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Tauberian theorems for sum sets

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Introduction. The sums formed from the set of non-negative powers of 2 are just the non-negative integers. It is easy to obtain "abelian" results to the effect that if a set is distributed like the powers of 2, then the sum set will be distributed like the non-negative integers. We will be concerned here with converse, or "Tauberian" results. The main theme of this paper is the following question: if the set of sums formed from a given set of positive real numbers resembles an arithmetic progression, how much must the original set resemble a set of constant multiples of powers of 2?

If we denote the given set by $k_0, k_1, k_2, \ldots$, arranged in ascending order, and let $S(x)$ count the number of those sums of distinct k_j that do not exceed x, our problem is, roughly, that of showing that k_n is close to 2^n if $S(x)$ is close to x. Our first result gives sharp bounds for liminf and limsup of $2^n/k_n$ in terms of liminf and limsup of $S(x)/x$. In the next section, we show that if $S(x)-x$ is bounded, then k_n-2^n is bounded, and furthermore, $\sum|k_n-2^n|<\infty$, so that if the k_n are integers, then $k_n = 2^n$ for all large n. We extend the method in the succeeding section to obtain estimates for k_n-2^n and $\sum_{n\leqslant N}|k_n-2^n|$ in terms of suitable bounds for $S(x)-x$, even if $S(x)-x$ is unbounded. Finally, on a slightly different note, we show that it is not possible for $S(x)$ to behave too much like x^a if $a<1$.

1. Asymptotic behavior. Let $K = k_0, k_1, k_2, \ldots,\ 0<k_0\leqslant k_1\leqslant k_2 \leqslant \ldots$, be any sequence of positive real numbers. Let $S(x)$ denote the number of choices of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ such that for each $j = 0, 1, 2, \ldots$, either $\varepsilon_j = 0$ or $\varepsilon_j = 1$, and such that $\varepsilon_0 k_0+\varepsilon_1 k_1+\ldots \leqslant x$. Let

$$A = \liminf_{x\to\infty} S(x)/x, \qquad \alpha = \liminf_{n\to\infty} 2^n/k_n,$$

$$B = \limsup_{x\to\infty} S(x)/x, \qquad \beta = \limsup_{n\to\infty} 2^n/k_n.$$

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A simple estimate shows that

$$\alpha \leqslant A \quad \text{and} \quad \beta \geqslant B. \tag{1}$$

We now give sharp inequalities in the opposite direction.

THEOREM 1. $\alpha = A$ *and*

$$B \geqslant \beta\left(2\frac{\alpha}{\beta} - \frac{\alpha^2}{\beta^2}\right). \tag{2}$$

Inequality (2) *is best possible in the sense that, given any* α *and* β *with* $\frac{1}{2} < \alpha/\beta \leqslant 1$, *there exists a sequence* K *for which equality holds, and given any* A *and* B *with* $\frac{2}{3} < A/B \leqslant 1$, *there exists a sequence* K *for which equality holds.*

Remarks. It follows immediately from the theorem that

$$\text{if} \quad \lim_{x\to\infty} S(x)/x = \theta \neq 0, \quad \text{then} \quad \lim_{n\to\infty} 2^n/k_n = \theta.$$

This result was proved by a different method in [1]. The question was raised in [1] whether the statement remains true for $\theta = 0$. The answer is no, as the following example shows. Let $k_{2^n} = 2^{2^n}$, and let $k_{2^n-r} = k_{2^n}$ for $0 \leqslant r < 2^{n-1}$. It is easy to see that $S(x) = o(x)$. On the other hand, $2^n/k_n = 1$ for infinitely many n. It is easy to modify the example so that the k_n are distinct, but it seems difficult to satisfy the additional condition, described in [1], that the sums of the k_n are all distinct.

It seems likely that our methods, if carried out in greater detail, would yield an estimate similar to (2), but taking account of the integral part of $\log_2 \beta/\alpha$, and that such an estimate would be best possible for any range of α/β, and not merely for $\alpha/\beta > \frac{1}{2}$. Finally, if we permit $\varepsilon_j = 0, 1, \ldots, N-1$, then it seems likely that our methods will yield analogous results for the lim sup and lim inf of N^n/k_n.

Proof of the estimates. First, $S(k_n - 1) \leqslant 2^n$, since if $\varepsilon_0 k_0 + \varepsilon_1 k_1 + \ldots + \varepsilon_n k_n + \ldots \leqslant k_n - 1$, then $\varepsilon_n = \varepsilon_{n+1} = \ldots = 0$, so that there are at most 2^n suitable choices of $\{\varepsilon_j\}$. Hence

$$\frac{S(k_n-1)}{k_n-1} \leqslant \frac{2^n}{k_n-1} = \frac{2^n}{k_n}\cdot\frac{k_n}{k_n-1},$$

and on letting $n \to \infty$, we get $A \leqslant \alpha$.

To obtain the estimate (2), we may suppose that $\alpha > 0$, since if $\alpha = 0$ then (2) is trivially true. We now choose any $a > 1/\alpha$, so that $k_n \leqslant 2^n a$ for all sufficiently large n. Without loss of generality, we shall suppose that $k_n \leqslant 2^n a$ for all $n = 0, 1, 2, \ldots$, because for any two sequences K and K' with $k_n = k'_n$ for $n \geqslant n_0$, it is easy to show that $A = A'$, $B =$

$= B'$, $a = a'$, $\beta = \beta'$. And given any b with $b > 1/\beta$ we will have $k_n \leqslant 2^n b$ for infinitely many n.

We now choose n large, with $k_n \leqslant 2^n b$, and estimate $S(2^n a)$. Clearly, $S(2^n a) \geqslant N_1 + N_2$, where N_1 is the number of choices of $\{\varepsilon_j\}$, $j = 0, 1, \ldots, n-1$, such that

$$\varepsilon_0 k_0 + \ldots + \varepsilon_{n-1} k_{n-1} \leqslant 2^n a \tag{3}$$

and N_2 is the number of choices of $\{\varepsilon_j\}$, $j = 0, 1, \ldots, n-1$, such that

$$\varepsilon_0 k_0 + \ldots + \varepsilon_{n-1} k_{n-1} + k_n \leqslant 2^n a. \tag{4}$$

But if $\varepsilon_0 2^0 a + \ldots + \varepsilon_{n-1} 2^{n-1} a \leqslant 2^n a$, then (3) holds, and therefore $N_1 \geqslant 2^n$. And if $\varepsilon_0 2^0 a + \ldots + \varepsilon_{n-1} 2^{n-1} a \leqslant 2^n (a-b)$, then (4) holds, so that

$$N_2 \geqslant \left[2^n\left(1 - \frac{b}{a}\right)\right] \geqslant 2^n\left(1 - \frac{b}{a}\right) - 1.$$

Hence

$$S(2^n a) \geqslant 2^n + 2^n(1 - b/a) - 1, \qquad \frac{S(2^n a)}{2^n a} \geqslant \frac{1}{a}\left(2 - \frac{b}{a}\right) - \frac{1}{2^n a}.$$

On letting $n \to \infty$ through a suitable sequence, we get

$$B \geqslant \frac{1}{a}\left(2 - \frac{b}{a}\right) = \frac{1}{b} \cdot \frac{b}{a}\left(2 - \frac{b}{a}\right).$$

We may now let $a \to 1/\alpha$ and $b \to 1/\beta$ to obtain (2).

The estimate is best possible. To show that (2) is best possible, we prove the first part, that given any α and β with $\frac{1}{2} < \alpha/\beta < 1$, there exists a K such that $B = \beta(2\alpha/\beta - \alpha^2/\beta^2)$. The second part then follows since $\varphi(\beta) = \beta(2A/\beta - A^2/\beta^2)$ is a continuous function of β, with $\varphi(A) = A$ and $\varphi(2A) = 3A/2$, so that if we are given A and B with $1 \leqslant B/A < 3/2$, we may apply the first part with $\alpha = A$ and β such that $\varphi(\beta) = B$. For the construction of K, let n_m be a sequence of positive integers that increases very rapidly to ∞. Let $a = 1/\alpha$ and $b = 1/\beta$ and define k_n by $k_n = 2^n a$ unless $n = n_m$ for some m, and $k_n = 2^n b$ if $n = n_m$ for some m. The point of the restriction $\alpha/\beta > \frac{1}{2}$ now appears; for the sequence K to be suitably defined, we need $2^n b \geqslant 2^{n-1} a$, or $b/a \geqslant \frac{1}{2}$.

A simple argument now shows that $B \geqslant \limsup_{n\to\infty} B_n$, where B_n is defined as follows. Let K^n be the sequence $\{k_j\}$, $j = 0, 1, 2, \ldots$, where $k_j = 2^j a$ for $j \neq n$ and $k_j = 2^j b$ if $j = n$. Let $S_n(x) = S(x : K^n)$ and let $B_n = \sup\big(S_n(x)/x\big)$, where the supremum is over all values of $x \geqslant x_0(n)$, where $x_0(n)$ is a function of n that tends very slowly to $+\infty$ as n tends to $+\infty$.

To determine $S_n(x)$, we must count those $\{\varepsilon_j\}$ for which

$$\varepsilon_0 2^0 a + \ldots + \varepsilon_{n-1} 2^{n-1} a + \varepsilon_n 2^n b + \varepsilon_{n+1} 2^{n+1} a + \ldots \leqslant x. \tag{5}$$

We now define $f_n(t)$ as the number of choices of $\{\varepsilon_j\}$ for which

$$\varepsilon_0 2^0+\varepsilon_1 2^1+\ldots+\varepsilon_{n-1}2^{n-1}+\varepsilon_{n+1}2^{n+1}+\ldots \leqslant t. \tag{6}$$

Now, considering in (5) the two cases $\varepsilon_n=0$ and $\varepsilon_n=1$, we see that

$$S_n(x)=f_n\left(\frac{x}{a}\right)+f_n\left(\frac{x-2^n b}{a}\right). \tag{7}$$

If we write $y=x/a$, then from (7) we get

$$\frac{S_n(ay)}{ay}=\frac{1}{a}\left\{\frac{f_n(y)}{y}+\frac{f_n(y-2^n b/a)}{y}\right\}, \tag{8}$$

so that

$$B_n=\frac{1}{a}\sup\left\{\frac{f_n(y)+f_n(y-2^n b/a)}{y}\right\}, \tag{9}$$

where the supremum is over the range $y\geqslant y_0$, where $y_0=y_0(n)=x_0(n)/a$. A computation shows that, writing $[t]$ for the integral part of t,

$$f_n(t)=2^n\left[\frac{t}{2^{n+1}}\right]+\min\left(2^n,1+[t]-2^{n+1}\left[\frac{t}{2^{n+1}}\right]\right) \quad \text{for} \quad t\geqslant 0, \tag{10}$$

and, of course, $f_n(t)=0$ for $t\leqslant 0$. For we may write $t=k2^{n+1}+s$, where k is a non-negative integer, and $0\leqslant s<2^{n+1}$. And $\varepsilon_0 2^0+\varepsilon_1 2^1+\ldots+\varepsilon_{n-1}2^{n-1}$ may be any non-negative integer $p<2^n$, while $\varepsilon_{n+1}2^{n+1}+\varepsilon_{n+2}2^{n+2}+\ldots$ may be any number $2^{n+1}q$, where q is any non-negative integer. Thus, we may rewrite (6) as

$$p+2^{n+1}q\leqslant k2^{n+1}+s \tag{11}$$

and $f_n(t)=f_n(k2^{n+1}+s)$ is the number of choices of p and q that make (11) valid. Now for $q=0,1,2,\ldots,k-1$ there are exactly 2^n choices of p that make (11) hold. So far we have accounted for $k\cdot 2^n$ choices. For $q=k+1,k+2,\ldots,$ there are no acceptable values of p. For $q=k$, if $s\geqslant 2^n-1$ then there are 2^n choices of p, while if $s<2^n-1$, then there are $[s+1]$ choices of p. Thus, we have

$$f_n(k\cdot 2^{n+1}+s)=2^n k+\min(2^n,[s+1]), \tag{12}$$

which is equivalent to (10).

Now, writing $y=2^{n+1}k+s$, with k a non-negative integer and $0\leqslant s<2^{n+1}$ as before, we get

$$f_n(y)=2^n k+\min(2^n,[s]+1), \tag{13}$$

$$f_n(y-2^n b/a)=\max\{0,2^n k+2^n[*]+\min(2^n,[s-2^n b/a]+1-2^{n+1}[*])\}, \tag{14}$$

where

$$* = \frac{s - 2^n b/a}{2^{n+1}}, \tag{15}$$

and we remark that $[*] = 0$ or -1 according as $s \geqslant 2^n b/a$ or $s < 2^n b/a$, respectively.

We now let

$$g(y) = g_n(y) = 2^n k + \min(2^n, s), \tag{16}$$

$$h(y) = h_n(y) = \max\{0, 2^n k + 2^n [*] + \min(2^n, s - 2^n b/a - 2^{n+1}[*])\}, \tag{17}$$

and let

$$B_n' = \frac{1}{a} \sup_{y \geqslant y_0(n)} \psi(y) \tag{18}$$

where

$$\psi(y) = \psi_n(y) = \frac{g_n(y) + h_n(y)}{y}. \tag{19}$$

Since $|B_n' - B_n| \leqslant \dfrac{2}{y_0(n)}$, we see that $B = \limsup\limits_{n\to\infty} B_n'$. We now compute B_n'.

Case 1. $s < 2^n b/a$. Here $[*] = -1$, $g(y) = 2^n k + s$, and $h(y) = \max\ \{0,\ 2^n k - 2^n + \min(2^n,\ s - 2^n b/a + 2^{n+1})\}$, but $2^{n+1} + s - 2^n b/a \geqslant 2^{n+1} - 2^n b/a = 2^n(2 - b/a) \geqslant 2^n$ since $b/a = \alpha/\beta \leqslant 1$, so that $h(y) = 2^n k$, and $\sup_1 \psi(y) = (2^{n+1}k + s)/(2^{n+1}k + s) = 1$. There are three more cases, in all of which $[*] = 0$ since $s \geqslant 2^n b/a$.

Case 2. $2^n b/a \leqslant s \leqslant 2^n$. Here $g(y) = 2^n k + s$ and $h(y) = 2^n k + \min(2^n, s - 2^n b/a) = 2^n k + s - 2^n b/a$ since $s - 2^n b/a \leqslant 2^n$. Hence

$$\psi(y) = \frac{2^{n+1}k + 2s - 2^n b/a}{2^{n+1}k + s},$$

and an elementary computation shows that

$$\sup_2 \psi(y) = \frac{g(2^n) + h(2^n)}{2^n} = 2 - \frac{b}{a}.$$

Case 3. $2^n \leqslant s \leqslant 2^n + 2^n b/a$. Here, $g(y) = 2^n k + 2^n$ and $h(y) = 2^n k + \min(2^n, s - 2^n b/a) = 2^n k + s - 2^n b/a$, and hence

$$\psi(y) = \frac{2^{n+1}k + 2^n + s - 2^n b/a}{2^{n+1}k + s} = 1 + \frac{2^n - 2^n b/a}{2^{n+1}k + s},$$

and an elementary computation shows that

$$\sup_3 \psi(y) = \frac{g(2^n)+h(2^n)}{2^n} = 2-\frac{b}{a}.$$

Case 4. $2^n+2^n b/a < s < 2^{n+1}$. Here $g(y) = 2^n k+2^n$ and $h(y) = 2^n k+\min(2^n, s-2^n b/a) = 2^n k+2^n$ so that

$$\psi(y) = \frac{2^{n+1}k+2^{n+1}}{2^{n+1}k+s},$$

and an elementary computation shows that

$$\sup_4 \psi(y) = \frac{g(2^n+2^n b/a)+h(2^n+2^n b/a)}{2^n+2^n b/a} = \frac{2}{1+b/a}.$$

So we must compare the three numbers $2-b/a, 1, 2(1+b/a)^{-1}$. Now each of them is $\geqslant 1$, and $2-b/a \geqslant 2(1+b/a)^{-1}$, as an elementary estimate shows. Hence $B'_n = (2-b/a)/a = \beta(2\alpha/\beta-\alpha^2/\beta^2)$, and the result follows, on letting $n \to \infty$.

2. Bounded error terms.

THEOREM 2. *If there are constants c_1 and c_2 so that for all $x > 0$ we have*

$$x-c_1 \leqslant S(x) \leqslant x+c_2, \tag{20}$$

then

$$k_n \leqslant 2^n+c_1 \quad \textit{for all } n \tag{21}$$

and

$$k_n \geqslant 2^n-(c_1+c_2) \quad \textit{if} \quad 2^{n-1} > nc_1+c_2. \tag{22}$$

Finally, we have

$$\sum_n |k_n-2^n| < \infty \tag{23}$$

so that if the k_n are integers, then $k_n = 2^n$ for all sufficiently large n.

Proof. As before, if $x < k_n$, then $S(x) \leqslant 2^n$. Thus $k_n-c_1 \leqslant 2^n$ and (21) is established. Now let

$$K_n = k_0+k_1+\ldots+k_{n-1}.$$

Then

$$K_n+c_2 \geqslant S(K_n) \geqslant 2^n. \tag{24}$$

We next prove that

$$2k_n > K_n$$

for all n satisfying $2^{n-1}-nc_1-c_2>0$. For suppose that $2k_n\leqslant K_n$. Then for each choice of $\varepsilon_0,\ldots,\varepsilon_{n-1}$, at least one of the sums

$$k_n+\sum_{j=0}^{n-1}\varepsilon_j k_j \quad \text{or} \quad k_n+\sum_{j=0}^{n-1}(1-\varepsilon_j)k_j$$

is less than K_n, so that in this case we would have $S(K_n)\geqslant 2^n+2^{n-1}$. According to (21) and (23), we would have

$$2^n+nc_1+c_2\geqslant K_n+c_2\geqslant S(K_n)\geqslant 2^n+2^{n-1}, \tag{25}$$

and the assertion is proved. Under the hypothesis of (22), we have $2k_n>K_n$. Now for each y with

$$0\leqslant y<2k_n-K_n$$

we have

$$K_n+y+c_2\geqslant S(K_n+y)\geqslant 2^n+S(K_n+y-k_n), \tag{26}$$

where the second term on the right counts the number of $\varepsilon_0,\ldots,\varepsilon_{n-1}$ for which

$$k_n+\sum_{j=0}^{n-1}\varepsilon_j k_j\leqslant K_n+y.$$

Hence

$$K_n+y+c_2\geqslant 2^n+K_n+y-k_n-c_1, \tag{27}$$

and (22) is established.

Now we choose p so that $2^{p-1}>c_1+c_2$. Then

$$K_n\leqslant 2^n+K_p \quad \text{for all large } n. \tag{28}$$

For, assume that n is so large that (22) holds, that $n>p$, and that $k_{n+1}>2^n$. Then if (28) fails, there would exist at least 2^p choices of $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{n-1}$ for which

$$\sum_{j=0}^{n-1}\varepsilon_j k_j>2^n,$$

namely all choices with $\varepsilon_{p+1}=\ldots=\varepsilon_{n-1}=1$. Since there are at most $S(c_1+c_2)$ sums not exceeding 2^n in which one of the summands is k_n, and no such sums in which one of the summands is k_{n+1} or larger, we obtain

$$2^n-c_1\leqslant S(2^n)\leqslant 2^n-2^p+S(c_1+c_2)\leqslant 2^n-2^p+c_1+c_2+c_2, \tag{29}$$

which is contrary to the hypothesis that $2^p>2c_1+2c_2$.

If $\sum|k_n-2^n|=\infty$, then according to (24),

$$\sum_{k_n>2^n}(k_n-2^n)=\infty. \tag{30}$$

Thus, we could choose $n_1 < n_2 < \ldots < n_r$, with n_1 so large that (22) and (28) hold for $n \geqslant n_1$, with

$$K_{n_1} > K_p + c_1 + c_2 \quad \text{and} \quad k_{n_j} > 2^{n_j}$$

and such that

$$A = \sum_{j=1}^{r} k_{n_j} > \sum_{j=1}^{r} 2^{n_j} + c_1 + 1 = B + 1 + c_1. \tag{31}$$

We now show that

$$\sum 2^{m_j} > B \quad \text{implies} \quad \sum k_{m_j} > A.$$

This is obvious if $\{n_j\}$ is a subset of $\{m_j\}$. If not, let n_s be the largest n not contained in $\{m_j\}$. It follows that

$$\begin{aligned} \sum k_{m_j} &\geqslant A + k_{n_s+1} - \sum_{j=1}^{s} k_{n_j} \geqslant A + k_{n_s+1} - K_{n_s+1} + K_{n_1} \\ &\geqslant A + 2^{n_s+1} - c_1 - c_2 - 2^{n_s+1} - K_p + K_{n_1} > A. \end{aligned} \tag{32}$$

Hence $S(A)$ is no greater than the number of sums of powers of 2 that do not exceed B, and this number is at most $B+1$. Hence $S(A) \leqslant B + 1 < A - c_1$, contrary to hypothesis.

COROLLARY. *If* $-c_1 \leqslant S(x) - \lambda x \leqslant c_2$ *for some positive constant* λ *and all* $x \geqslant 0$, *then*

$$k_n \leqslant \lambda^{-1} 2^n + c_1 \quad \textit{for all } n, \tag{33}$$

$$k_n \geqslant \lambda^{-1} 2^n - (c_1 + c_2) \quad \textit{if} \quad \lambda^{-1} 2^{n-1} > n c_1 - c_2, \tag{34}$$

and

$$\sum |k_n - \lambda^{-1} 2^n| < \infty. \tag{35}$$

This result follows by applying Theorem 2 to the sequence $\{\lambda k_n\}$.

COROLLARY. *If the* k_n *are integers, then the only constants* λ *that can occur above have the form* $\lambda = 2^N/M$, *where* $N \geqslant 0$ *and* $M > 0$ *are integers, and then* $k_n = \lambda^{-1} 2^n$ *for all sufficiently large* n.

The proof is a simple application of (35), and we omit it.

3. Unbounded error terms. The methods of the preceding section can be extended to the case where $S(x) - x$ is unbounded.

THEOREM 3. *Suppose that*

$$x - f_1(x) \leqslant S(x) \leqslant x + f_2(x) \quad \textit{for all } x \geqslant 0,$$

where the f_i *are continuous, positive, non-decreasing functions, not both* *bounded, such that* $f_i(x)/x \to 0$ *as* $x \to \infty$, *and such that* $x - f_1(x)$ *and* $x + f_2(x)$ *are strictly increasing. Let* φ_1 *and* φ_2 *be the inverse functions defined by*

$$x = y - f_1(y) \Leftrightarrow y = x + \varphi_1(x),$$

$$x = y + f_1(y) + f_2(y) \Leftrightarrow y = x - \varphi_2(x),$$

so that the φ_i *are non-decreasing,* $\varphi_i(x)/x \to 0$ *as* $x \to \infty$, *and* $x + \varphi_1(x)$, $x - \varphi_2(x)$ *are strictly increasing for sufficiently large* x. *Then*

$$k_n \leqslant 2^n + \varphi_1(2^n) \quad \text{for all } n, \tag{37}$$

$$k_n \geqslant 2^n - \varphi_2(2^n) \quad \text{for all large } n. \tag{38}$$

Let φ_3 *be the inverse function defined by*

$$x = y + f_2(y) \Leftrightarrow y = x - \varphi_3(x).$$

Then

$$K_n \geqslant 2^n - \varphi_3(2^n) \quad \text{for all } n \tag{39}$$

and

$$K_n \leqslant 2^n + \varphi_4(2^n) \quad \text{for all large } n, \tag{40}$$

where

$$\varphi_4(x) = 2f_1(x) + 2f_2(x) + 2\varphi_2(x).$$

Finally, if we set

$$\psi(x) = \max[f_1(2^{x+1}), \varphi_2(2^x) + \varphi_3(2^x) + \varphi_4(2^x)\},$$

then

$$\sum_{0}^{N} |k_n - 2^n| = O\big(\psi(N)\big). \tag{41}$$

For example, if $f_1(x)$ and $f_2(x)$ are both x^a for large x, then the $\varphi_i(x)$ are each asymptotic to a suitable constant multiple of x^a.

To prove (37), use the inequality $S(x) < 2^n$ if $x < k_n$, as before. To prove (39), use the inequality

$$K_n + f_2(K_n) \geqslant S(K_n) \geqslant 2^n,$$

also as before. By a method entirely analogous to that of the preceding section, it follows that $2k_n > K_n$ for all sufficiently large n. And for such n, proceeding again as before, we have, for each y with $0 \leqslant y < 2k_n - K_n$,

$$\begin{aligned} K_n + y + f_2(K_n + y) &\geqslant S(K_n + y) \geqslant 2^n + S(K_n + y - k_n) \\ &\geqslant 2^n + K_n + y - k_n - f_1(K_n + y - k_n), \end{aligned} \tag{42}$$

so that

$$k_n \geqslant 2^n - f_1(K_n + y - k_n) - f_2(K_n + y) \geqslant 2^n - f_1(k_n) - f_2(k_n),$$

which implies (38).

In order to prove (40), we suppose that

$$K_n > 2^n + K_p,$$

where n is so large that $k_{n+1} > 2^n$ and $k_n > 2^{n-1}$. Then, as in the proof of Theorem 2, we get

$$2^n - f_1(2^n) \leqslant S(2^n) \leqslant 2^n - 2^p + S(2^n - k_n)$$
$$\leqslant 2^n - 2^p + 2^n - 2^n + \varphi_2(2^n) + f_2(2^n)$$

or

$$2^p \leqslant f_1(2^n) + f_2(2^n) + \varphi_2(2^n) \tag{43}$$

so that in view of (37), we have

$$K_n \leqslant 2^n + K_p \leqslant 2^n + 2^{p+1} \leqslant 2^n + 2f_1(2^n) + 2f_2(2^n) + 2\varphi_2(2^n) \tag{44}$$

for all sufficiently large n.

Now assume that

$$\limsup \sum_{k=0}^{N} |k_n - 2^n| / \psi(N) = \infty.$$

In view of (39), this implies that

$$\limsup \sum_{k=0}^{N} (k_n - 2^n)^+ / \psi(N) = \infty,$$

where

$$(k_n - 2^n)^+ = \max(0, k_n - 2^n).$$

We first prove that if $\chi(N) = \log_2\big(\varphi_2(2^N) + \varphi_3(2^N) + \varphi_4(2^N)\big)$, then

$$\sum_{\chi(N) < n < N} (k_n - 2^n)^+ / \big(\psi(N) + 1\big) \leqslant 1. \tag{45}$$

For, let $n_1, n_2, \ldots, n_r$ be the values of n for which

$$k_n > 2^n, \quad \chi(N) < n_1 < n_2 < \ldots < n_r < N.$$

Let

$$A = \sum_{j=1}^{r} k_{n_j} > \sum_{j=1}^{r} 2^{n_j} + \psi(N) + 1 = B + \psi(N) + 1.$$

Then, as in the proof of Theorem 2, $\sum 2^{m_j} > B$ implies that $\sum k_{m_j} > A$.

This is obvious if $\{n_j\}$ is a subset of $\{m_j\}$. If not, let n_s be the largest element of $\{n_i\}$ not contained in $\{m_j\}$, so that

$$\begin{aligned}\sum k_{m_j} &\geqslant A+k_{n_s+1}-K_{n_s+1}+K_{n_1}\\ &\geqslant A+2^{n_s+1}-\varphi_2(2^{n_s+1})-2^{n_s+1}-\varphi_4(2^{n_s+1})+2^{n_1}-\varphi_3(2^{n_1})\\ &> A+2^{\chi(N)}-\varphi_2(2^N)-\varphi_3(2^N)-\varphi_4(2^N)>A.\end{aligned}$$

Hence $S(A)$ is no greater than the number of sums of powers of 2 that do not exceed B, so that $S(A)\leqslant B+1$. It follows that

$$A-f_1(A)\leqslant S(A)\leqslant B+1<A-\psi(N),$$

which leads to a contradiction, since $A\leqslant K_N<2^{N+1}$ for all large N.

Now we have $1+\psi(x)<\frac{1}{2}2^x$ for all $x>x_0$, and therefore, from (45), if we let

$$\chi_0(N)=N \quad \text{and} \quad \chi_{m+1}(N)=\chi\big(\chi_m(N)\big),$$

then, provided $\chi_{m+1}(N)>x_0$, we have

$$\sum_{\chi_{m+1}(N)<n<\chi_m(N)}(k_n-2^n)^+\leqslant 1+\psi\big(\chi_m(N)\big)<\tfrac{1}{2}2^{\chi_m(N)}.$$

But

$$2^{\chi_m(N)}\leqslant\psi\big(\chi_{m-1}(N)\big),$$

so that if $\chi_{m+1}(N)>x_0$, we have

$$\sum_{\chi_{m+1}(N)<n<\chi_m(N)}(k_n-2^n)^+\leqslant\frac{1}{2^m}\psi(N). \tag{46}$$

On adding the inequalities (46) for all suitable m, we get

$$\sum_{n=0}^{N}(k_n-2^n)^+\leqslant 2\psi(N)+O(1),$$

which proves (41) by contradiction.

4. Irregularity of $S(x)$. We say that a function f is *slowly oscillating* to mean that for each positive constant a, $f(ax)/f(x)\to 1$ as $x\to\infty$.

THEOREM 4. *It is impossible to have* $S(x)\sim x^a f(x)$, *where* $0<a<1$, *and* $f(x)$ *is a continuous positive slowly oscillating function such that* $x^a f(x)$ *is strictly increasing.*

Proof. Define the inverse function g by

$$y=x^a f(x)\Longleftrightarrow x=y^{1/a}g(y).$$

Then g is also a continuous positive slowly oscillating function. From $S(k_n-) \leqslant 2^n$, we get

$$k_n < (1+\varepsilon)2^{n/a}g(2^n)$$

for any $\varepsilon > 0$ and all sufficiently large n, so that

$$K_n = \sum_{m=0}^{n-1} k_m < (1+\varepsilon)\sum_{m=0}^{n-1} 2^{m/a}g(2^m).$$

On the other hand, we have $S(K_n) \geqslant 2^n$, so that

$$K_n > (1-\varepsilon)2^{m/a}g(2^n),$$

and hence

$$(47) \qquad \frac{1-\varepsilon}{1+\varepsilon} < \sum_{m=0}^{n-1} 2^{(m-n)/a}\frac{g(2^m)}{g(2^n)}$$

for all sufficiently large n. We show now that (47) is impossible for small ε. We use the result [2] that there is a function $h(x)$ with $h(x) \sim cg(x)$ as $x \to \infty$, where c is a positive constant, such that $h(x)$ has the representation

$$(48) \qquad h(x) = \exp\int_1^x \beta(t)\,t^{-1}dt$$

where

$$(49) \qquad \beta(t) = o(1) \quad \text{as} \quad t \to \infty.$$

It follows from (49) that $h(x) \geqslant x^{-\delta}$ for any $\delta > 0$, for all sufficiently large x, and the same inequality consequently holds for g. It follows that the values of $g(x)$ when x is small do not affect the inequality (47) for large n, and that to contradict (47), it is enough to contradict the corresponding inequality for h, which by (48) may be written as

$$(50) \qquad \frac{1-\varepsilon}{1+\varepsilon} < \sum_{m=0}^{n-1} \exp\left(-\int_{2^m}^{2^n}\left\{\frac{1}{a}+\beta(t)\right\}\frac{dt}{t}\right).$$

If we now choose γ so that $1 < \gamma < 1/a$, then for all sufficiently large t,

$$1/a + \beta(t) > \gamma,$$

and by the above remarks, there is no loss in assuming this for all t. We then have

$$(51) \qquad \frac{1-\varepsilon}{1+\varepsilon} < \sum_{m=0}^{n-1} 2^{\gamma(m-n)} = \sum_{r=1}^{n} 2^{-\gamma r} < \frac{1}{2^\gamma - 1},$$

which is a contradiction if ε is small enough, and the theorem is proved.

On the other hand, it is possible, for each positive integer a, to have $S(x) \sim cx^a$, where c is any positive constant. For example, if we let K consist of a copies of $\{2^n\}$, $n = 0, 1, \ldots$, then a simple computation shows that $S(x) \sim c_a x^a$. Perhaps, then, it is impossible to have $S(x) \sim f(x)x^a$, where $a > 0$, $f(x)$ is slowly oscillating, and $x^a f(x)$ is strictly increasing, unless a is an integer. We outline here a proof of a partial result in this direction, namely that if $1 < a < a_0$, for a certain a_0 (with $1 < a_0 < 2$) then $S(x) \sim f(x)x^a$ is impossible. We treat the case $f(x) = 1$ for all x; the general case is similar.

In this case, we apply arguments like those above to get $(2-\delta)k_n > K_n$ for some $\delta > 0$ and infinitely many n. But then we have

$$S\big((\tfrac{1}{2}+\varepsilon)K_n\big) = 2^n - S\big((\tfrac{1}{2}-\varepsilon)K_n\big),$$

provided only that ε is chosen so small that

$$k_n > (\tfrac{1}{2}+\varepsilon)K_n,$$

since for every sum

$$\sum_{i=0}^{n-1} \varepsilon_i k_i < (\tfrac{1}{2}-\varepsilon)K_n,$$

we have

$$\sum_{i=0}^{n-1} (1-\varepsilon_i)k_i > (\tfrac{1}{2}+\varepsilon)K_n.$$

But the asymptotic relation

$$(\tfrac{1}{2}+\varepsilon)^a K_n^a \sim 2^n - (\tfrac{1}{2}-\varepsilon)^a K_n^a$$

cannot hold identically in ε unless $a = 1$, which is excluded, and the result is proved.

References

[1] Basil Gordon and L. A. Rubel, *On the density of sets of integers possessing additive bases,* Illinois J. Math. 4 (1960), pp. 367-369.

[2] J. Korevaar, T. van Aardenne-Ehrenfest and N. G. de Bruijn, *A note on slowly oscillating functions,* Nieuw. Arch. Wisk. (2), 23 (1949), pp. 77-86.

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