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AN INTERSECTION PROPERTY OF SETS WITH POSITIVE MEASURE

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1. If $A_1, A_2, \ldots$ are Lebesgue-measurable sets of real numbers in the interval $I = [0, 1]$ with measures satisfying

$$\mu(A_r) > \eta > 0, \quad r = 1, 2, \ldots,$$

the set

$$\bigcap_{n\geqslant 1} \bigcup_{r\geqslant n} A_r$$

is measurable with measure at least η. So it is certainly possible to choose a sequence $n_1 < n_2 < \ldots$ such that the intersection $\bigcap_{r=1}^{r=\infty} A_{n_r}$ is non-empty. But (see the example in § 2) there may be no such sequence for which the intersection has positive measure. However, we show that the subsequence can be chosen to ensure that the intersection is uncountable. More precisely, we prove (see §§ 3 and 4)

THEOREM 1. *Suppose η is a positive number and $A_1, A_2, \ldots$ are Lebesgue-measurable subsets of the interval $[0, 1]$ with $\limsup \mu(A_r) \geqslant \eta$. Then there is a Borel set S with $\mu(S) \geqslant \eta$, and a sequence $q_1 < q_2 < \ldots$ such that every point of S is a point of condensation of the set*

$$\bigcup_{j\geqslant 1} \bigcap_{r\geqslant j} A_{q_r},$$

so that every open set containing points of S also contains a perfect subset of $A_{q_j} \cap A_{q_{j+1}} \cap \ldots$ for some j.

We arrange our proof so that it can be trivially generalized (see § 5).

It is natural to ask if, under the conditions of Theorem 1, one can say anything about Hausdorff measures of the set

$$\bigcap_{j\geqslant 1} A_{q_j}$$

for suitably chosen sequences $q_1, q_2, \ldots$ As far as we can see, it may be that, for every strictly increasing continuous function $\varphi(t)$ with $\varphi(0) = 0$, there is a sequence of sets $A_1, A_2, \ldots$ satisfying the conditions of Theorem 1 and such that, φ-m denoting the Hausdorff measure generated by φ, we have

$$\varphi\text{-}m\Big(\bigcap_{j\geqslant 1} A_{q_j}\Big) = 0$$

for every sequence $q_1, q_2, \ldots$ But, on the other hand, it may be that, for every such φ (provided that $\varphi\text{-}m(I) = \infty$) and every sequence of sets satisfying the conditions of Theorem 1, there will be a sequence $q_1, q_2, \ldots$ such that

$$\varphi\text{-}m\Big(\bigcap_{j\geqslant 1} A_{q_j}\Big) = \infty.$$

Perhaps it is most likely that the truth lies between these two extremes and depends in some way on the value of the parameter η between 0 and 1 (**P 442**) (*).

2. Before proving the theorem, we discuss a special example. Let K_q denote the set of all numbers of the form

$$a_1 \cdot 2^{-1} + a_2 \cdot 2^{-2} + \ldots + a_n \cdot 2^{-n} + \ldots$$

with $a_q = 0$ and $a_n = 0$ or 1 for all other values of n. Clearly $\mu(K_q) = \frac{1}{2}$ and the intersection of any N sets K_q has measure 2^{-N}. Hence the intersection of any infinite subsequence of the sets has measure zero, and so has the set

$$\bigcup_{j\geqslant 1} \bigcap_{r\geqslant j} K_{q_r} \quad \text{for any sequence } q_1 < q_2 < \ldots$$

In this instance we may verify the theorem by taking $q_r = 2r$ and $S = [0, 1]$, since an open subset of $[0, 1]$ contains, for some suitable integers j and m, the perfect set of all numbers of the form

$$m \cdot 2^{-(2j-1)} + \sum_{r=j}^{\infty} b_r \cdot 2^{-(2r+1)},$$

where $b_r = 0$ or 1 for $r \geqslant j$, and this perfect set is contained in $\bigcap_{r=j}^{r=\infty} K_{2r}$. The set

$$\bigcup_{j\geqslant 1} \bigcap_{r\geqslant j} K_{2r}$$

is the set of numbers of the form $\sum_{r=1}^{\infty} a_r \cdot 2^{-r}$ with $a_r = 0$ or 1 for all r, and $a_{2r} = 0$ for all sufficiently large r.

(*) Added in proof. The second extreme turned out to hold true; see P. Erdös and S. J. Taylor, *The Hausdorff measure of the intersection of sets of positive Lebesque measure*, Mathematika 10 (1963), p. 1-9.

3. It will be convenient to introduce the following conventions:

(a) $\mathcal{N}$, with or without a suffix, will denote an infinite set of positive integers;

(b) if $E_1, E_2, \ldots$ are sets, then $\mathcal{N}\{E_n\}$ will denote $\bigcap_{n\in\mathcal{N}} E_n$;

(c) if A and B are subsets of I, we say that A *avoids* B if $\mu(A \cap B) = 0$.

We prove

LEMMA 1. *Suppose that $E_1, E_2, \ldots$ are measurable subsets of $I = [0, 1]$ with $\liminf \mu(E_r) = \eta > 0$. Then there is a Borel subset D of I with $\mu(D) \geqslant \eta$, and a set $\mathcal{N}$, such that every Borel subset of D which has positive measure avoids only a finite number of E_n with n in $\mathcal{N}$.*

Proof. Suppose the lemma is false. This implies that

(1) if A is any Borel subset of I with $\mu(A) \geqslant \eta$, and $\mathcal{N}$ is any infinite set of positive integers, then A contains a Borel set with positive measure which avoids E_n for infinitely many n in $\mathcal{N}$.

Applying (1) with $A = I$, we see that I contains a Borel set T, with $\mu(T) > 0$, which avoids E_n for infinitely many n. Take T_1 to be such a set T, chosen from among the possible sets T so that all the other possible sets T have measure less than $2\mu(T_1)$. Let $\mathcal{N}_1$ be the set of n such that E_n avoids T_1. Suppose that, for some $k \geqslant 1$, disjoint Borel subsets $T_1, T_2, \ldots, T_k$ of I, and sets $\mathcal{N}_1 \supset \mathcal{N}_2 \supset \ldots \supset \mathcal{N}_k$, have been chosen so that $T_1 \cup T_2 \cup \ldots \cup T_k$ avoids E_n for all n in $\mathcal{N}_k$. Then $I-(T_1 \cup \ldots \cup T_k)$ contains almost all points of some sets E_n with n arbitrarily large, and so its measure is at least η. We apply (1) with $A = I-(T_1 \cup T_2 \cup \ldots \cup T_k)$ and $\mathcal{N} = \mathcal{N}_k$, and choose a Borel set T_{k+1} contained in I and disjoint from $T_1, T_2, \ldots, T_k$, and a subset $\mathcal{N}_{k+1}$ of $\mathcal{N}_k$, such that T_{k+1} avoids E_n for all n in $\mathcal{N}_{k+1}$, but all Borel sets T contained in I and disjoint from $T_1, T_2, \ldots, T_k$, which avoid E_n for infinitely many n in $\mathcal{N}_k$, have measure less than $2\mu(T_{k+1})$. Then $T_1 \cup T_2 \cup \ldots \cup T_k \cup T_{k+1}$ avoids E_n for all n in $\mathcal{N}_{k+1}$. Since the conditions are satisfied when $k = 1$, we may suppose that $T_1, T_2, \ldots$ and $\mathcal{N}_1, \mathcal{N}_2, \ldots$ have been chosen inductively in this way.

Since

$$\mu\big(I-(T_1 \cup T_2 \cup \ldots \cup T_k)\big) \geqslant \eta,$$

for all k, we have

$$\mu\big(I-(T_1 \cup T_2 \cup \ldots)\big) \geqslant \eta.$$

So we may apply (1) with $A = I-(T_1 \cup T_2 \cup \ldots)$ and $\mathcal{N} = \mathcal{N}_0$, defined to be the set $n_1, n_2, \ldots$, where n_1 is the least integer in $\mathcal{N}_1$, n_2 is the least in $\mathcal{N}_2$ which exceeds n_1, and so on. There will be a Borel set F contained in A, with $\mu(F) > 0$, which avoids E_n for infinitely many n in $\mathcal{N}_0$. Now, if we choose any positive integer k, all but a finite number of integers in $\mathcal{N}_0$ are in $\mathcal{N}_k$, and so F avoids E_n for infinitely many n

in $\mathcal{N}_k$, and at the same time $F \subset I-(T_1 \cup T_2 \cup \ldots \cup T_k)$. Hence $\mu(F) < 2\mu(T_{k+1})$. Since $T_1, T_2, \ldots$ are disjoint Borel subsets of I, and $\mu(I) = 1$, it follows that $\mu(T_{k+1}) \to 0$ as $k \to \infty$, and this contradicts $\mu(F) > 0$.

4. Proof of Theorem 1. Since $\limsup \mu(A_r) \geqslant \eta$, and we are concerned with the existence of a subsequence with a certain property, we may without loss of generality suppose that $\liminf \mu(A_r) \geqslant \eta$. For each r we may choose K_r, a closed subset of A_r, with

$$\mu(K_r) \geqslant \mu(A_r) - (1/r).$$

Then $\liminf \mu(K_r) \geqslant \eta$. So, by the lemma, there is a Borel set D with $\mu(D) \geqslant \eta$ and a set $\mathcal{N}$ such that every Borel subset of D with positive measure avoids K_n for only a finite number of n in $\mathcal{N}$. Let $I_1, I_2, \ldots$ be a countable base for the open subsets of I; for example, take $I_1, I_2, \ldots$ to be an enumeration of the open subintervals of I with rational end-points. Take

$$S = D - \bigcup{}' I_r,$$

the union being taken over all r for $\mu(D \cap I_r) = 0$. Then S is a Borel set with

$$\mu(S) \geqslant \mu(D) - \sum_{\mu(D \cap I_r)=0} \mu(D \cap I_r) = \mu(D) \geqslant \eta,$$

and every open set which meets S does so in a set of positive measure.

Now let G be an open set with $G \cap S \neq \emptyset$. Then $\mu(G \cap S) > 0$, and $G \cap S$ avoids K_n for at most a finite number of n in $\mathcal{N}$. Also, as $\mu(G \cap S) > 0$, we can choose two disjoint closed subsets H_0 and H_1 of G, each intersecting S in a set of positive measure (see § 5). Then $H_a \cap S$ avoids K_n for at most a finite number of n in $\mathcal{N}$, for $a = 0$ or 1. Thus we can choose ν^1 in $\mathcal{N}$ so that both

$$\mu(H_0 \cap S \cap K_{\nu_1}) > 0 \quad \text{and} \quad \mu(H_1 \cap S \cap K_{\nu_1}) > 0.$$

By repeating this argument, we see that there exist four disjoint closed sets, H_{00} and H_{01} in H_0, and H_{10} and H_{11} in H_1, and an integer ν^2, larger than ν_1, in $\mathcal{N}$ such that

$$\mu(S \cap H_{\alpha\beta} \cap K_{\nu_1} \cap K_{\nu_2}) > 0$$

for all four closed sets $H_{\alpha\beta}$, $\alpha, \beta = 0$ or 1. It follows, by induction, that for each integer $k \geqslant 2$ we can choose a system of 2^k disjoint closed sets

(1) $$H_{a_1 a_2 \ldots a_k}, \quad a_1, a_2, \ldots, a_k = 0 \text{ or } 1,$$

and an integer ν_k in $\mathcal{N}$, so that $\nu_k > \nu_{k-1}$,

$$H_{a_1 a_2 \ldots a_k} \subset H_{a_1 a_2 \ldots a_{k-1}}, \quad a_1, a_2, \ldots, a_k = 0 \text{ or } 1,$$

and

$$\mu(S \cap H_{a_1a_2\ldots a_k} \cap K_{\nu_1} \cap K_{\nu_2} \cap \ldots \cap K_{\nu_k}) > 0,$$

for $a_1, a_2, \ldots, a_k = 0$ or 1. For each infinite sequence $a_1, a_2, \ldots$ of 0's and 1's, write

$$X_k = H_{a_1a_2\ldots a_k} \cap K_{\nu_1} \cap K_{\nu_2} \cap \ldots \cap K_{\nu_k},$$

for $k = 1, 2, \ldots$ Then the sets $X_1, X_2, \ldots$ are closed and non-empty and they decrease. So their intersection contains at least one point. As the sets (1) are disjoint, for each fixed k, it follows that disjoint sets $\bigcap X_k$ correspond to distinct sequences $a_1, a_2, \ldots$ If $\mathscr{N}'$ is the set of the integers $\nu_1, \nu_2, \ldots$, the closed intersection

$$\mathscr{N}'\{K_n\} = K_{\nu_1} \cap K_{\nu_2} \cap \ldots$$

contains this uncountable system of disjoint non-empty subsets of G, and therefore contains a perfect subset of G.

Let $I_1, I_2, \ldots$ be a countable base for the open sets of I, and let $G_1, G_2, \ldots$ be an enumeration of those sets of the base that meet S. By the last paragraph, $\mathscr{N}$ contains a subset $\mathscr{N}_1$ such that $\mathscr{N}_1\{K_n\} \cap G_1$ contains a perfect set. Similarly $\mathscr{N}_1$ contains $\mathscr{N}_2$ such that $\mathscr{N}_2\{K_n\} \cap G_2$ contains a perfect set. Continuing in this way, we obtain a decreasing sequence $\mathscr{N}_1 \supset \mathscr{N}_2 \supset \ldots$ such that $\mathscr{N}_r\{K_n\} \cap G_r$ contains a perfect subset for $r = 1, 2, \ldots$ Take $\mathscr{N}$ to be the set $q_1, q_2, \ldots$, where q_1 is the least in $\mathscr{N}_1$, and q_{r+1} is the least in $\mathscr{N}_{r+1}$ which exceeds q_r, for $r = 1, 2, \ldots$ Now the sequence $q_1, q_2, \ldots$ and the set S satisfy the conditions of the theorem. For, if G is any open set which meets S at a point, x say, there is a set I_r of the base with $x \in I_r$ and $I_r \subset G$. So for some j we have $I_r = G_j$. Hence

$$G \cap \{A_{q_j} \cap A_{q_{j+1}} \cap \ldots\} \supset G_j \cap \mathscr{N}_j\{K_n\}$$

and so contains a perfect set.

5. Theorem 2. *Let X be a compact set. Suppose the topology in X has a countable base. Let μ be a Carathéodory outer measure on X with the properties*

(a) $\mu(X) = 1$,

(b) $\mu((x)) = 0$ *for each x in X*,

(c) *Borel sets in X are μ-measurable*,

(d) *if E is μ-measurable and $\varepsilon > 0$, then there is an open set G with $E \subset G$ and $\mu(G) < \mu(E) + \varepsilon$.*

Suppose η is a positive number and $A_1, A_2, \ldots$ are μ-measurable subsets of X with $\limsup \mu(A_r) \geqslant \eta$. *Then there is a Borel set S in X with $\mu(S) \geqslant \eta$, and a sequence $q_1 < q_2 < \ldots$, such that every point of S is a point* *of condensation of the set*

$$\bigcup_{j\geqslant 1}\bigcap_{r\geqslant j} A_{q_r},$$

and every open set containing a point of S also contains a perfect subset of $A_{q_j}\cap A_{q_{j+1}}\cap\ldots$ for some j.

Proof. It is clear how nearly all the steps in the proof of Theorem 1 have to be modified to provide a proof of Theorem 2; the only difficulty is in the choice of the disjoint closed subsets H_0 and H_1 and the subsequent choice of the subsets (1) for $k=2,3,\ldots$ These choices are justified by the following lemma, which we prove by using one of the ideas we have already used:

LEMMA. *Under the conditions of Theorem 2, if A is a μ-measurable set with $\mu(A)>0$, we can choose two disjoint closed subsets H_0 and H_1 of A with $\mu(H_0)>0$, $\mu(H_1)>0$.*

Proof. As A is μ-measurable and $\mu(A)>0$, we can choose a closed set B contained in A with $\mu(B)>0$. Let $X_1, X_2, \ldots$ be a countable base for the open sets of X. Take

$$C = B - {\bigcup}' X_r,$$

the union being taken over all the integers r for which $\mu(B\cap X_r)=0$. Then C is closed and

$$\mu(C) = \mu(B) - \sum_{\mu(B\cap X_r)=0}\mu(B\cap X_r) = \mu(B) > 0.$$

Hence C contains at least one point, c say. As $\mu((c))=0$, we can choose an open set G with $c\in G$ and $\mu(G)<\mu(C)$. Choose r so that $c\in X^r$ and $X_r\subset G$. Then, as $c\in X_r$, we have $\mu(B\cap X_r)>0$, so that

$$\mu(C\cap G)\geqslant \mu(B\cap X_r) > 0.$$

Finally, take H_0 to be a closed subset of $C\cap G$ with $\mu(H_0)>0$, and take $H_1 = C\cap(X-G)$. It is easy to verify that these sets satisfy our requirements.

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