sum_{\substack{z/p(n) \leq d \leq z \\ d|n'}} f(d) \Big| \leq \tau(n; z, p(n)). \tag{3.1}$$

Introducing the notations

$$C(n) = \max_z \Big| \sum_{\substack{d|n \\ d \leq z}} f(d) \Big|, \; T_B(n) = \max_z \tau(n; z, B),$$

from (3.1) we have

$$C(n) \leq T_B(n) \text{ if } p(n) \leq B. \tag{3.2}$$

By choosing $A = B^2$, we have

$$T_B^2(n) \leq \max_k \tau^2(n; A^k, A) \leq \sum_{A^k \leq x} \tau^2(n;\; A^{k+1} A),$$

and hence, by Lemma 1

$$\Sigma' T_B^2(n) \leq cx\; 2^R x_2 \log A. \tag{3.3}$$

Let $B = x_1$. From (3.3) $T_B(n) \leq 2^{R/2} x_2^2$ for all $n \leq x$ except perhaps some the number of which smaller than $cxx_2^{-2} = o(x)$.
Since

$$p(n) \leq B \text{ and } |\Omega(n) - x_2| < \varepsilon x_2$$

for almost all n, therefore, from (3.2)

$$C(n) \leq 2^{R/2} x_2^2 \leq 2^{\frac{1+\varepsilon}{2} x_2} x_2^2 \leq 2^{\frac{1}{2}\frac{1+\varepsilon}{1-\varepsilon}\omega(n)} x_2^2$$

for almost all n.

Using the arbitrariness of ε we obtain the assertion of the theorem.

It is probable that our theorem is nearly best possible. We conjecture that for every $\varepsilon > 0$ and almost all n $M(n) > n^{1/2-\varepsilon}$.

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