SOME PROBLEMS ON THE PRIME FACTORS OF CONSECUTIVE INTEGERS

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In this note we are going to discuss some elementary questions on consecutive integers. Though the problems are all quite elementary we are very far from being able to solve them.

For n a positive integer and k a non-negative integer let

$$v(n; k) = \sum_{p \mid n+k,\ p>k} 1.$$

In other words $v(n; k)$ denotes the number of prime factors of $n + k$ which do not divide $n + i$ for $0 \leq i < k$. Put

$$v_0(n) = \max_{0 \leq k < \infty} v(n; k).$$

One would expect that $v_0(n) \to \infty$ as $n \to \infty$ but we are very far from being able to prove this. We can only show that $v_0(n) > 1$ for $n \geq 17$. In fact we can show the following result: $v_0(n) > 1$ for all n except $n = 1, 2, 3, 4, 7, 8, 16$.

It is easy to see that $v_0(n) = 1$ for the above values of n. In general if $k > 1$ then $v(n; k) \leq 1$ for $k^2 + 3k > n - 3$. In fact for $k \geq n$ we have $v(n; k) = 1$ if and only if $n + k$ is a prime.

Clearly $v_0(n) = 1$ implies $n = p^\alpha$. Assume first p odd. $v_0(n) = 1$ implies $\nu(p^\alpha + 1) = 1$ or $p^\alpha + 1 = 2^\beta$. (Here $\nu(m)$ denotes the number of distinct prime factors of m.) $p = 3$ is impossible for $n > 3$ since $3^\alpha + 1 = 2^\beta$ is impossible for $\alpha > 1$. But then $n + 2 \equiv 0 \pmod 3$ and $n + 2 = 2^\beta + 1 = 3^\gamma$, but this is also known to be impossible for $\beta > 3$ i.e., for $n > 7$. If n is even then $n = 2^\alpha$, $2^\alpha + 1 = g^\beta$, $g \neq 3$ since $\alpha > 3$. Thus $2^\alpha + 2 = 2 \cdot 3^\gamma$ which is impossible since $\alpha > 4$. Thus our result is proved.

Put

$$v_l(n) = \max_{l \leq k < \infty} v(n; k).$$

It seems certain that

$$\lim_{n=\infty} v_l(n) = \infty$$

for every l, but unfortunately we have not even been able to prove that $v_1(n) = 1$ has only a finite number of solutions, though this certainly must be true. Probably the greatest n for which $v_1(n) = 1$ is $n = 330$, but we have no method of proving this. In fact, $v_1(n) = 1$ for $n =$ 1–4, 6–8, 10, 12, 15, 16, 18, 22, 24, 26, 30, 36, 42, 46, 48, 60, 70, 78, 80, 96, 120, 190, 222, 330, and for no other values of $n < 2500$.

A slight modification of this problem might be more amenable to attack.

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Denote by $V(n; k)$ the number of those p for which p^α divides $n + k$ and $p^\alpha > k$. That is, $V(n; k)$ is the number of primes p with $p^\alpha \parallel n + k$, $p^\alpha \nmid (n + i)$ for all $0 \leq i < k$, and put

$$V_l(n) = \max_{l \leq k < \infty} V(n; k).$$

It should be easier to prove that $V_1(n) = 1$ has only a finite number of solutions, but we did not quite succeed in showing this; probably $n = 80$ is the largest solution of $V_1(n) = 1$. In fact $V_1(n) = 1$ for $n =$ 1–4, 6–8, 12, 15, 16, 24, 30, 48, 80, and for no other values of $n < 2500$. The largest n such that $V_0(n) = 2$ is probably very large. From factor tables we notice that $V_0(94491) = V_0(94492) = V_0(99387) = V_0(99741) = 2$ and that $V_0(n) > 2$ for all other n, $94000 < n < 10^5$.

Put

$$f(n) = \max_{0 \leq k < \infty} \frac{1}{k+1} \sum_{i=0}^{k} v(n; i).$$

It seems probable that $f(n) \to \infty$ as $n \to \infty$ but this will be probably very difficult to prove. It easily follows from a well known result of Hardy and Ramanujan that for almost all n we have $f(n) \leq (1 + o(1)) \log \log n$, and it is not impossible that for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$ we have

$$f(n) > (1 - \varepsilon) \log \log n.$$

It is well known and easily follows from the prime number theorem that

$$\limsup_{n=\infty} \nu(n) \frac{\log \log n}{\log n} = 1.$$

One could conjecture that for every k

$$\limsup_{n=\infty} \sum_{i=0}^{k} \nu(n + i) \frac{\log \log n}{\log n} = 1,$$

but this if true will be difficult. In the other direction we can not even prove that

$$\limsup_{n=\infty} (\max_{1 \leq m \leq n} (\nu(m) + \nu(m+1)) - \max_{1 \leq m \leq n} \nu(m)) = \infty.$$

The analogous conjecture for other number theoretic functions is often not too hard to prove e.g. we can show without too much difficulty that for every k we have [1]

$$\limsup_{n=\infty} (\max_{1 \leq m \leq n} \textstyle\sum_{i=1}^{k} \sigma(m + i))/\max_{1 \leq m \leq n} \sigma(m) = 1$$

and

$$\limsup_{n=\infty} (\max_{1 \leq m \leq n} \textstyle\sum_{i=1}^{k} d(m + i))/\max_{1 \leq m \leq n} d(m) = 1.$$

The proofs follow without too much difficulty from the fact that $d(n)$ and $\sigma(n)$ are large if n is composed of all the very small primes but then $n + 1$

[1] In fact it is not hard to prove that

$$\lim_{n=\infty} (\max_{1 \leq m \leq n-1} (\sigma(m) + \sigma(m+1)) - \max_{1 \leq m \leq n} \sigma(m))/n = 1.$$

can not have small prime factors, but the details of the proof are somewhat messy and will not be given here. Because of the slow growth of $\nu(n)$ this method of proof breaks down for $\nu(n)$.

One could try to investigate the lower bound of $\sum_{i=0}^{k-1} \nu(n+i)$. A well known theorem of Pólya easily implies that

$$\text{(1)} \qquad \liminf_{n=\infty} \sum_{i=0}^{k-1} \nu(n+i) \geq k + \pi(k) - 1.$$

First of all $\prod_{i=0}^{k-1} (n+i)$ is divisible by all the primes not exceeding k. Pólya's theorem states that if $a_1^{(k)} < a_2^{(k)} < \cdots$ is the sequence of integers composed of the primes not exceeding k then $a_{i+1}^{(k)} - a_i^{(k)} \to \infty$. Hence, if n is sufficiently large, every integer $n, n+1, \cdots, n+k-1$ with one possible exception has a prime factor greater than k and this implies (1). It seems to us that

$$\text{(2)} \qquad \liminf_{n=\infty} \sum_{i=0}^{k-1} \nu(n+i) \leq k + \pi(k)$$

for every k (perhaps the sign of equality always holds in (2)).

Let $p_1 = 2 < p_2 < \cdots$ be the sequence of consecutive primes. Schinzel deduces from Pólya's theorem that with the possible exception of a finite number of cases amongst $p_1 \cdots p_{k-1} p_{k+1}$ consecutive integers there is always one which has more than k prime factors. It seems possible that $p_1 \cdots p_k$ is the right value, but even for $k = 2$ we can not improve the value of Schinzel.

Reference

A. Schinzel, *Problem 31*, Elem. Math., vol. 14 (1959), pp. 82–83.

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