SOLUTION OF A PROBLEM OF DIRAC

P. ERDÖS and T. GALLAI

In the graph-theoretic colloquium at Smolenice, DIRAC conjectured (see Problem 5.) that the chromatic number of a proper regular subgraph of a complete n-gon is $\leqq 3n/5$. We shall prove this conjecture. In fact we shall prove the following theorem ($G^{(n)}$ always denotes a graph with n vertices).

Theorem. *Let $G^{(n)}$ be a regular graph of valency $r < n - 1$ and chromatic number k. Then*

$$k \leqq \frac{3n}{5},$$

with equality if and only if the components of the complementary graph $\bar{G}^{(n)}$ of $G^{(n)}$ are pentagons.

Proof. Clearly $\bar{G}^{(n)}$ is regular of valency $\bar{r} = n - 1 - r > 0$. The fact that the chromatic number of $G^{(n)}$ is k is equivalent with the statement that the minimal number of complete subgraphs of $\bar{G}^{(n)}$ covering all of its vertices is k (i.e. that the *covering number* of $\bar{G}^{(n)}$ is k). We shall call a graph *point-critical with respect to covering* if the omission of any of its vertices decreases its covering number. For these graphs the following theorem holds (T. GALLAI, Publ. Math. Inst. Hung. Acad. Sci. 8 (1963)): Let $G_*^{(n_1)}$ be point-critical with respect to covering, let its covering number be k, let $n_1 \leqq \frac{5}{3}k$. Then the number of its isolated vertices s satisfies

$$s \geqq \frac{3}{2}\left(\frac{5}{3}k - n_1\right) = \frac{5k - 3n_1}{2}, \tag{1}$$

with equality if and only if the components of $G_*^{(n_1)}$ are $\frac{1}{2}(5k - 3n_1)$ isolated vertices and $\frac{1}{2}(n_1 - k)$ pentagons.

Now assume $k \geqq \frac{3}{5}n$ (i.e. $n \leqq \frac{5}{3}k$). Evidently there exists a subset A of the vertices of $\bar{G}^{(n)}$ so that if we omit from $\bar{G}^{(n)}$ the vertices of A and all the incident edges, we obtain a graph $G_*^{(n_1)}$ which has covering number k and is point-critical with respect to covering. Let s be the number of isolated vertices of $G_*^{(n_1)}$, and $n - n_1 = j$. By (1),

$$s \geqq \frac{5k - 3n_1}{2} = \frac{5k - 3n + 3j}{2} \geqq \frac{3}{2}j. \tag{2}$$

If $s = 0$ then by (2) $j = 0$ (i.e. $G_*^{(n_1)} = \bar{G}^{(n)}$) and $5k = 3n$; furthermore all components of $\bar{G}^{(n)}$ are pentagons.

Now we show that $s > 0$ leads to a contradiction. Denote by B the set of isolated vertices of $G_*^{(n_1)}$. By our assumption B is non-empty. Since $\bar{G}^{(n)}$ is a regular graph of valency $\bar{r} > 0$, every vertex of B is connected with precisely $\bar{r}$ vertices of A. Now A has fewer vertices than $B (s \geqq \frac{3}{2}j > 0)$, therefore A has at least one vertex x which is connected with more than $\bar{r}$ vertices of B. Clearly the valency of this vertex x in $\bar{G}^{(n)}$ is greater than $\bar{r}$, which contradicts the regularity of $\bar{G}^{(n)}$; thus the proof of our theorem is complete.