On a Problem of B. Jónsson

by

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B. Jónsson has asked: Is there an algebra of power α with no proper subalgebra of power α? H. J. Keisler and F. Rowbottom proved that the answer is affirmative for every α, provided Gödel's constructibility axiom holds [1].

The main aim of this paper is to prove, using the generalized continuum hypothesis (G.C.H. in what follows), that the answer is affirmative for each non limit cardinal (see Theorem 1). This result can be extended for locally finite algebras, too (see Theorem 3). Without using G.C.H. we prove that the answer to the problem is "yes", provided $\alpha = \omega_n$, $n < \omega$ (see Theorem 2). We also prove without using G.C.H. that for every α there is an algebra $\langle A, f\rangle$ of power α with one ω-ary operation which has no proper subalgebra of power α. (See Theorem 5).

Finally, we state another problem and some results relevant to Jónsson's problem.

THEOREM 1. *Assume G.C.H.*

Then for each infinite cardinal α there exists an algebra $\langle A, f\rangle$ of power α^+ witц one binary operation such that it has no proper subalgebra of power α^+.

THEOREM 2. *For each finite n there exists an algebra $\langle A, f\rangle$ of power ω_n with one binary operation which has no proper subalgebra of power ω_n.*

Proof of Theorem 1. Put $A = \alpha^+$. By Theorem 17 of [2] there exists a function $f \in {}^{\alpha^+ \times \alpha^+} \alpha^+$ satisfying the following condition

$$(1) \qquad \text{If} \quad B \subseteq \alpha^+, \quad |B| = \alpha^+, \quad \text{then} \quad f(B \times B) = \alpha^+.$$

The algebra $\langle A, f\rangle$ obviously satisfies the requirement of Theorem 1.

Note that if β is a singular limit cardinal and the G.C.H. holds, there is no function f satisfying (1) with β instead of α^+ (See Theorem 20 of [2]).

We do not know if (1) can be satisfied if we replace α^+ by a strongly incompact inaccessible cardinal β.

Proof of Theorem 2. We prove the weaker statement, namely that there exists an algebra $\langle A, f\rangle$ with one $n+1$-ary operation satisfying the requirement.

We omit the slightly technical but easy proof that this implies the stronger statement of Theorem 2. (See our Lemma 2).

We prove the Theorem by induction on n.

For $n=0$ the Theorem is well-known. We assume that it is true for n and prove it for $n+1$.

Put $A=\omega_{n+1}$. By the induction hypothesis for each $\omega_n \leqslant \xi < \omega_{n+1}$ there exists an algebra $\langle \xi, f_\xi \rangle$ with one $n+1$-ary operation which has no proper subalgebra of power ω_n. We define an $n+2$-ary operation $f(\xi_0, ..., \xi_{n+1})$ on ω_{n+1} as follows.

If

$$\xi_i \geqslant \xi_{n+1} \quad \text{for some} \quad i < n+1 \quad \text{or} \quad \xi_{n+1} < \omega_n \tag{2}$$

put $f(\xi_0, ..., \xi_{n+1}) = 0$.

If

$$\xi_i < \xi_{n+1} \quad \text{for} \quad i < n+1 \quad \text{and} \quad \xi_{n+1} \geqslant \omega_n \tag{2a}$$

put $f(\xi_0, ..., \xi_{n+1}) = f_{\xi_{n+1}}(\xi_0, ..., \xi_n)$.

Assume $B \subseteq \omega_{n+1}$, $|B| = \omega_{n+1}$, $\xi < \omega_{n+1}$.

We prove that ξ belongs to the subalgebra generated by B. In fact, there is $\xi' \in B$, $\xi' > \xi \cup \omega_n$ such that $|\xi' \cap B| = \omega_n$. ξ belongs to the subalgebra generated by $\xi' \cap B$ in $\langle \xi', f_{\xi'} \rangle$, hence, by (2a), it belongs to the subalgebra generated by B in $\langle \omega_{n+1}, f \rangle$.

Definition 1. Let $\langle A, ... \rangle$ be an algebra, $B \subseteq A$. We denote by $C(B)$ the subalgebra generated by B in A. The algebra $\langle A, ... \rangle$ is said to be *locally finite* if $C(B)$ is finite for every finite B.

The following generalization of Jónsson's problem has been suggested to us by J. Mycielski.

Does there exist a locally finite algebra of power α with no proper subalgebra of power α?

As a generalization of Theorem 1 we can prove

Theorem 3. *Assume G.C.H. Then for each infinite cardinal α there exists a locally finite algebra $\langle A, f \rangle$ of power α^+ with one binary operation f which has no proper subalgebra of power α^+.*

Proof in outline. Put $A = \alpha^+$. Instead of theorem 17 of [2] used in the proof of Theorem 1 we need the following

Lemma 1. *Assume G.C.H. There exists a function $f \in {}^{\alpha^+ \times \alpha^+}\alpha^+$ satisfying the following conditions:*

For each

$$\xi, \eta < \alpha^+ \quad f(\xi, \eta) < \xi \cap \eta \tag{3}$$

provided $\xi \cap \eta > 0$ and

$$f(\xi, \eta) = 0 \quad \textit{provided } \xi \cap \eta = 0. \tag{3a}$$

If

$$(4)\qquad B,C \subseteq \alpha^+, \quad |B| = \alpha, \quad |C| = \alpha^+, \quad \zeta < \alpha^+, \quad B \cap \zeta = 0$$

then there exist $\xi \in B$, $\eta \in C$ *such that* $f(\xi, \eta) = \zeta$.

Theorem 17/A of [2] states that there exists a function $f \in {}^{\alpha^+ \times \alpha^+}\alpha^+$ satisfying condition (4) without the additional assumption $B \cap \zeta = 0$. With a trivial modification of the proof given in [2], conditions (3) and (4) can be satisfied simultaneously. We omit the details.

The fact that f satisfies (4) trivially implies that $\langle \alpha^+ f \rangle$ has no proper subalgebra of power α^+. On the other hand, (3) implies that the algebra is locally finite as follows.

Let B be a finite set, $B_0 = B$ and $B_{k+1} = \{f(\xi, \eta)\colon \xi, \eta \in \bigcup_{i \leqslant k} B_i\}$ for $k < \omega$. Then each B_k is finite and $C(B) = \bigcup_{k<\omega} B_k$. Put $C_{k+1} = B_{k+1} - \bigcup_{i \leqslant k} B_k$. If C_k and $C_{k'}$ are non empty for $k < k' < \omega$ then by (3) Max $C_k >$ Max $C_{k'}$. It follows that there is a $k_0 < \omega$ such that $C(B) = \bigcup_{i \leqslant k_0} B_i$.

Note that the proof given in [2] implies that f can be supposed to be a commutative operation, too.

We mention that as an easy consequence of theorem 7 of [3] we have the following

THEOREM 4. *Assume α is a measurable cardinal and $\langle A, \ldots, \rangle$ is an algebra of power α with less than α finitary operations. Then there exists a proper subalgebra B of power α.*

On the other hand, we prove

THEOREM 5. *For each infinite cardinal α there exists an algebra $\langle A, f \rangle$ of power α with one ω-ary operation such that A has no proper subalgebra of power α.*

Theorem 5 is an easy consequence of a result of [3]. To make clear the connection of problems considered in this paper and in [3] we need a definition and a lemma.

DEFINITION 2. Let $\langle A, \ldots, \rangle$ be an algebra. A subset $B \subseteq A$ is said to be an *independent subset* of A if for every $x \in B$, $x \notin C(B - \{x\})$.

It is obvious that if all the operations are finitary, B is an independent subset iff for every finite $C \subseteq B$, $x \in B$, $x \notin C$ we have $x \notin C(B)$.

An element $x \in A$, $x \notin B \subseteq A$ *is said to be independent of B if* $x \notin C(B)$.

LEMMA 2. A) *Let $\langle A, f_1, \ldots, f_n, \ldots \rangle$ be an algebra of power α with denumerably many finitary operations which has no proper subalgebra of power α. Then there exists an algebra $\langle A^*, f \rangle$ of power α with one binary operation which has no proper subalgebra of power α.*

B) *If $\langle A, f_1, \ldots, f_n, \ldots \rangle$ is an algebra of power α with denumerably many ω-ary operations which has no proper subalgebra of power α, then there exists an algebra $\langle A^*, f \rangle$ of power α with one ω-ary operation which has no proper subalgebra of power α.*

We omit the proofs.

Proof of Theorem 5 in outline. Theorem 1 of [3] states — in another terminology — that for each infinite cardinal α there exists an algebra $\langle A, f\rangle$ of power α with one ω-ary operation such that there is no free subset of power ω. Moreover, $f(x_1, \dots, x_n, \dots)$ depends only on the set $\{x_1, \dots, x_n, \dots\}$. Assume now that Theorem 5 is false. Then there exists an element $x_0 \in A$ and a subset $A_0 \subseteq A$ of power α such that x_0 is independent of A_0. Assume that the elements $x_0, \dots, x_n$ and the subsets $A_0, \dots, A_n$ have been already defined in such a way that A_n has power α.

For each $B \subseteq \{x_0, \dots, x_n\}$ define on A_n an ω-ary operation $f_B(X)$, for each denumerable subset X of A_n by the stipulation

$$f_B(X) = f(B \cup X).$$

If Theorem 5 is false, then, by Lemma 2B), there is an element $x_{n+1} \in A_n$ and a subset $A_{n+1} \subseteq A_n$ of power α such that x_{n+1} is independent of A_{n+1} in the algebra $\langle A_n, f_B\rangle_{B \subseteq \{x_0, \dots, x_n\}}$. Then $x_0, \dots, x_n, \dots$ is defined by induction and the infinite set $\{x_0, \dots, x_n, \dots\}$ is an independent subset of the algebra $\langle A, f\rangle$. This is a contradiction, hence Theorem 5 is true.

We state the following

PROBLEM. For what cardinals α does there exist an algebra $\langle A, \dots,\rangle$ with finitely many finitary operations such that it has no independent subset of power ω?

Quite similarly as in the proof outlined above, Lemma 2A) gives that if for some α the answer is "yes", then the answer is "yes" for Jónsson's problem for the same α.

For ω_n, $n < \omega$ our next theorem gives an answer but we do not know the answer for any $\alpha \geqslant \omega_\omega$. For $\alpha = \omega_\omega$ the Problem is equivalent to Problem 1 of [3].

We state three more theorems relevant to the Problem. The proof of Theorem 6 we preserve for later publication, the others are corollaries of known results.

THEOREM 6.

A) *Assume $2 \leqslant n < \omega$. There exists an algebra $\langle A, f\rangle$ of power ω_n with one binary commutative operation such that there is no independent set of $n+1$ elements.*

B) *There exists an algebra $\langle A, f\rangle$ of power ω_1 with one binary commutative operation such that each subset of three elements contains at least two elements which belong to the subalgebra generated by the other two elements.*

C) *There exists an algebra $\langle A, f\rangle$ of power $\omega_0 = \omega$ with one binary commutative operation such that each subset of two elements generates the whole algebra.*

It is easy to see that part B) (and, obviously, C)) is best possible in the sense that each algebra of power ω_1 with finitely many finitary operations contains three elements x, y, z such that $x \notin C(\{y, z\})$. It is also easy to see that each algebra of power ω_2 with finitely many finitary operations contains three elements x, y, z such that $x \notin C(\{y, z\})$ and $y \notin C(\{x, z\})$.

Part A) of Theorem 6 is best possible in the following sense:

THEOREM 7. *Assume $n < \omega$ and let $\langle A, \ldots \rangle$ be an algebra of power ω_n with finitely many finitary operations.*

Then there exists a free subset B of at least n elements.

Theorem 7 is a trivial consequence of a theorem of [4].

THEOREM 8. *Let α be an infinite cardinal.*

A) *Then there exists an algebra $\langle A, \ldots \rangle$ of power α^+ with α unary operations such that A does not contain two independent elements and has no proper subalgebra of power α^+.*

B) *Let $\beta^+ < \alpha$. Each algebra $\langle A, \ldots \rangle$ of power α with at most β unary operations contains an independent subset of power α, and consequently has a proper subalgebra of power α.*

Part A) of Theorem 8 is trivial. Part B) is corollary of theorem 1 of [5] in case $\beta \geqslant \omega$. In case $\beta < \omega$ the proof can be reduced to the same theorem using the fact that then $\langle A, \ldots \rangle$ contains a locally finite subalgebra of power α.

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