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AN EXTREMAL GRAPH PROBLEM

By

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Throughout this paper graphs are supposed not to contain loops and multiple edges. G^n denotes a graph of n vertices but only if n is an upper index. $e(G)$ denotes the number of edges, $v(G)$ denotes the number of vertices, $\chi(G)$ denotes the chromatic number of G. $G_1 \times \cdots \times G_d$ or $\bigtimes_{i=1}^{d} G_i$ denotes the product of the G_i's, i.e. the graph obtained from the graphs $G_1, \ldots, G_d$ by joining any two vertices belonging to different G_i's. Here the graphs $G_1, \ldots, G_d$ are supposed to be vertex-independent. $K_d(r_1, \ldots, r_d)$ denotes the complete d-chromatic graph with r_i vertices of the i^{th} colour, i.e. $K_d(r_1, \ldots, r_d) = \bigtimes_{i=1}^{d} G_i$ where $e(G_i)=0$, $v(G_i)=r_i$. If E is any set, $|E|$ denotes the number of its elements.

Introduction

P. TURÁN proved in 1941 [1] that if $K^n = \bigtimes_{i=1}^{p-1} G^{n_i}$ where $n_i = \left[\frac{n}{p-1}\right]$ or $n_i = \left[\frac{n}{p-1}\right]+1$, and $e(G^{n_i})=0$ then K^n does not contain a complete p-graph and if G^n is an arbitrary other graph not containing a complete p-graph, then $e(G^n) < e(K^n)$.

This is the source of the following problems:

PROBLEM 1. Let $G_1, \ldots, G_l$ be given graphs. What is the maximum number of edges a graph can have if it does not contain any G_j as a subgraph?

Putting

$$(1) \qquad f(n; G_1, \ldots, G_l) = \max\{e(G^n) : G_i \not\subseteq G^n, \quad i = 1, \ldots, l\}$$

the problem can be rephrased:

Determine the function $f(n; G_1, \ldots, G_l)$ for given graphs $G_1, \ldots, G_l$.

PROBLEM 2. The graphs attaining the maximum in (1) are called extremal graphs. Determine the structure of the extremal graphs for given $G_1, \ldots, G_l$ and n.

The answer for these problems is fairly similar to the answer for TURÁN's original problem:

I. We have proved [2] that

$$f(n; G_1, \ldots, G_l) = \binom{n}{2}\left(1-\frac{1}{d}+o(1)\right) \tag{2}$$

$$\text{where}\quad d+1 = \min_{1\le i\le l} \chi(G_i). \tag{3}$$

(2) and (3) express that $f(n; G_1, \ldots, G_l)$ depends very loosely on the structure of the graphs $G_1, \ldots, G_l$, its order of magnitude is already determined by the minimal chromatic number.

II. Later we proved independently [3], [4] that the structure of the extremal graphs is also fairly independent of the G_i's. Our most interesting results connected with Problem 2 can be summarized as follows:

Let $G_1, \ldots, G_l$ be given graphs, K^n be an extremal graph for $G_1, \ldots, G_l$ and n be large enough. Then there exists an integer $r>0$ (depending on some colouring properties of G_i's) such that

A) K^n can be obtained from a graph-product $\mathop{\times}\limits_{i=1}^{d} N_i$ by omitting $O\left(n^{2-\frac{1}{r}}\right)$ edges from and adding $O\left(n^{2-\frac{1}{r}}\right)$ new edges to it. Here

$$d+1 = \min \chi(G_i).$$

B) The components of the product are of almost equal size:

$$n_i = v(N_i) = \frac{n}{d}+O\left(n^{1-\frac{1}{r}}\right)$$

C) Each vertex $x\in K^n$ has valency greater than $\frac{n}{d}(d-1)-c_1 n^{1-\frac{1}{r}}$ where c_1 is a suitable constant.

D) Let $\varepsilon>0$ be fixed. There is a constant K_ε such that the number of vertices of N_i joined to at least εn_i vertices of N_i is less than K_ε.

These assertions have asymptotic character. They illustrate that the extremal graphs are very similar to that one in TURÁN's original theorem. They are the best possible in a certain way. The theorem we prove in this paper has "exact character" but the graphs G_i are more special.

Here we have to remark, that this theorem is the first one, which describes the structure of rather complicated extremal graphs fairly well.

THEOREM. *Let $r_1=1$, 2 or 3. $r_1\leqq r_2\leqq\cdots\leqq r_{d+1}$ be given integers. If n is large enough, then each extremal graph K^n for $K_{d+1}(r_1, \ldots, r_{d+1})$ is a graph product:*

$$K^n = \mathop{\times}_{i=1}^{d} N_i$$

where

1) $n_i = v(N_i) = \frac{n}{d}+o(n)$;

2) *N_1 is an extremal graph for $K_2(r_1, r_2)$;*

3) *$N_2, \ldots, N_d$ are extremal graphs for $K_2(1, r_2)$.*

Conversely, if $\hat{N}_1, \ldots, \hat{N}_d$ *are given graphs such that*

4) *there exists an extremal graph* $\bigtimes_{i=1}^{d} N_i$ *satisfying* 1), 2), 3) *such that* $v(\hat{N}_i) = v(N_i)$;

5) $\hat{N}_1$ *is an extremal graph for* $K_2(r_1, r_2)$;

6) $\hat{N}_i$ *is an extremal graph for* $K_2(1, r_2)$, $K_2(2, 2)$, $K_3(1, 1, 1)$ $(i \neq 1)$, *then* $\hat{K}^n = \bigtimes_{i=1}^{d} \hat{N}_i$ *is an extremal graph for* $K_{d+1}(r_1, \ldots, r_{d+1})$.

REMARK 1. Our theorem does not characterize the extremal graphs for $K_{d+1}(r_1, \ldots, r_{d+1})$ completely. First of all, we do not know the extremal graphs for $K_2(r_1, r_2)$ sufficiently well. Further, just because of this lack of knowledge about the extremal graphs we do not know the exact values of n_i for given n. The extremal graphs are those among the described ones which have the maximum number of edges. As far as we know this can occur for many different choices of the n_i.

REMARK 2. For $r_1 = 1$ [4] proves the statement. We shall prove it only for $r_1 = 3$. The case $r_1 = 2$ can be treated similarly.

REMARK 3.

$$f(n; K_2(r_1 - 1, r_2)) = o(f(n; K_2(r_1, r_2))) \quad \text{if} \quad r_1 \leqq r_2 \tag{4}$$

probably always holds, but we do not know it for $r_1 \geqq 4$. This is why we can prove the theorem only for $r_1 < 4$. (4) can be proved for $r_1 = 2$ as follows: T. KŐVÁRI, V. T. SÓS and P. TURÁN [5] and independently P. ERDŐS (unpublished) proved that

$$f(n; K_2(p, q)) = O\left(n^{2-\frac{1}{p}}\right) \quad \text{if} \quad p \leqq q. \tag{5}$$

P. ERDŐS, A. RÉNYI and V. T. SÓS proved for $p = 2$, BROWN for $p = 2, 3$ that (5) can not be improved [6], [7]:

$$f(n; K_2(2, 2)) = \frac{1}{2} n^{3/2} + o(n^{3/2}) \tag{5a}$$

and

$$\underline{\lim} f(n; K_2(3, 3))/n^{5/3} > 0 \quad \text{if} \quad n \to \infty. \tag{5b}$$

Now, (5a), (5b) and (5) imply (4) if $r_1 = 3$.

Trivially (5b) gives a lower estimation for $f(n; K_2(4, 4))$. We do not know any better lower estimation for it.

REMARK 4. In a forthcoming paper M. SIMONOVITS is going to prove some generalizations, based on Remark 3.

Proofs

First we prove two lemmas.

LEMMA 1. *Let G_1 be a graph not containing $K_2(r_1, r_2)$, let G_i $(i=2, \ldots, d)$ be graphs not containing $K_2(1, r_2)$, $K_2(2, 2)$, $K_3(1, 1, 1)$, where $r_1 \leqq r_2 \leqq \cdots \leqq r_{d+1}$ are given positive integers. Then $\bigtimes_{i=1}^{d} G_i$ does not contain $K_{d+1}(r_1, \ldots, r_{d+1})$.*

PROOF. It is sufficient to consider only the case $r_2 = r_3 = \cdots = r_{d+1}$. We prove, that if G_d does not contain any of $K_2(r_1, r_2)$, $K_2(2, 2)$, $K_3(1, 1, 1)$ and G does not contain any $K_d(r_1, r_2, r_2, \ldots, r_2)$, then $G \times G_d$ does neither contain any $K_{d+1}(r_1, r_2, r_2, \ldots, r_2)$. From this the lemma follows immediately by mathematical induction.

First we remark, that $K_{d+1}(r_1, r_2, \ldots, r_2)$ has the following property: If we omit some vertices $x_1, x_2, \ldots, x_\lambda$ from it and either all these vertices belong to the same class or $x_2, x_3, \ldots, x_\lambda$ belong to the same class and $\lambda < r_2$, then the remaining graph contains a $K_d(r_1, r_2, \ldots, r_2)$. This assertion is trivial if all the vertices belong to the same class. In the other case let us denote by $U_1, \ldots, U_{d+1}$ the classes of $K_{d+1}(r_1, r_2, \ldots, r_2)$ and suppose that $x_1 \in U_j$, $x_2, \ldots, x_\lambda \in U_k$. Let V be the empty set if $U_k = \{x_2, \ldots, x_\lambda\}$ and a set containing exactly one vertex of $U_k - \{x_2, \ldots, x_\lambda\}$ otherwise. Then one can easily show that the classes U_i $(i \neq j, k)$ and $U_j \cup V - \{x_1\}$ span a graph containing $K_d(r_1, r_2, \ldots, r_2)$.

Let us consider now $G \times G_d$ and suppose that it contains a $K_{d+1}(r_1, r_2, \ldots, r_2)$ the classes of which are $U_1, U_2, \ldots, U_{d+1}$. We show, that either G_d contains only vertices of one U_j or it contains one vertex from a U_j and at most $r_2 - 1$ vertices belonging to another U_k.

Indeed, if there were $x, y, z \in G_d$ belonging to different U_j's then they would determine a $K_3(1, 1, 1) \subseteq G_d$ contradicting our assumptions. Thus $G_d \cap U_j$ is empty for all but at most two values of j. If there existed $u_1, u_2 \in U_j \cap G_d$, $v_1, v_2 \in U_k \cap G_d$ then they would determine a $K_2(2, 2) \subseteq G_d$ contradicting our assumptions. Thus, $G_d \cap K_{d+1}(r_1, r_2, \ldots, r_2)$ contains vertices, belonging to the same U_j or a vertex $x \in U_k$ and at most $r_2 - 1$ other vertices belonging to the same U_j indeed. ($|U_j \cap G_d| < r_2$ since G_d does not contain a $K_2(1, r_2)$.) Because of this there is a $K_d(r_1, \ldots, r_2)$ determined by the other vertices of $K_{d+1}(r_1, \ldots, r_2)$ which is contained by $G \times G_d - G_d = G$. This contradiction proves Lemma 1.

LEMMA 2. *Let G^v, r be given, $r \geqq 3$. There exists a constant $c_{\delta, r} > 0$ depending only on δ and r such that if G^v is a graph not containing $K_2(3, r)$ and $x \in G^v$ is a vertex of valency greater than δv in it then*

$$e(G^v) \leqq f(v; K_2(3, r)) - c_{\delta, r} \cdot v^{5/3} \tag{6}$$

PROOF. Let C be a subclass of vertices of G^v consisting of $\approx \delta v$ vertices, each of which is joined to x. Then for no $p_1, \ldots, p_r \in C$, $u, v \in G^v - \{x\}$ the set of these vertices determines a $K_2(2, r)$ the first class of which is $\{u, v\}$; otherwise $\{x, u, v\}$ and $\{p_1, \ldots, p_r\}$ would determine a $K_2(3, r) \subseteq G^v$. Therefore the graph determined by the edges both endpoints of which belong to C, does not contain a $K_2(2, r)$. Similarly the bipartite graph, determined by the edges one endpoint of which belongs to C, the other to $G^v - C - \{x\}$, does neither contain a $K_2(2, r)$ the second class of which is in C. Therefore the number of these edges is $O(v^{3/2})$. (The proof in [5] also gives this.) The remaining edges of G^v have both their endpoints in $G^v - C$, thus the number of these edges is at most $f((1-\delta)v; K_2(3, r))$. Thus

$$e(G^v) \leqq f((1-\delta)v; K_2(3, r)) + O(v^{3/2}).$$

Since the disjoint union of two extremal graphs for $K_2(3, r)$ does not contain a $K_2(3, r)$ either,

$$f(v_1 + v_2; K_2(3, r)) \geqq f(v_1; K_2(3, r) + f(v_2; K_2(3, r)). \tag{7}$$

Thus

$$e(G^v) \leqq f(v; K_2(3, r)) + O(n^{3/2}) - f(\delta v; K_2(3, r)). \tag{8}$$

Since $f((\delta v; K_2(3, r)) \geqq c_r(\delta v)^{5/3}$, (8) implies (6).

PROOF OF THEOREM. Let K^n be an extremal graph for $K_{d+1}(r_1, \dots, r_{d+1})$ and colour it by d colours so that the number of edges, having endpoints of the same colour be minimal. Then there exist an integer r and graphs $N_1, \dots, N_d$ so that A), B), C), D) hold (see Introduction and [4], [3]). We shall use them only in the following weaker form:

α) C_i denotes the class of vertices of N_i, $|C_i| = n_i = \dfrac{n}{d} + o(n)$.

β) All the vertices have valency greater than $\dfrac{n}{d}(d-1) - o(n)$.

γ) Let $\varepsilon > 0$ be a small constant (fixed only later). Let us denote the class of vertices of C_i, joined to at most εn vertices of the same C_i by C_i'. Then there exists a constant K_ε depending only on ε and $r_1, \dots, r_{d+1}$ such that $|C_i - C_i'| < K_\varepsilon$. The vertices of $C_i - C_i'$ will be called exceptional vertices, and γ) expresses that their number is bounded. Clearly, if $x \in C_i'$, then x is joined to at most εn vertices of C_i but if $n > n_0(\varepsilon)$ it is joined to at least $|C_j| - 2\varepsilon n$ vertices of C_j because of α) and β) $(i \neq j)$.

I. Let $E = \sum_{1 \leqq i < j \leqq d} n_i n_j$. Trivially, E is the number of pairs of vertices in K^n belonging to different classes.

Lemma 1 implies that

$$f(n; K_{d+1}(r_1, \dots, r_{d+1})) = e(K^n) \geqq E + f(n_1; K_2(3, r_2)) + \sum_{i=2}^{d} f(n_i; K_2(1, r_2)). \tag{9}$$

Indeed, if G^{n_1} is an extremal graph for $K_2(r_1, r_2)$, $G^{n_1}, \dots, G^{n_d}$ are extremal graphs for $\{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}$, then $G^n = \bigtimes_{i=1}^{d} G^{n_i}$ does not contain a $K_{d+1}(r_1, \dots, r_{d+1})$, thus $e(K^n) \geqq e(G^n)$. It is easy to see that the extremal graphs for $\{K_2(1, r_2)$ $K_2(2, 2)$, $K_3(1, 1, 1)\}$ are also extremal graphs for $K_2(1, r_2)$, if n is large enough. If $n_i(r_2 - 1)$ is even, the extremal graphs for $K_2(1, r_2)$ are regular graphs of degree $r_2 - 1$. If $n_i(r_2 - 1)$ is odd, such graphs do not exist, the extremal graphs have $n_i - 1$ vertices of valency $r_2 - 1$ and one vertex of valency $r_2 - 2$. If n_i is large enough, among these graphs there exist graphs not containing either $K_2(2, 2)$ or $K_3(1, 1, 1)$. This and

$$f(n; K_2(1, r_2)) \geqq f(n; K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1))$$

prove that

$$f(n; K_2(1, r_2)) = f(n; K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1))$$

for large values of n_i. This implies, that each extremal graph for $\{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}$ is also an extremal graphs for $K_2(1, r_2)$. Therefore, the right hand side of (9) equals to $e(G^n) \leqq e(K^n)$. Thus (9) holds.

II. First we remark, that C_i' does not contain a $K_2(3, r_2)$; for if it contained a $K_2(3, r_2)$, we could find a $K_{d-1}(r_3, \dots, r_{d+1})$ in the graph spanned by the other classes so that $K_2(3, r_2) \times K_{d-1}(r_3, \dots, r_{d+1}) = K_{d+1}(3; r_2, \dots, r_{d+1})$ would be contained by K^n.

Now we prove that if C_1' contains a $K_2(2, r_2)$, then for $i \geqq 2$, C_i' does not contain a $K_2(1, r_2)$. Let us denote by B_j $(j=2, \dots, d)$ the class of vertices of C_j' $(j=2, \dots, d)$ joined to all vertices of the fixed $K_2(2, r_2) \subseteqq C_1'$. If there were a $u \in B_j$ and $v_1, \dots, v_{r_3} \in B_j$ joined to u $(j \geqq 2)$, then these r_3+1 vertices and the fixed $K_2(2, r_2) \subseteqq C_1'$ and $r_4, \dots, r_{d+1}$ suitable vertices of $B_3, \dots, B_d$ (if $d \geqq 3$) would determine a $K_{d+1}(3, r_2, \dots, r_{d+1})$ in K^n if ε is small enough. (The expression "suitable" means: the other vertices must determine a $K_{d-3}(r_4, \dots, r_{d+1})$ each vertex of which is joined to each vertex of the fixed $K_2(2, r_2)$ and to $u, v_1, \dots, v_{r_3}$.) Therefore the set $\{u, v_1, \dots, v_{r_3}\}$ can not exist. Thus B_j contains $O(n)$ edges. Let us consider the j^{th} class, $j \geqq 2$. The number of edges in $C_j' - B_j$ is $O(m_j^{5/3})$, where

$$m_j = |C_j' - B_j| \leqq (2+r_2) \cdot 2\varepsilon n.$$

The remaining edges of K^n in C_j' join $C_j' - B_j$ to B_j. Their number is $O(nm_j^{2/3})$.[1]

Let us divide B_j into classes of $\approx m_j$ vertices. Each of these classes together with $C_j' - B_j$ determines a graph of $\approx 2m_j$ vertices, not containing $K_2(3, r_2)$. Therefore each of them has $O(m_j^{5/3})$ edges and their number is $\approx \dfrac{n}{dm_j}$. Thus C_j' contains

$$O(n) + O(m_j^{5/3}) + O(nm_j^{2/3}) = \varepsilon^{2/3} \cdot O(n^{5/3})$$

edges and the same bound holds for C_j. Thus $C_2, \dots, C_d$ contain $\varepsilon^{2/3} \cdot O(n^{5/3})$ edges.

Let us suppose now that C_2' contains a $K_2(1, r_2)$ and let A_1 denote the set of vertices of C_1' joined to this $K_2(1, r_2)$. Clearly, A_1 does not contain any $K_2(2, r_3)$, otherwise $C_2' \cup A_1$ would contain a $K_2(2, r_3) \times K_2(1, r_2) \supseteqq K_3(3, r_2, r_3)$ and taking suitable vertices from the other classes we could complete this $K_3(3, r_2, r_3)$ into a $K_{d+1}(3, r_2, r_3, \dots, r_{d+1}) \subseteqq K^n$. Therefore, the method used above gives that C_1' contains only $\varepsilon^{2/3} O(n^{5/3})$ edges. The same bound is valid for C_1, thus

$$e(K^n) \leqq E + \varepsilon^{2/3} O(n^{5/3}). \tag{10}$$

Now we fix ε so that (10) should contradict (9). Thus C_2' does not contain $K_2(1, r_2)$ and generally, C_j' $(j \geqq 2)$ also does not contain it.

In general it could happen that C_1' did not contain $K_2(2, r_2)$. But if no C_j' contained a $K_2(2, r_2)$, then

$$e(K^n) \leqq E + d \cdot O(n^{3/2}) + O(n)$$

would hold contradicting (9). Thus we may assume that C_1' does contain a $K_2(2, r_2)$ and $C_2', \dots, C_d'$ do not contain any $K_2(1, r_2)$.

[1] This can also be derived directly from the proof of [5].

III. Now we show that if n is sufficiently large, then there exist no exceptional vertices: $C_i'=C_i$. Actually we prove that if $\varepsilon' = \frac{1}{2}r_{d+1}\cdot d\cdot\varepsilon$ and n is sufficiently large, then K^n contains no vertices joined to at least $\varepsilon' n$ vertices of each class. Since ε is an arbitrarily small positive number, this gives that the maximal valency in N_i is $o(n)$. This, of course, implies that $C_i'=C_i$ for $n>n_0$.

Let us suppose that $x\in K^n$ is joined to at least $\varepsilon' n$ vertices of each class. Then the graph G^* spanned by x and C_1' can not contain a $K_2(3, r_2)$. Indeed, since C_1' does not contain a $K_2(3, r_2)$, if G^* does, then x must be a vertex of this $K_2(3, r_2)$. Since each non-exceptional vertex is joined to all the vertices of the other classes but at most εn, we may select succesively $r_3, \dots, r_{d+1}$ vertices of $C_2', \dots, C_d'$ so that the selected vertices span a $K_{d-1}(r_3, \dots, r_{d+1})$ and are joined to each vertex of the fixed $K_2(3, r_2)$. Thus K^n contains a $K_{d+1}(3, r_2, \dots, r_{d+1})$. This contradiction proves that G^* can not contain any $K_2(3, r_2)$. Thus C_1' (and C_1 as well) contain $f(n_1; K_2(3, r_2))-$ $-cn^{5/3}$ edges (Lemma 2) where $c>0$. Since C_i' ($i\geqq 2$) does not contain any $K_2(1, r_2)$,

$$e(K^n) \leqq E+f(n_1; K_2(3, r_2))-cn^{5/3}+O(n). \tag{12}$$

But (12) contradicts (9). This proves that K^n has no exceptional vertices: $C_i'=C_i$. Thus C_1 does not contain $K_2(3, r_2)$, $C_2, \dots, C_d$ do not contain $K_2(1, r_2)$ and consequently

$$e(K^n) \leqq E+f(n_1; K_2(3, r_2))+\sum_{i=2}^{d} f(n_i, K_2(1, r_2)). \tag{13}$$

(13) and (9) proves that

$$e(K^n) = E+f(n_1, K(3, r_2))+\sum_{i=2}^{d} f(n_i, K_2(1, r_2)). \tag{14}$$

Since C_1 does not contain $K_2(3, r_2)$, the graph spanned by it must be an extremal graph for $K_2(3, r_2)$, otherwise the "equal" of (14) would be "definitely less". Similarly, the graphs spanned by $C_2, \dots, C_d$ are extremal graphs for $K_2(1, r_2)$ and if they are denoted by $N_1, N_2, \dots, N_d$, then $K^n = \bigtimes_{i=1}^{d} N_i$, i.e. every two vertices are joined, if they belong to different C_i's.

The second part of the Theorem is trivial now: If $\hat{N}_i$ satisfies our conditions, $\hat{K}^n = \bigtimes_{i=1}^{d} \hat{N}_i$ has the same number of edges as K^n and according to Lemma 1 it does not contain a $K_{d+1}(r_1, \dots, r_{d+1})$. Therefore it is an extremal graph for it. This completes our proof.

REMARK 5. An easy discussion shows that if $r_1\geqq 2$, $r_2\geqq 3$, $\{K_2(1, r_2), K_2(2, 2), K_3(1, 1, 1)\}$ can be replaced by $\{K_2(1, r_2), K_2(2, 2)\}$ but it cannot be replaced by $K_2(1, r_2)$.

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