ON SOME APPLICATIONS OF GRAPH THEORY TO GEOMETRY

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To Professor H. S. M. Coxeter on his sixtieth birthday

Let $[P_n^{(k)}]$ be the class of all subsets $P_n^{(k)}$ of the k-dimensional Euclidean space consisting of n distinct points and having diameter 1. Denote by $d_k(n, r)$ the maximum number of times a given distance r can occur among points of a set $P_n^{(k)}$. Put

$$D_k(n) = \max_r d_k(n, r).$$

In other words $D_k(n)$ denotes the maximum number of times the same distance can occur between n suitably chosen points in k-dimensional space.

Lenz showed that $D_4(n) > \frac{1}{4}n^2 + cn$, and by using the method of Lenz and a graph-theoretic result of Stone and myself **(1)** I proved **(2)** that

$$\lim_{n=\infty} D_k(n)\bigg|_{n^2} = \frac{1}{2} - \frac{1}{2[\frac{1}{2}k]}. \tag{1}$$

Denote by $G(n; l)$ a graph of n vertices and l edges and denote by $m(n; p)$ the largest integer for which there exists a $G(n; m(n, p))$ which contains no complete graph of p vertices K_p. Turán **(6)** proved that

$$m(n; p) = \frac{p-2}{2(p-1)}(n^2 - r^2) + \binom{r}{2} \qquad \text{if } n \equiv r \pmod{p-1}.$$

In this note we prove the following sharpening of (1):

THEOREM 1. *Let* $k = 2l, n \equiv 0 \pmod{2k}, n > n_0(k)$. *Then*

$$D_k(n) = m(n; l) + n = \frac{n^2}{2}\frac{l-1}{2} + n. \tag{2}$$

Further, for every $n > n_0(k)$,

$$m(n; l) + n - l \leqslant D_k(n) \leqslant m(n; l) + n.$$

For odd k I cannot substantially improve the results stated in **(2)**. I have not been able to disprove that for every k and n

$$D_k(n) = n^2\left(\frac{1}{2} - \frac{1}{2[\frac{1}{2}k]}\right) + O(n) \tag{3}$$

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holds. (3) is certainly false unless the following result holds. Let there be given n points on the surface of the two-sphere; then the same distance can occur at most cn times between them.

Denote by (x_i, x_j) the distance between x_i and x_j. We outline the proof of the following theorem.

THEOREM 2. *To every s there is a c_s so that if $x_1, \ldots, x_n$ $(n > n_0(s))$ are n distinct points in four-dimensional Euclidean space. Then there are at least s distinct numbers amongst any $\frac{1}{4}n^2 + c_s n$ of the (x_i, x_j).*

Theorem 2 has some interest in view of the fact that, by Theorem 1, for $c_s = 1$ all the distances can be equal.

It seems to be very difficult to obtain a good estimation for $D_2(n)$ and $D_3(n)$. It is known **(3)** that

$$n^{1+c/\log\log n} < D_2(n) < n^{3/2}. \tag{4}$$

The lower bound in (4) is probably close to being best possible, but I could not even prove that $D_2(n) = o(n^{3/2})$.

First we prove that

$$D_k(n) \leqslant m(n; l) + n. \tag{5}$$

Denote by $K_r(p_1, \ldots, p_r)$ the complete r-chromatic graph which has p_i vertices of the ith colour. To prove (5) we need the following lemma.

LEMMA. *Every $G(n; m(n; l) + n + 1)$ contains a $K_{l+1}(1, 3, \ldots, 3)$ for $n > n_0(l)$.*

This lemma is due to Simonovits and myself **(5)**.

Now let $x_1, \ldots, x_n$ be n points in k-dimensional space for which

$$d_k(n; r) \geqslant m(n; l) + n + 1$$

for some r. Define a graph G whose vertices are $x_1, \ldots, x_n$; we join two vertices x_i and x_j if their distance is r. By our lemma this $G(n; d_k(n; r))$ contains a $K_{l+1}(1, 3, \ldots, 3)$, i.e. there are $3l + 1$ points

$$x_1^{(1)}, \text{ and } x_i^{(s)} \ (2 \leqslant s \leqslant l + 1, i = 1, 2, 3)$$

so that the distance between any two $x_i^{(s)}$ for different values of s is always r. But this is easily seen to be impossible in $k = 2l$-dimensional space since the points $x_i^{(s)}$ $(2 \leqslant s \leqslant l + 1, i = 1, 2, 3)$ determine l planes which must be mutually orthogonal, and then clearly $x_1^{(1)}$ cannot be equidistant from all these points (the $x_i^{(s)}$, $i = 1, 2, 3$, for $2 \leqslant s \leqslant l + 1$, must all lie on circles with equal radius and common centre, which is the intersection of the orthogonal planes). This completes the proof of (5).

Next we show that for $n \equiv 0 \pmod{2k}$

$$D_k(n) \geqslant m(n; l) + n. \tag{6}$$

Our proof is substantially identical with that of Lenz.

Consider l mutually orthogonal planes in $k = 2l$-dimensional space. In each of these planes consider a circle of radius $\frac{1}{2}$ and assume that all these circles have a common centre. On each of these circles choose $n/l = 4r$ points which form r squares of side $1/\sqrt{2}$. Clearly the distance between any two of these points which are on different circles is $1/\sqrt{2}$ and this gives

$$m(n;l) = n^2(l-1)/2l$$

pairs of points whose distance is $1/\sqrt{2}$; the points on the circles clearly give the remaining n pairs of points at distance $1/\sqrt{2}$. This completes the proof of (6). (5) and (6) prove (2). The same method which proved (6) gives

$$D_k(n) \geqslant m(n;l) + n - l.$$

This completes the proof of Theorem 1.

Now we outline the proof of Theorem 2. We define a $[G(n; \frac{1}{4}n^2 + c_s n]$ as follows: The vertices are our $x_1, \ldots, x_n$. x_i and x_j are joined if and only if x_i and x_j belong to the $\frac{1}{4}n^2 + c_s n$ selected pairs. Let $l = l(k)$ be sufficiently large; it will be determined later. It follows from **(5)** that for sufficiently large c_s our graph contains a $K_3(1, l, l)$. Denote the vertices of this $K_3(1, l, l)$ by $x_1; y_1, \ldots, y_l; z_1, \ldots, z_l$. If there are at least s distinct numbers amongst the (y_i, z_j), Theorem 2 is proved. If not, then the same distance—say r—occurs at least l^2/s times. Join y_i and z_j if and only if $(y_i, z_j) = r$. It easily follows from a theorem of Kövári, Sós, and Turán **(4)** that for $l > l_0(s)$ this graph contains $K_2(2s-1, 2s-1)$. Without loss of generality, denote the vertices of this graph by $y_1, \ldots, y_{2s-1}; z_1, \ldots, z_{2s-1}$. Since

$$(y_i, z_j) = r \qquad (1 \leqslant i, j \leqslant 2s-1),$$

it easily follows that the x's and y's are on two orthogonal planes, and on these planes they are on circles which have a common centre—the intersection of the two planes. x_1 is in the 4-dimensional space spanned by these two planes; hence if we drop a perpendicular from x_1 onto these planes, the foot of at least one of them cannot be the intersection of these planes (i.e. the common centre of our two circles). Without loss of generality we can assume that the foot of the perpendicular dropped from x_1 onto the plane of the y's is not the centre of the circle containing the y's. But then at most two y's are equidistant from x_1; hence there are at least s distinct distances amongst the (x_1, y_j), $j = 1, \ldots, 2s-1$. This completes the proof of Theorem 2.

Finally we state the following without proof.

THEOREM 3. *Let there be given n points $x_1, \ldots, x_n$ in 4-dimensional space. Then there is an absolute constant c and an $n_0 = n_0(\epsilon, c)$ so that for $n > n_0(\epsilon, c)$ there are more than n^c distinct numbers amongst any $\frac{1}{4}n^2(1+\epsilon)$ of the (x_1, x_j).*

The proof of Theorem 3 is similar to that of Theorem 2; no doubt both are special cases of a more general theorem which gives an estimation of the number of distinct numbers among $\frac{1}{4}n^2 + nf(n)$ numbers (x_i, x_j).

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