ON CHROMATIC NUMBER OF INFINITE GRAPHS

by

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§ 1. Introduction. Notations

In our paper [1] using a special set-theoretical construction assuming the generalized continuum hypothesis (G.C.H. in what follows) we proved that the topological product of $\aleph_k$ discrete topological spaces of power $\aleph_0$ is not k-compact for every $k < \omega$. Since then several related or equivalent problems were independently discussed in the literature. General results of A. TARSKI, P. HANF and H. J. KIESLER made it possible to prove that (assuming G.C.H.) a similar result holds for a very extensive class of cardinals $\aleph_\alpha$ (see e.g. [2], [3], [4]). J. MYCIELSKI proved without assuming G.C.H. that the result holds for every $\aleph_\alpha$ less than the first weakly inaccessible cardinal $> \aleph_0$.

On the other hand in [1] we stated several problems depending on a parameter $\aleph_\alpha$ of the type that a positive answer for them would imply the corresponding incompactness result, but the general method for obtaining the incompactness results does not work for them. One of them is the following:

Does there exist a graph $\mathcal{G}$ of $\aleph_\alpha$ vertices such that $\mathcal{G}$ has chromatic number $> \aleph_0$ but all subgraphs $\mathcal{G}'$ of $\mathcal{G}$ spanned by less than $\aleph_\alpha$ vertices have chromatic number $\leq \aleph_0$?

As far as we know, the general methods mentioned above do not help to solve this problem, and all nontrivial instances of the problem are unsolved for $\aleph_\alpha > \aleph_1$.

In this paper (using G.C.H.) we are going to give a partial solution of this problem for $\aleph_\alpha < \aleph_\omega$. In fact we prove a more general theorem (see Theorem 2) which gives information on the problem involved without using G.C.H.

We obtain our result by using a rather special partition theorem. This will be proved in § 2. In § 3 we prove the main result already mentioned and we are going to define universal graphs $\mathcal{G}_{\aleph_\alpha, \aleph_0}$ which for every regular $\aleph_\alpha$ have the property that they contain all graphs $\mathcal{G}$ of $\aleph_\alpha$ vertices such that every subgraph $\mathcal{G}'$ of $\mathcal{G}$ spanned by less than $\aleph_\alpha$ vertices has chromatic number at most $\aleph_0$. We will discuss some special properties of these graphs as well. In § 4 we deal with some partition problems related to the one considered in § 2.

In what follows we are going to use the notations introduced in our paper [6]. These are mostly the usual notations of set-theory. We mention that ordinals are defined so that each ordinal ξ is the set of all smaller ordinals. The cardinal $\aleph_\xi$ is identified with ω_ξ.

§ 2. A special partition problem

DEFINITION 2.1. Let γ be a cardinal ≥ 2, and let k be an integer. A sequence $\mathscr{H}_\xi = \langle h, H_\xi \rangle$, $\xi < \gamma$ of type γ of uniform set-systems $\mathscr{H}_\xi$ with $\varkappa(H_\xi) = k$ is said to be a k-partition of type γ of the set h if[1]

$$\mathscr{S}_k[h] = \bigcup_{\xi<\gamma} H_\xi .$$

Let $\mathcal{G}$ be a graph and let $\prec$ be an ordering of the set g. We say that a path $\mathscr{P}(x_0, \ldots, x_i) \subseteq \mathcal{G}$ is an *increasing path of length i* (with respect to ordering $\prec$) if $x_0 \prec \ldots \prec x_i$. We are going to generalize the concept of an increasing path for more general set-systems.

DEFINITION 2.2. Let $\mathscr{H} = \langle h, H \rangle$ be a uniform set-system, with $\varkappa(H) = k$, $1 \leq k < \omega$. Let $\prec$ be an ordering of the set h. A subsystem $\mathscr{P} \subseteq \mathscr{H}$ is said to be an *increasing path of length i* ($i \geq 1$) if $\mathscr{P} = \langle p, P \rangle$ and the following conditions hold

$$p = \{x_0, \ldots, x_{i+k-2}\}, \qquad x_0 \prec \ldots \prec x_{i+k-2}$$

and

$$P = \big\{\{x_j, x_{j+1}, \ldots, x_{j+k-1}\}\big\}_{j<i}\ ^2.$$

DEFINITION 2.3. To have a brief notation we write $\Theta \to [i]^k_\gamma$ to denote that the following statement is true.

Let h be a set and $\prec$ an ordering of it, type $h(\prec) = \Theta$. Let further $\mathscr{H}_\xi$, $\xi < \gamma$ be a k-partition of type γ of h. Then there exists a $\xi < \gamma$ and an increasing path $\mathscr{P}$ of length i such that $\mathscr{P} \subseteq \mathscr{H}_\xi$. As usual $\Theta \nrightarrow [i]^k_\gamma$ denotes the negation of the statement.

DEFINITION 2.4. Let γ be an infinite cardinal. We put

$$\exp_0(\gamma) = \gamma, \quad \exp_{i+1}(\gamma) = 2^{\exp_i(\gamma)}.$$

We need the following

LEMMA 1. (A. TARSKI)[3] *Let γ be an infinite cardinal, $|A| = \gamma$. Then there exists a set $\mathscr{A} \subseteq \mathscr{S}_\gamma[A]$ such that $|\mathscr{A}| = 2^\gamma$ and $B \nsubseteq C$ for an arbitrary pair $B \neq C \in \mathscr{A}$.*

Using this we prove

LEMMA 2. *Let $\gamma \geq \omega$, $1 \leq k < \omega$. Then there exists a function $f(\xi_0, \ldots, \xi_{k-1})$ of k variables defined for $\xi_i < \exp_{k-1}(\gamma)$, $i < k-1$ with $\mathscr{R}(f) \subseteq \gamma$ satisfying the condition*

$$f(\xi_0, \ldots, \xi_{k-1}) \neq f(\xi_1, \ldots, \xi_k)$$

whenever

$$\xi_i \neq \xi_{i+1} \qquad \textit{for} \qquad i < k.$$

[1] A uniform set-system $\mathscr{H}$ consists of sets A for which $|A| = \varkappa(H)$.

[2] This changes slightly the definition given in 2. 13 [6] where in the special case $k = 2$ such a path was said of length $i+1$.

[3] See [7].

PROOF. For $k = 1$ the theorem is trivial. Let $k = 2$. Let $\mathscr{A}$ be a system of subsets of γ satisfying Lemma 1, and let $g \in {}^{\exp_1(\gamma)}\mathscr{A}$ be a one-to-one mapping of $\exp_1(\gamma)$ onto $\mathscr{A}$.

For an arbitrary pair $\xi_0 \neq \xi_1 < \exp_1(\gamma)$ put $f(\xi_0, \xi_1) = \min\big(g(\xi_0) \setminus g(\xi_1)\big)$. f satisfies the requirement since if $\xi_0 \neq \xi_1$, $\xi_1 \neq \xi_2 < \exp_1(\gamma)$ then

$$f(\xi_0, \xi_1) \notin g(\xi_1) \quad \text{and} \quad f(\xi_1, \xi_2) \in g(\xi_1)\ [4].$$

Let now $k > 2$ and assume that Lemma 2 is true for $1 \leq k' < k$.

Then there exist functions g, h satisfying the following conditions:

(1) $g(\xi_0, \xi_1)$ is defined for $\xi_0, \xi_1 < \exp_{k-1}(\gamma)$.
$\mathscr{R}(g) \subseteq \exp_{k-2}(\gamma)$ and for $\xi_0 \neq \xi_1$, $\xi_1 \neq \xi_2 < \exp_{k-1}(\gamma)$ we have

$$g(\xi_0, \xi_1) \neq g(\xi_1, \xi_2).$$

(2) $h(\eta_0, \ldots, \eta_{k-2})$ is defined for $\eta_i < \exp_{k-2}(\gamma)$, $i < k-1$,
$\mathscr{R}(h) \subseteq \gamma$ and for $\eta_i \neq \eta_{i+1}$, $i < k-1$, $\eta_i < \exp_{k-2}(\gamma)$ we have

$$h(\eta_0, \ldots, \eta_{k-2}) \neq h(\eta_1, \ldots, \eta_{k-1}).$$

We define a function $f(\xi_0, \ldots, \xi_{k-1})$ for $\xi_i < \exp_{k-1}(\gamma)$, $i < k$ by the stipulation

$$f(\xi_0, \ldots, \xi_{k-1}) = h\big(g(\xi_0, \xi_1), \ldots, g(\xi_{k-2}, \xi_{k-1})\big).$$

Then by (1) and (2) f is defined whenever $\xi_i < \exp_{k-1}(\gamma)$ for $i < k$ and $\mathscr{R}(f) \subseteq \gamma$. Assume now that $\xi_0, \ldots, \xi_k$ is a sequence of $k+1$-ordinals less than $\exp_{k-1}(\gamma)$ such that $\xi_i \neq \xi_{i+1}$ for $i < k$. Then by (1) $\eta_i = g(\xi_i, \xi_{i+1})$ $i < k$ is a sequence of k ordinals less than $\exp_{k-2}(\gamma)$ such that $\eta_i \neq \eta_{i+1}$ for $i < k-1$.

Hence by (2) we have

$$f(\xi_0, \ldots, \xi_{k-1}) = h(\eta_0, \ldots, \eta_{k-2}) \neq h(\eta_1, \ldots \eta_{k-1}) = f(\xi_1, \ldots, \xi_k).$$

We prove

THEOREM 1. *Let $\gamma \geq \omega$, $k \geq 1$, $i \geq 2$. Then $\Theta \to [i]^k_\gamma$ holds iff $|\Theta| > \exp_{k-1}(\gamma)$.*

PROOF. Assume $|\Theta| > \exp_{k-1}(\gamma)$. Then by Theorem 39 of [13] we know, that if type $h(<) = \Theta$, hence $|h| > \exp_{k-1}(\gamma)$ then for an arbitrary k-partition $\mathscr{H}_\xi$, $\xi < \gamma$ of type γ of h there exists a $\xi < \gamma$ and a subset $h' \subseteq h$, $|h'| = \gamma^+$ such that $\mathscr{S}_k[h'] \subseteq \mathscr{H}_\xi$, hence by Definition 2.3. we have $\Theta \to [i]^k_\gamma$.

Assume $|\Theta| = \exp_{k-1}(\gamma)$. It is sufficient to prove $\Theta \nrightarrow [2]^k_\gamma$.

Let typ $h(<) = \Theta$ and let $g \in {}^h\exp_{k-1}(\gamma)$ be a one-to-one mapping of h onto $\exp_{k-1}(\gamma)$. Let f be a function satisfying the requirements of Lemma 2.

We define a k-partition $\mathscr{H}_\xi$, $\xi < \gamma$ of type γ of h as follows. Let $x \in \mathscr{S}_k[h]$, $x = \{x_0, \ldots, x_{k-1}\}$

$$x_i < x_{i+1} \quad \text{for} \quad i < k-1.$$

Put $x \in H_\xi$ iff $f(g(x_0), \ldots, g(x_{k-1})) = \xi$.

[4] Note that case $k = 2$ of Lemma 2 and Lemma 1 are in fact equivalent.

It follows from Lemma 2 and Definition 2.2. that $\mathscr{H}_\xi$ contains no increasing path of length 2.

In § 4 we will give an independent proof of case $k = 2$ of the positive part of Theorem 1. There we will also discuss the case of finite γ's, (see Theorem 5 and Problem 8).

§ 3. Proof of the main result

THEOREM 2. *Let $\gamma \geq \omega$, $1 \leq k < \omega$. There exists a graph $\mathcal{G} = \langle g, G\rangle$ satisfying the following conditions* (a), (b), (c):

$$\text{(a)}\quad \alpha(\mathcal{G}) = |g| = \exp_{k-1}(\gamma)^+$$

$$\text{(b)}\quad \mathrm{Chr}(\mathcal{G}) > \gamma$$

$$\text{(c)}\quad \textit{for every}\ \ g' \subseteq g\ \ |g'| \leq \exp_{k-1}(\gamma)$$
$$\mathrm{Chr}(\mathcal{G}(g')) \leq \gamma.$$

PROOF. Put $\exp_{k-1}(\gamma)^+ = \alpha$, $g = \mathscr{S}_k[\alpha]$.

Let X, Y be two elements of $\mathscr{S}_k[\alpha] = g$.

$$X = \{\xi_i\}_{i<k},\quad Y = \{\eta_i\}_{i<k},\quad \xi_i < \xi_{i+1} < \alpha,\ \eta_i < \eta_{i+1} < \alpha$$

for $i < k-1$.

Put $\{X, Y\} \in G$ iff $\xi_{i+1} = \eta_i$ for $i < k-1$.

Then $\mathcal{G}$ satisfies the requirements of the theorem.

Put $g^\tau = \mathscr{S}_k[\tau]$ for $\tau \leq \alpha$. Then $g^\tau \subseteq g$. Let g_ξ, $\xi < \gamma$ be a sequence of type γ of subsets of g^τ such that $g^\tau = \bigcup_{\xi<\gamma} g_\xi$. Then the statement that $\mathcal{G}(g_\xi)$ contains no edge is equivalent to the statement that the set-system $\langle \tau, g_\xi\rangle$ contains no increasing path of length 2. Hence by Theorem 1 $\mathrm{Chr}\big(G(g^\tau)\big) \leq \gamma$ holds iff $\tau < \alpha$. Considering that α being regular $g' \subseteq g$, $|g'| \leq \exp_{k-1}(\gamma)$ implies that $g' \subseteq g^\tau$ for some $\tau < \alpha$ and taking into account that $g^\alpha = g$, this implies that condition (b) and (c) both hold. (a) is true since $|\mathscr{S}_k[\alpha]| = \alpha$.

Note that the graph defined in the proof of Theorem 2 was already used by the authors for other purposes, see e.g. [9] Theorems 6 and 7.

COROLLARY 1. Assume G.C.H. Then for every ξ and for every $1 \leq k < \omega$ there exists a graph $\mathcal{G}$ with $\alpha(\mathcal{G}) = \omega_{\xi+k}$ and of chromatic number $\omega_{\xi+1}$, all whose subgraphs spanned by a set of vertices of power at most $\omega_{\xi+k-1}$ have chromatic number $\leq \omega_\xi$.

Assume $\alpha(\mathcal{G}) = \omega_\omega$ and $\mathrm{Chr}\big(\mathcal{G}(g')\big) = \omega$ for every $g' \subseteq g$, $|g'| < \omega_\omega$. Then we obviously have $\mathrm{Chr}(\mathcal{G}) = \omega$. Hence the simplest unsolved problem we have here is

PROBLEM 1. Assume G.C.H. Does there exist a graph $\mathcal{G}$ with $\alpha(\mathcal{G}) = \omega_{\omega+1}$ such that $\mathrm{Chr}\big(\mathcal{G}(g')\big) \leq \omega$ for each $g' \subseteq g$, $|g'| \leq \omega_\omega$ but $\mathrm{Chr}(\mathcal{G}) > \omega$?

REMARK. In view of the results formulated in [3] and the above remark one may conjecture that there is a positive answer for Problem 1 if we replace $\omega_{\omega+1}$ by a regular α, and assume G.C.H. and $\alpha \in C_0$, $[\omega_1, \alpha) \subseteq C_1$ where C_0, C_1 are the classes of cardinals defined in [2]. Or it is quite conceivable that one can prove with a refinement of the method of [5] and without assuming G.C.H. that the same result holds for every regular α less than the first weakly inaccessible cardinal $> \omega$, but we did not succeed in proving these.

There is a problem of another type which remains open.

PROBLEM 2. Assume G.C.H. Does there exist a graph $\mathcal{G}$ with $\alpha(\mathcal{G}) = \mathrm{Chr}(\mathcal{G}) = \omega_2$ such that for every $g' \subseteq g$, $|g'| < \omega_2$ we have

$$\mathrm{Chr}(\mathcal{G}(g')) \leq \omega\,?$$

This should be compared with Theorem 1.1.3 and Problem 11.4. of [6].

Note that Corollary 1 gives us a graph with $\alpha(\mathcal{G}) = \omega_2$, $\mathrm{Chr}(\mathcal{G}) = \omega_1$ satisfying the last condition. It is possible that the answer for Problem 2 is positive even if we replace ω_2 by a regular α, and we assume that α is not too large.

On p. 88 we prove an implication relevant to Problem 2.

For every infinite cardinal α and for every γ we define a graph $\mathcal{G}_{\alpha,\gamma}$ as follows.

DEFINITION 3.1. Put $g_{\alpha,\gamma} = {}^{\alpha}\gamma$.

Let $f \neq h \in {}^{\alpha}\gamma$.
Put $\{f, h\} \in G_{\alpha,\gamma}$ iff there is a $\xi < \alpha$ such that

1) $$f(\zeta) \neq h(\zeta) \text{ for every } \zeta,\ \xi \leq \zeta < \alpha.$$

$$\mathcal{G}_{\alpha,\gamma} = \langle g_{\alpha,\gamma}, G_{\alpha,\gamma}\rangle.$$

For every $\{f, h\} \in G_{\alpha,\gamma}$ put $\xi_{\alpha,\gamma}(f, h)$ for the least ξ satisfying (1).

DEFINITION 3.2. The graph $\mathcal{G}$ is said to have property $P(\alpha, \gamma)$ if each subgraph of $\mathcal{G}$ spanned by a set of vertices of power less than α has chromatic number at most γ.

We prove

THEOREM 3. (A): *Assume $\alpha \geq \omega$. Then $\mathcal{G}_{\alpha,\gamma}$ has property $P(cf(\alpha), \gamma)$.*
(B): *Assume $\alpha(\mathcal{G}) = \alpha$ and $\mathcal{G}$ has property $P(\alpha, \gamma)$, $\gamma \geq 2$. Then there is a $\mathcal{G}' \subseteq \mathcal{G}_{\alpha,\gamma}$ such that $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic.*

PROOF OF PART (A). Let $g' \subseteq g_{\alpha,\gamma}$, $|g'| < cf(\alpha)$.
Put $A = \{\xi < \alpha : \xi = \xi_{\alpha,\gamma}(f, h)$ for some $f, h \in g'$, $\{f, h\} \in G_{\alpha,\gamma}\}$. Then $|A| < cf(\alpha)$ hence there is a $\xi_0 < \alpha$ such that $\xi < \xi_0$ for every $\xi \in A$. Put $g_\eta = \{f \in g' : f(\xi_0) = \eta\}$ for $\eta < \gamma$. Then g_η, $\eta < \gamma$ is a partition of type γ of g'. Assume $f, h \in g_\eta$ for some $\eta < \gamma$. Then $f(\xi_0) = h(\xi_0) = \eta$ hence $\{f, h\} \notin G_{\alpha,\gamma}$ by the definition of A. It follows that $\mathrm{Chr}\big(\mathcal{G}_{\alpha,\gamma}(g')\big) \leq \gamma$, hence $\mathcal{G}_{\alpha,\gamma}$ possesses property $P(\alpha, \gamma)$.

PROOF OF PART (B). Assume that $\mathcal{G}$ satisfies the conditions of part (B) of our theorem and $\gamma \geq \omega$. We may assume $g = \alpha$. By the assumption that

$\mathcal{G}$ has property $P(\alpha, \gamma)$, for each $\xi < \alpha$ we have $\mathrm{Chr}(\mathcal{G}(\xi)) \leq \gamma$. Let $g_{\xi,\eta}$ $0 < \eta < \gamma$ be a disjoint partition of type γ of ξ such that the graphs $\mathcal{G}(g_{\xi,\eta})$ $\xi < \alpha$, $o < \eta < \gamma$ have no edges. For every $\zeta < \alpha$ let f_ζ be the element of $g_{\alpha,\gamma} = {}^{\alpha}\gamma$ defined by the following stipulation.

$$f_\zeta(\xi) = \begin{cases} 0 & \text{if} \quad \xi \leq \zeta \\ \eta & \text{if} \quad \zeta < \xi \quad \text{and} \quad \zeta \in g_{\xi,\eta}. \end{cases} \tag{1}$$

It is obvious that $\zeta_1 \neq \zeta_2 < \alpha$ implies $f_{\zeta_1} \neq f_{\zeta_2}$.

Assume now that $\{\zeta_1, \zeta_2\} \in G$ then $\zeta_1 \in g_{\xi,\eta_1}$, $\zeta_2 \in g_{\xi,\eta_2}$ implies $\eta_1 \neq \eta_2$ for $\xi > \max(\zeta_1, \zeta_2)$, hence $\{f_{\zeta_1}, f_{\zeta_2}\} \in G_{\alpha,\gamma}$ by (1). Put $\mathcal{G}' = \langle g', G' \rangle$, where $g' = \{f_\zeta : \zeta < \alpha\}$, $G' = \{\{f_{\zeta_1}, f_{\zeta_2}\} : \{\zeta_1, \zeta_2\} \in G\}$ Then $\mathcal{G}' \subseteq \mathcal{G}_{\alpha,\gamma}$ and $\mathcal{G}'$ is isomorphic to $\mathcal{G}$. In cases $2 \leq \gamma < \omega$ a slight modification of this proof gives the result. We omit the details.

By a theorem of P. ERDŐS and N. G. DE BRUIJN [10] each graph $\mathcal{G}$ possessing property $P(\alpha, \gamma)$ for some finite γ, has chromatic number at most γ. As a corollary of part A of Theorem 3 we have

COROLLARY 2. Chr $(\mathcal{G}_{\alpha,\gamma}) = \gamma$ for every finite γ and for every α.

Theorem 3 shows that the determination of Chr $(\mathcal{G}_{\alpha,\gamma})$ would be decisive to answer the problem formulated in the introduction. Considering that each graph has property $P(\gamma^+, \gamma)$ for every γ, the relevant cases are those where $\alpha > \gamma^+$. As a corollary of Theorems 2 and 3 we have Chr $(\mathcal{G}_{\omega_{\xi+k},\omega_\xi})$ $\geq \omega_{\xi+1}$ provided G.C.H. holds. The simplest unsolved problem is

PROBLEM 3. Assume G.C.H. Is Chr $(\mathcal{G}_{\omega_2,\omega}) = \omega_1$? Or $= \omega_2$ or $= \omega_3$?
The following problem seems to be strongly related to Problem 3.

PROBLEM 4. Assume G.C.H. Does there exist a subset $\mathcal{A} \subseteq {}^{\omega_2}\omega_1$ satisfying the following conditions $|\mathcal{A}| = \omega_3$ and for every pair $f \neq g \in \mathcal{A}$ there is a $\xi(f, g) < \omega_2$ such that $f(\zeta) \neq g(\zeta)$ for every $\xi < \zeta < \omega_2$.

Problem 4 is well known and the answer to it is affirmative provided the same is true for the special case $\xi = 1$ of the general KUREPA problem, i.e. if there exists a family $\mathcal{F}$, $|\mathcal{F}| = \omega_{\xi+2}$, $\mathcal{F} \subseteq \mathcal{S}(\omega_{\xi+1})$ such that for every $\varrho < \omega_{\xi+2}$, $\mathcal{F} \upharpoonright \varrho = \{F \cap \varrho : F \in \mathcal{F}\}$ has cardinal at most ω_ξ.
(The special case $\zeta = 0$ is usually known as KUREPA's problem).

It has been proved recently that a positive answer to these problems is consistent with the usual axiom systems of set theory. We cannot give the exact reference.

We will outline the proof of the following

THEOREM 4. *Assume G.C.H. and assume that there exists a set $\mathcal{A}$ satisfying the conditions of Problem 4. Then* Chr $(\mathcal{G}_{\omega_2,\omega}) \geq \omega_2$.

PROOF. We prove that $\mathcal{G}_{\omega_2,\omega}$ contains a subgraph $\mathcal{G}'$ homomorphic to the graph $\mathcal{G} = \langle g, G \rangle$ defined by the stipulations $g = \mathcal{S}_2[\omega_3]$, if $x = \{\xi_0, \xi_1\}$, $y = \{\eta_0, \eta_1\}$, $\xi_0 < \xi_1$ and $\eta_0 < \eta_1$, ξ_i, $\eta_i < \omega_3$, then $\{x, y\} \in G$ iff $\xi_1 = \eta_0$.

$\mathcal{G}$ is the graph constructed for the proof of Theorem 2 in case $k = 2$, $\alpha = \omega_3$, $\gamma = \omega_1$. By Theorem 2 we have $\mathrm{Chr}(\mathcal{G}) = \omega_2$.

Let $\mathscr{A}$ be a set satisfying the conditions of Problem 4.

Let φ be one-to-one mapping of ω_3 onto $\mathscr{A}$. By case $\gamma = \omega$, $k = 2$ of Theorem 1 there exists a function $f(\eta_0, \eta_1)$ defined for $\eta_0, \eta_1 < \omega_1$ with $\mathscr{R}(f) \subseteq \omega$ such that

$$(1) \qquad f(\eta_0, \eta_1) \neq f(\eta_1, \eta_2) \quad \text{for} \quad \eta_0 \neq \eta_1, \ \eta_1 \neq \eta_2.$$

We define a function ψ on g as follows. Let $\{\xi_1, \xi_2\}$ $\xi_1 < \xi_2 < \omega_3$ be an arbitrary element of $g = \mathscr{S}_2[\omega_3]$.

Let $\psi(\{\xi_1, \xi_2\}) = \psi$ be the element of ${}^{\omega_2}\omega$ defined by the stipulation

$$\psi(\zeta) = f(\varphi_{\xi_1}(\zeta), \varphi_{\xi_2}(\zeta)) \quad \text{for} \quad \zeta < \omega_2.$$

φ_{ξ_1} and φ_{ξ_2} are elements of $\mathscr{A}$, hence $\varphi_{\xi_1}(\zeta)$, $\varphi_{\xi_2}(\zeta)$ are ordinals $< \omega_1$ for $\zeta < \omega_2$. Thus $\psi(\zeta) < \omega$ for $\zeta < \omega_2$ and $\psi \in {}^{\omega_2}\omega = g_{\omega_2,\omega}$. Let now

$$\{\{\xi_0, \xi_1\}, \quad \{\xi_1, \xi_2\}\} \qquad \xi_0 < \xi_1 < \xi_2$$

be an edge of $\mathscr{G}$. Put $\psi(\{\xi_0, \xi_1\}) = \psi_0$, $\psi(\{\xi_1 \ \xi_2\}) = \psi_1$. By the definition of $\mathscr{A}$ there is an ordinal $\zeta_0 < \omega_2$ such that $\varphi_{\xi_0}(\zeta) \neq \varphi_{\xi_1}(\zeta)$ and $\varphi_{\xi_1}(\zeta) \neq \varphi_{\xi_2}(\zeta)$ for $\zeta_0 < \zeta < \omega_2$. Hence by (1) $\psi_0(\zeta) \neq \psi_1(\zeta)$ for $\zeta_0 < \zeta < \omega_2$.

It follows that if $\{x, y\}$ is an edge of $\mathscr{G}$ then $\{\psi(x), \psi(y)\}$ is an edge of $\mathscr{G}_{\omega_2,\omega}$.

Hence if $\mathscr{G}'$ is the subgraph of $\mathscr{G}_{\omega_2,\omega}$ spanned by $\psi(g)$ then $\mathrm{Chr}(\mathscr{G}) \leq \mathrm{Chr}(\mathscr{G}')$. It results that $\mathrm{Chr}(\mathscr{G}_{\omega_2,\omega}) \geq \omega_2$.

It is obvious that using the general form of Kurepa's problem a more general result can be formulated. Since we do not know if the assumption of a positive solution of Kurepa's problem is really necessary we did not bother to formulate the general statement.

We mention that it would be more natural to use the special case $\xi = 0$ of Kurepa's problem (instead of the case $\xi = 1$) for the proof of Theorem 4 but we did not succeed in doing this.

§ 4. Results and problems related to the special partition problem of § 2.

Definition 4.1. The ordered pair $\mathscr{G}^* = \langle g, G^* \rangle$ is said to be a directed graph if G^* is a subset of $g \times g$ not containing pairs of the form $\langle x, x \rangle$. If $\mathscr{G}^*$ is a directed graph, we denote by $\mathscr{G}$ the corresponding graph $\langle g, G \rangle$ where $G = \big\{\{x, y\} : \text{for which } \langle xy \rangle \in G^* \text{ or } \langle yx \rangle \in G^*\big\}$. We assume that the reader is familiar with the concept of directed path.

We put $\mathrm{Chr}(\mathscr{G}^*) = \mathrm{Chr}(\mathscr{G})$.

Note that if $\mathscr{G}$ is a graph and $\prec$ is an ordering of g, we can associate to it a directed graph $\mathscr{G}^*$ by the stipulation $\langle xy \rangle \in G^*$ iff $\{x, y\} \in G$ and $x \prec y$.

We need the following

Lemma 3. Let $\mathscr{H} = \langle h, H \rangle$ be a set-system where H consists of sets of at least two elements. Let $\mathscr{H}_\xi$, $\xi < \alpha$ be a sequence of sub-set systems of $\mathscr{H}$ forming an "edge-partition" of $\mathscr{H}$ i.e. $h_\xi = h$ for $\xi < \alpha$ and $H = \bigcup_{\xi<\alpha} H_\xi$.

Assume further that $\mathrm{Chr}(\mathscr{H}_\xi) \leq \alpha_\xi$ for $\xi < \alpha$. Then

$$\mathrm{Chr}(\mathscr{H}) \leq \prod_{\xi<\alpha} \alpha_\xi.$$

Lemma 3 is well known and it is an obvious corollary of the definition. We omit the proof.

We will use

LEMMA 4. (Theorem of GALLAI [11]) Let $\mathcal{G}^*$ be a directed graph, k an integer ≥ 1. If $\mathcal{G}^*$ does not contain a directed path of length k then $\mathrm{Chr}(\mathcal{G}^*) \leq k$.

We prove

THEOREM 5. *Let $\mathcal{G}^*$ be a directed graph, γ a cardinal ≥ 1 (finite or infinite). Let i be an integer. Assume $\mathrm{Chr}(\mathcal{G}^*) > i^\gamma$ and let $\mathcal{G}^*_\xi = \langle g, G^*_\xi \rangle, \xi < \gamma$ be an edge-partition of $\mathcal{G}^*$, i.e.*

$$G^* = \bigcup_{\xi<\gamma} G^*_\xi.$$

Then at least one of the graphs $\mathcal{G}^*_\xi$ contains a directed path of length i. If γ is infinite then one of the graphs $\mathcal{G}^*_\xi$ contains a directed path of length i for every $i < \omega$.

PROOF. There is a directed graph $\mathcal{G}^*_\xi$ with chromatic number $> i$, (or with chromatic number $> 2^\gamma$ if $\gamma > \omega$). For if not then by Lemma 3

$$\mathrm{Chr}(\mathcal{G}^*) \leq i^\gamma \quad \text{or} \quad \mathrm{Chr}(\mathcal{G}^*) \leq \prod_{\xi<\gamma} 2^\gamma = 2^\gamma,$$

respectively. Hence by Lemma 4 this $\mathcal{G}^*_\xi$ contains a directed path of length i or a directed path of length i for every $i < \omega$, respectively.

Easy examples show (see e.g. the graph defined in [13] that it is no longer true even for $\gamma > \omega$ that one of the graphs $\mathcal{G}^*_\xi$ contains an infinite directed path. It is also easy to see (e.g. by induction on γ) that for finite γ the number i^γ is best possible. We omit the easy proof.

It is obvious by the remark made after the Definition 4.1. that Theorem 5 is a generalization of case $k = 2$ of Theorem 1, since the chromatic number of a complete graph is equal to its cardinality. Thus in this case the proof of Theorem 5 gives a simple proof of the positive part of Theorem 1 not referring to the involved argument of [12]. A corresponding generalization of Theorem 5 for uniform set-systems of $\varkappa(H) \geq 3$ is no longer true. We will give a very strong counterexample in our Theorem 7.

First we prove a theorem which will serve as a lemma in the proof of Theorem 7.

THEOREM 6. *Assume G.C.H. Let $\alpha \geq \omega$ be regular and let $\mathcal{G} = \langle g, G \rangle$ be a graph with $\alpha(\mathcal{G}) = \alpha^+$ satisfying the following condition.*

(1) *For every A, $B \subseteq g$ $|A| = \alpha$, $|B| = \alpha^+$ there exist $a \in A$, $b \in B$ such that $\{a, b\} \in G$.*

Assume further that t is an integer and A_i, $i < t$ and B are subsets of g such that $|A_i| = \alpha$ for $i < t$ and $|B| = \alpha^+$.

Then there exist subsets $A'_i \subseteq A_i$ for $i < t$ and $B' \subseteq B$ such that $|A'_i| = \alpha$ for $i < t$ and $|B'| = \alpha$ satisfying the following condition.

(2) *For every* $a \in \bigcup_{i<t} A'_i$ *the set*

$$\{b \in B' : \{a, b\} \notin G\}$$

has cardinal less than α^+.[5]

One can prove Theorem 6 with a ramification argument frequently applied in [8]. However, here we need a very simple and special form of it and it seems to be worth to give it in details.

PROOF. We may assume that the A_i are disjoint.

Let $f_i \in {}^{\alpha\times\alpha}A_i$ be a one-to-one mapping of $\alpha\times\alpha$ onto A_i for $i < t$.

Put $\{f_i(\xi, \eta)\}_{\eta<\alpha} = A_{i,\xi}$ for $i < t$, $\xi < \alpha$. Then $|A_{i,\xi}| = \alpha$ for $\xi < \alpha$.

Put $C_{i,\xi} = \{b \in B$: there is an $\eta < \alpha$ for which $\{f_i(\xi, \eta), b\} \in G\}$. By (1) we have

(3) $|B - C_{i,\xi}| < \alpha^+$ for every $\xi < \alpha$ and $i < t$ otherwise $A_{i,\xi}$, $B - C_{i,\xi}$ do not satisfy (1).

Put

$$C = \bigcap_{\xi<\alpha} \bigcap_{i<t} C_{i,\xi}$$

By (3) we have

$$|C| = \alpha^+ \tag{4}$$

We may assume that for every $\xi < \alpha$ and for every $i < t$ $\{C_\eta^{i,\xi}\}_{\eta<\alpha}$ is a disjoint partition of $C \subseteq C_{i,\xi}$ such that

$$\{\{f_i(\xi, \eta)\}, b\} \in G \quad \text{for every} \quad b \in C_\eta^{i,\xi} \quad \eta < \alpha. \tag{5}$$

Hence

$$C = \bigcap_{\xi<\alpha} \bigcap_{i<t} (\bigcup_{\eta<\alpha} C_\eta^{i,\xi}).$$

Applying the distributive law we obtain

$$C = \bigcup_{\varphi \in {}^{t\times\alpha}\alpha} D_\varphi. \tag{6}$$

where

$$D_\varphi = \bigcap_{\xi<\alpha} \bigcap_{i<t} C_{\varphi(i,\xi)}^{i,\xi}.$$

Put briefly

$$D_\psi = \bigcap_{\xi<\zeta} \bigcap_{i<t} C_{\psi(i,\xi)}^{i,\xi}$$

for every

$$\psi \in \bigcup_{\zeta<\alpha} {}^{t\times\xi}\alpha, \qquad D(\psi) = t\times\zeta.$$

(7) Put $C_1 = \{x \in C :$ there is a $\psi \in \bigcup_{\xi<\alpha} {}^{t\times\xi}\alpha,\ x \in D_\psi,\ |D_\psi| < \alpha^+\}$.

Considering that by G.C.H. and α being regular, $|\bigcup_{\xi<\alpha} {}^{t\times\xi}\alpha| \leq \alpha$ we have

$$|C_1| \leq \alpha.$$

It follows from (4) that there is an element x of $C - C_1$.

[5] This should be compared with the results and problems stated for polarized partitions in [8] especially with Theorem 43 and Problem 12.

Then by (6) and (7) $x \in D_\varphi$ for some $\varphi \in {}^{t\times\alpha}\alpha$ and $|D_{\varphi|\zeta}| = \alpha^+$ for every $\zeta < \alpha$. It results that there exists a one-to-one sequence $\{b_\zeta\}_{\zeta<\alpha}$ satisfying

(8) $$b_\zeta \in D_{\varphi|\zeta} \quad \text{for} \quad \zeta < \alpha.$$

Put $B' = \{b_\zeta\}_{\zeta<\alpha}$ and $A'_i = \{f_i(\xi, \varphi(i, \xi)) : \xi < \alpha\}$. Then $|B'| = \alpha$ since b_ζ is one-to-one. $|A'_i| = \alpha$ for $i < t$ since f_i is one-to-one. Let $a \in A'_i$ for some $i < t$ then $a = f_i(\xi, \varphi(i, \xi))$ for some $\xi < \alpha$. Assume $\zeta > \xi$. Then by (8) $b_\zeta \in D_{\varphi|\zeta}$ hence by (6) $b_\zeta \in C^{i,\xi}_{\varphi(i,\xi)}$.

It follows from (5) that then $\{a, b_\zeta\} \in G$. Thus (2) is fulfilled and the theorem is proved.

Remark. It is easy to see from the proof that Theorem 6 remains true if t is replaced by any cardinal $\gamma < \alpha$, but we do not need this in this paper.

Definition 4.2. Let g be an ordered set, and let $\prec$ be an ordering of g. In what follows if we write some $X \in \mathscr{S}_k[g]$ in the form $\{x_0, \ldots, x_{k-1}\}$ we always assume $x_0 \prec \ldots \prec x_{k-1}$. We will say that $X, Y \in \mathscr{S}_k[g]$ are in *similar position* if $a \in X \cap Y$ implies $a = x_i$, $a = y_i$ for some $i < k$.

It was shown by E. Milner (personal communication) that for every $\alpha \geq \omega$ there exists a uniform set-system $\mathscr{H} = \langle h, H\rangle$ with $\varkappa(H) = 3$, $\alpha(\mathscr{H}) = \alpha^+$ such that $\mathscr{H}$ does not contain an increasing path of length 2, and every $h' \leq h$, $|h'| = \alpha^+$ contains a triplet of H. As a corollary of this last statement we have $\mathrm{Chr}\,(\mathscr{H}) = \alpha^+$.

Using his idea we prove Theorem 7.

Theorem 7. *Assume G.C.H. Let $k < \omega$ and $\alpha \geq \omega$, α regular.*

There exists a uniform set-system $\mathscr{H} = \langle \alpha^+, H\rangle$ with $\varkappa(H) = k$ with the natural ordering $<$ of the ordinals $< \alpha^+$ satisfying the following conditions 1., 2., 3.

1. *For each $h' \subseteq \alpha^+$, $|h'| = \alpha^+$ there is an $X \in H$, $X = \{\xi_0, \ldots, \xi_{k-1}\}$ such that $\xi_i \in h'$* for $i < k$. *As a corollary of this*

$$\mathrm{Chr}(\mathscr{H}) = \alpha(\mathscr{H}) = \alpha^+.$$

2. *If $X, Y \in H$ have at least two elements in common then they are in the same position.*

3. *If $X = \{\xi_0, \ldots, \xi_{k-1}\}$, $Y = \{\eta_0, \ldots, \eta_{k-1}\} \in H$, $X \neq Y$ and $\xi_{k-1} = \eta_{k-1}$ then $X \cap Y = \{\xi_{k-1}\}$.*

Proof. By the assumption that G.C.H. holds, $\mathscr{S}_\alpha[\alpha^+]$ has cardinal α^+. Put $\mathscr{S}_\alpha[\alpha^+] = S$.

(1) Let $\mathscr{A} \in {}^{\alpha^+}S$ be a one-to-one mapping of α^+ onto the set S.

Put

(2) $S_\xi = \{\mathscr{A}_\zeta : \mathscr{A}_\zeta \subseteq \xi$ and $\zeta < \xi\}$ for $\xi < \alpha^+$. Then $|S_\xi| \leq \alpha$ for $\xi < \alpha^+$.

(3) For every $\xi < \alpha^+$ let $\{A^\xi_\eta\}_{\eta<\alpha}$ be a sequence of type α containing all the elements of the set S_ξ.

(4) Let $f \in {}^{\mathscr{S}_2[k]}\binom{k}{2}$ be a one-to-one mapping of the set $\mathscr{S}_2[k]$ onto the integer $\binom{k}{2}$. We briefly write $f(i, j) = f(\{i, j\})$ for $i < j < k$. As a corollary of

Theorem 17/A of [8] there exists a *disjoint* partition $\mathscr{S}_2[\alpha^+] = \bigcup_{l<\binom{k}{2}} I$ of type $\binom{k}{2}$ of the set $\mathscr{S}_2[\alpha^+]$ satisfying the following condition.

(5) If $B, C \subseteq \alpha^+$, $|B| = \alpha$, $|C| = \alpha^+$ then for every $l < \binom{k}{2}$ there exist $\xi_0, \xi_1 < \alpha^+$ such that $\xi_0 \in B$, $\xi_1 \in C$ and $\{\xi_0, \xi_1\} \in I_l$.

First we define a uniform set system $\mathscr{H}' = \langle \alpha^+, H' \rangle$ with $\varkappa(H') = k$ by the following stipulation.

(6) Let $X = \{\xi_0, \ldots, \xi_{k-1}\} \in \mathscr{S}_k[\alpha^+]$. Then $X \in H'$ iff for every $i < j < k$ $\{\xi_i, \xi_j\} \in I_l$, where $l = f(i, j)$.

Now we are going to define the uniform set-system $\mathscr{H} = \langle \alpha^+, H \rangle$ with $\varkappa(H) = k$ by the following stipulations.

(7) Let $\xi < \alpha^+$ be fixed. We will define the set H_ξ of those elements $X = \{\xi_0, \ldots, \xi_{k-1}\}$ of H for which $\xi_{k-1} = \xi$.

We define H_ξ as a sequence $\{X^\xi_{\eta'}\}_{\eta'<\varphi_\xi}$ of elements of $\mathscr{S}_k[\alpha^+]$, where $X^\xi_{\eta'} = \{\xi^{\eta'}_0, \ldots, \xi^{\eta'}_{k-1}\}$ and $\varphi_\xi \leq \alpha$.

We define the sequence by transfinite induction on η' as follows.

Assume $\eta' < \alpha$ and $X^\xi_{\eta''}$ is defined for every $\eta'' < \eta'$.

If there exists an $\eta' \leq \eta < \alpha$ such that there is an $X \in \mathscr{S}_k[\alpha^+]$, $X = \{\xi_0, \ldots, \xi_{k-1}\}$ satisfying the conditions

(8) $\xi_{k-1} = \xi, \{\xi_0, \ldots, \xi_{k-2}\} \subseteq A^\xi_{\eta}, X \cap X^\xi_{\eta''} = \{\xi\}$ for $\eta'' < \eta'$, $X \in \mathscr{H}$ then let η be the least ordinal of this kind and let X^ξ_{η} be a k-tuple satisfying (8) with this η.

If no such η exists we put $\eta' = \varphi_\xi$ and $X^\xi_{\eta'}$ will not be defined.

We put $H_\xi = \{X^\xi_{\eta'}\}_{\xi\eta'<\varphi_\xi}$ and $H = \bigcup_{\xi<\alpha^+} H_\xi$, $\mathscr{H} = \langle \alpha^+, H \rangle$.

We will prove that $\mathscr{H}$ satisfies the requirements of our theorem. It is obvious that $\alpha(\mathscr{H}) = \alpha^+$. By the definition of $\mathscr{H}$ given in (7) and (8) $\mathscr{H}$ obviously satisfies requirement 3. of the theorem. We will now show that even $\mathscr{H}'$ satisfies the second requirement.

(9) Assume $X_0 \neq X_1 \in H'$ and $|X_0 \cap X_1| \geq 2$. Let a, b $a < b$ be two elements of $X_0 \cap X_1$. If $X_0 = \{\xi^0_0, \ldots, \xi^0_{k-1}\}$ $X_1 = \{\xi^1_0, \ldots, \xi^1_{k-1}\}$, then $a = \xi^0_{i_0} = \xi^1_{i_1}$ $b = \xi^0_{j_0} = \xi^1_{j_1}$ for some $i_0 < j_0$, $i_1 < j_1$. We have to prove $i_0 = i_1$, $j_0 = j_1$.

In fact by (5) there is exactly one $l < \binom{k}{2}$ such that $\{a, b\} \in I_l$. By definition (6) we have $f(i_0, j_0) = f(i_1, j_1) = l$. f being one-to-one, by (4), we have $i_0 = i_1$, $j_0 = j_1$.

Considering that $\mathscr{H} \leq \mathscr{H}'$. (9) implies that Condition 2 of our theorem is fulfilled.

We briefly write $A < B$ for $A, B \leq \alpha^+$ if $\xi < \eta$ for every $\xi \in A$, $\eta \in B$. Now we prove

(10) Let $A \subseteq \alpha^+$, $|A| = \alpha^+$ and let t be an integer $1 \leq t \leq k - 1$. There exist sets A_i, $i < t$ such that $A_i \subseteq A$, $A_0 < \ldots < A_{t-1}$, $|A_i| = \alpha$ for $i < t$ which satisfy the condition:

(11) For each $i < j < t$ and for every $a \in A_i$ the set $\{b : b \in A_j$ and $\{a, b\} \notin I_{f(i,j)}\}$ has cardinal $< \alpha$.

We prove (10) by induction on t. For $t = 1$ the statement is trivial. Assume $t > 1$, and let the set $A'_0, \ldots, A'_{t-2}$ satisfy the requirements of (10) for $t-1$. Let B be a set $A_{t-2} < B$, $|B| = \alpha^+$, $B \subseteq A$. Let $\mathcal{G} = \langle g, G\rangle$ where $g = A'_0 \cup \ldots \cup A'_{t-2} \cup B$ and $\{a, b\} \notin G$ iff $a \in A_i$, $b \in B$ and $\{a, b\} \notin I_{f(i,t-1)}$.

Applying Theorem 6 for $\mathcal{G}$, A'_i and B, we obtain sets $A_i \subseteq A'_i$ for $i < t-1$ and a set $B' \subset B$. Put $B' = A_{t-1}$. $A_i, i < t$ obviously satisfy the requirements. Now we prove that $\mathscr{H}$ satisfies the first requirement of the theorem.

Let $A \subseteq \alpha^+$, $|A| = \alpha^+$.

Let A_i, $i < k-1$ be sets satisfying (10) with $t = k-1$. Let $\xi \in A$ and put

(12) $B_{i,\xi} = \{\zeta \in A_i,\ \{\zeta, \xi\} \in I_{f(i,k-1)} \text{ for } i < k-1\}$.

Let $A' = \{\xi \in A : |B_{i,\xi}| = \alpha \text{ for every } i < k-1\}$.
We prove

(13) $|A'| = \alpha^+$,

If $|A'| < \alpha^+$ then there is a $\xi_0 < \alpha^+$ such that for every $\xi_0 < \xi < \alpha^+$, $\xi \in A$ there is an $i(\xi) < k-1$ for which $|B_{i(\xi),\xi}| < \alpha$. Hence there is an $A'' \subseteq A'$, $|A''| = \alpha^+$ and an $i < k-1$ such that $|B_{i,\xi}| < \alpha$ for every $\xi \in A''$

Using G.C.H. and that α is regular then there exist a $B \subseteq A_i$ and an $A''' \subseteq A''$, $|A'''| = \alpha^+$ such that $B = B_{i,\xi}$ for every $\xi \in A'''$. But then by (12) the sets $A_i - B$ and A''' do not satisfy (5) for I_l with $l = f(i,k-1)$. Hence (13) is true.

Let now $D = \bigcup_{i<k-1} A_i$. Then $|D| = \alpha$, hence $D = \mathscr{A}_\zeta$ for some $\zeta < \alpha^+$ where $\mathscr{A}$ is the well ordering given in (1), hence if $A_{k-1} < \{\xi\}$ then by (2) and (3) $D = A^\xi_\eta$ for some $\eta < \alpha$.

By (13) there is an $A_{k-1} < \{\xi\}$, $\xi \in A'$ (hence $\xi \in A$). But briefly $B_i = B_{i,\xi}$ for $i < k-1$. Then $|B_i| = \alpha$, $B_i \subseteq A_i$ for $i < k-1$.

First we prove that there is a sequence $Z_\varrho = \{\zeta^\varrho_0, \ldots, \zeta^\varrho_{k-1}\}_{\varrho<\alpha}$ of type α of elements of $\mathscr{H}'$ such that $\zeta^\varrho_{k-1} = \xi$ for every $\varrho < \alpha$, $\zeta^\varrho_i \in B_i$ for $i < k-1$ and the sequence $Z'_\varrho = \{\zeta^\varrho_0, \ldots, \zeta^\varrho_{k-2}\}$ is disjointed.

We define Z'_ϱ by induction on ϱ. Assume that $Z'_{\varrho'}$ is defined for every $\varrho' < \varrho$ for some $\varrho < \alpha$. Put $C_{i,\varrho} = B_i - \bigcup_{\varrho'<\varrho} Z'_{\varrho'}$. Then $|C_{i,\varrho}| = \alpha$ for $i < k-1$. We define $Z'_\varrho = \{\zeta^\varrho_i\}_{i<k-1}$ by induction on i as follows. Assume ζ^ϱ_j is defined for every $j < i$ for some $i < k-2$. Then considering that $|C_{i,\varrho}| = \alpha$ it follows from (10) and (11) that there is a $\zeta^\varrho_i \in C_{i,\varrho}$ such that $\{\zeta^\varrho_j, \zeta^\varrho_i\} \in I_{f(j,i)}$ for every $j < i$. Thus $\{\zeta^\varrho_i\}$ is defined for $i < k-1$. Put $Z'_\varrho = \{\zeta^\varrho_0, \ldots, \zeta^\varrho_{k-2}\}$.

Considering (12) it follows from the construction that $Z_\varrho \in \mathscr{H}$ for every $\varrho < \alpha$.

Now we prove that there is an $\eta' \leq \eta$ such that $X^\xi_{\eta'} = \{\xi^{\eta'}_0, \ldots, \xi^{\eta'}_{k-1}\}$ and $\xi^{\eta'}_i \in A^\xi_\eta$ for $i < k-1$.

If this is not true for any $\eta'' < \eta$ then there is a $\varrho < \alpha$ such that Z'_ϱ is disjoint from each $X^\xi_{\eta''}$ for $\eta'' < \alpha$ hence $\eta = \eta'$ is the minimal ordinal $\geq \eta'$ satisfying (8) hence the statement is true by definition (7).

Considering that $\xi \in A$, $\xi^{\eta'}_i \in B_i \subseteq A$ for $i < k-1$ it follows that $X^\xi_{\eta'} \in H$, $X^\xi_{\eta'} \subseteq A$ hence $\mathscr{H}$ satisfies the first requirement of our theorem as well.

REMARKS. 1. Both Theorem 6 and Theorem 7 remain true for singular α as well. In case of Theorem 6 the proof can be carried out by improving the given proof using the idea of the proof of Theorem 38 of [8]. In case of Theorem 7 the regularity of α was used only in reference to Theorem 6 and in the proof of (13), where in case of singular α, Theorem 34 of [8] can be applied. Several problems arise if we replace α^+ with a limit number. We omit the discussion of them.

2. Both proofs make use of G.C.H. heavily. This seems to be natural in case of Theorem 6. On the other hand, one can hope that it can be avoided in the proof of Theorem 7, at least in some special cases.

This problem remains open even for $\alpha = \omega$.

3. Theorem 7 is best possible of its kind as is shown by the following assertions.

4.1. Let $\mathscr{H} = \langle h, H\rangle$ be a uniform set system with $\varkappa(H) = k \geq 2$ and let $<$ be an ordering of h. Assume that for $X = \{x_0, \ldots, x_{k-1}\}$, $Y = \{y_0, \ldots, y_{k-1}\} \in H$ $x_i \neq y_j$ for some fixed $i \neq j < k$. Then $\mathrm{Chr}\,(\mathscr{H}) \leq 2$. As a corollary of this if $|X \cap Y| \geq 1$, $X, Y \in H$ implies that X and Y are in the same position then $\mathrm{Chr}(\mathscr{H}) \leq 2$.

PROOF. Put $h_0 = \{a \in h : a \neq x_i \text{ for any } X \in H\}$

$$h_1 = \{a \in h : a \neq x_j \text{ for any } X \in H\}.$$

Then $h = h_0 \cup h_1$ and $H(h_0)$, $H(h_1)$ are empty.

4.2. Let $\mathscr{H} = \langle \alpha, H\rangle$ be a uniform set-system with $\varkappa(H) \geq 3$. If there are no $X = \{x_0, \ldots, x_{k-1}\}$, $Y = \{y_0, \ldots, y_{k-1}\} \in H$ with $x_i = y_i$ for $i < k-2$ then $\mathrm{Chr}\,(\mathscr{H}) \leq \omega$. 4.2. is a corollary of Theorem 12.1 of [6].

Theorem 7 shows that if $\mathscr{H}$ is a uniform set system with $\varkappa(H) \geq 3$ with a given ordering $<$ of h, the fact that $\mathscr{H}$ does not contain an increasing path of length 2 does not give a bound for the chromatic number of $\mathscr{H}$. However, this fact can be proved without using G.C.H. That is why we state separately the following

THEOREM 8. (MILNER). *For every $\alpha \geq \omega$ there exists a uniform set-system $\mathscr{H} = \langle \alpha^+, H\rangle$ of $\varkappa(H) = 3$ with the natural ordering of the ordinals $< \alpha^+$ such that*

1. *$\mathscr{H}$ does not contain an increasing path of length 2*
2. *Every $A \subseteq \alpha^+$, $|A| = \alpha^+$ contains an element of H as a subset.*

We only outline the PROOF. As a corollary of Theorem 7 of [8] we have $\alpha^+ \nrightarrow (\alpha^+, \alpha^+)^2$ i.e. there is a graph $\mathscr{G} = \langle \alpha^+, G\rangle$ satisfying the conditions

(1) if $A \subseteq \alpha^+$, $|A| = \alpha^+$ then $\mathscr{G}(A)$ is not complete

(2) if $A \subseteq \alpha^+$, $|A| = \alpha^+$ then $\mathscr{G}(A)$ has an edge.

We define the set system as follows. Let $X = \{\xi_0, \xi_1, \xi_2,\}$ be an arbitrary element of $\mathscr{S}_3[\alpha^+]$ $X \in H$ iff $\{\xi_0, \xi_1\} \in G$, $\{\xi_1, \xi_2\} \notin G$, $\mathscr{H}$ obviously satisfies requirement 1. Let $A \subseteq \alpha^+$, $|A| = \alpha^+$. Then there are a $\xi_0 \in A$, $A' \subseteq A$, $|A'| = \alpha^+$ such that $\{\xi_0, \xi\} \in G$ for every $\xi \in A'$ for if not then (2) is obviously false.

We may assume $\{\xi_0\} < A'$. On the other hand by (1) there are $\xi_1 < \xi_2$ such that $\xi_1, \xi_2 \in A'$, $\{\xi_1\ \xi_2\} \notin G$. Hence 2 is fulfilled as well.

Note that interesting new phenomena arise if we investigate the possible generalizations of Theorems 7 and 8 for $k \geq 3$, in case we replace the cardinal α^+ by a limit cardinal β. Both theorems are false if $\beta \notin C_0$, i.e. if $\beta \rightarrow (\beta, \beta)^2$ holds.

Theorem 8 remains true for every $\beta \in C_0$, i.e. if $\beta \nrightarrow (\beta, \beta)^2$ holds. We do not know if the same is true for Theorem 7 for strongly inaccessible β's, even if we only require conditions 1 and 2.

Assuming G.C.H. we can discuss the case of other β's where $cf(\beta)$ is not inaccessible using the methods of [8]. We preserve the details for a later publication.

Another problem arises in connection with the negative part of Theorem 1. One could hope that if h is a set ordered by a relation $\prec$, $\gamma \geq \omega$ and $|h| = \exp_{k-1}(\gamma)$ then there exists a k-partition H_ξ, $\xi < \gamma$ of type γ of h such that the set systems $\mathscr{H}_\xi$ not only do not contain increasing paths of length 2, but satisfy some stronger restrictions as those imposed on $\mathscr{H}$ in Theorem 7. For $k=2$ conditions 2, 3 are not really stronger. It is obvious that for $k \geq 3$ the $\mathscr{H}_\xi$'s cannot satisfy condition 3 of Theorem 7 even if $|h| > \gamma^+$. Hence the real problem is if there is a sequence satisfying condition 2 of Theorem 7. This remains unsolved even in the following simplest case.

PROBLEM 5. Let $h = \alpha$, with the natural ordering of ordinals, $\exp_1(\omega) < \alpha \leq \exp_2(\omega)$. Does there exist a 3-partition $\mathscr{H}_\xi$, $\xi < \omega$ of type ω of h such that for every $\xi < \omega$ $X, Y \in \mathscr{H}_\xi$, $|X \cap Y| \geq 2$ imply that X and Y are in the same position?

REMARK. It is easy to see that if $\alpha = \exp_1(\omega)$ then such 3-partition really exists, since by Theorem 1 there exists a 2-partition $\mathcal{G}_n$, $n < \omega$ of α such that $\mathcal{G}_n$ does not contain an increasing path of length 2. But then $\mathscr{H}_{(n_1, n_2, n_3)}$, $n_1, n_2, n_3 < \omega$ defined by the stipulation $X = \{\xi_1, \xi_2, \xi_3\} \in H_{(n_1, n_2, n_3)}$ iff $\{\xi_1, \xi_2\} \in G_{n_1}$, $\{\xi_2, \xi_3\} \in G_{n_2}$ and $\{\xi_1\ \xi_3\} \in G_{n_3}$ satisfies the requirement.

We mention some other special problems which arose in connection with Theorems 6, 7, 8. We only formulate the simplest unsolved forms of them.

Similarly, as in the proof of Theorem 8 it is easy to see that the following assertion is true.

4.3. Let $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2$ be a disjoint 2-partition of ω_1 satisfying the requirement:

(α) Whenever $A, B \subseteq \omega_1$, $|A| = |B| = \omega_1$ then for every $i < 3$ there is a $\{\xi, \eta\} \in G_i$ such that $\xi \in A$, $\eta \in B$.

Then there is an $X \in \mathscr{S}_3[\omega_1]$ such that $\mathscr{S}_3[X] \cap G_i \neq 0$ for every $i < 3$.

On the other hand, using the ideas of the proofs of Theorems 6 and 7 one can easily see that the following assertion is true.

4.4. Let $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2$ be a disjoint 2-partition of ω_1 satisfying the condition

(β) Whenever $A, B \subseteq \omega_1$ $|A| = \omega$, $|B| = \omega_1$, then for $i < 3$ there are $\xi \in A$, $\eta \in B$ such that $\{\xi, \eta\} \in G_i$.

Then for every $h \in {}^{\mathscr{S}_2[3]}3$ there is an $X \in \mathscr{S}_3[\omega_1]$ $X = \{\xi_0, \xi_1, \xi_2\}$ such that $\{\xi_i, \xi_j\} = G_{h(\{i,j\})}$ for every $i < j < 3$. We cannot solve the following

PROBLEM 6. (A) Is 4.4 true under the weaker condition (α) of 4.3 instead of (β) of 4.4?

(B) Are 4.3 or 4.4 true under the weaker condition

(γ) For every $A \subseteq \omega_1$, $|A| = \omega_1$ $\mathcal{G}_i(A)$ has edges for $i = 0, 1, 2$.

Note that we cannot prove the existence of a partition satisfying (α), (β) or (γ) without using the continuum hypothesis. See the remark made after the proof of Theorem 9 in [9].

We mention a problem of different type. Theorem 7 implies (using G.C.H.) that there is a uniform set-system $\mathscr{H} = \langle \omega_1, H \rangle$ with $\varkappa(H) = 3$ such that

(1) $\mathrm{Chr}\,(H) = \omega_1$

(2) If $h' \subseteq \omega_1$, $|h'| = n < \omega$ for some h' then $|H(h')| < \frac{n^2}{4} + o(n)$.

PROBLEM 7. Does the above statement remain true under the stronger condition $|H(h')| \leq o(n^2)$ or even cn^2 where $c < \frac{1}{4}$?

Note that there is a uniform set-system $\mathscr{H} = \langle \omega H \rangle$ with $\varkappa(H) = 3$ such that $\mathrm{Chr}(\mathscr{H}) = \omega$ and satisfying the condition $h' \subseteq \omega$, $|h'| = n$ implies that $|H(h')| < n.\ f(n)$ where $f(n)$ is a function tending to infinity as slowly as we please.

Finally we mention a problem of finite type connected to Theorem 2. By RAMSEY's theorem we know that for every i, t, k there is a least integer $m = m(i, t, k)$ such that for every $m' > m$ and for every k-partition $\mathscr{H}_\xi$, $\xi < t$ of type t of m' there is a $\xi < t$ such that $\mathscr{H}_\xi$ contains an increasing path of length i. Theorem 5 implies $m(i, t, 2) = i^t$.

PROBLEM 8. $m(i, t, k) = ?$ for $k \geq 3$.

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