ON TRIGONOMETRIC SUMS WITH GAPS

by

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A well known theorem states as follows:[1]

Let $n_1 < n_2 < \ldots, n_{k+1} / n_k > A > 1$ be an infinite sequence of real numbers and $\sum_{k=1}^{\infty} (a_k^2 + b_k^2)$ a divergent series satisfying

$$\lim_{N=\infty} (a_N^2 + b_N^2)^{1/2} \left(\sum_{k=1}^{N} a_k^2 + b_k^2 \right)^{-1/2} = 0 . \tag{1}$$

Then

$$\lim_{n\to\infty} \left| \underset{t}{\mathsf{E}} \left\{ \sum_{k=\infty}^{N} (a_k \cos 2\pi n_k t + b_k \sin 2\pi n_k t) < \right.\right. \tag{2}$$
$$\left.\left. < \omega \left(\frac{1}{2} \sum_{k=1}^{N} (a_k^2 + b_k^2) \right)^{1/2} \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2}\, du .$$

($\left| \underset{t}{\mathsf{E}} \{ \quad \} \right|$ denotes the Lebesgue measure of the set in question).

In the present paper I shall weaken the lacunarity condition $n_{k+1}/n_k > A > 1$. In fact I shall prove the following

Theorem 1. *Let $n_1 < n_2 < \ldots$ be an infinite sequence of integers satisfying*

$$n_{k+1} > n_k \left(1 + \frac{c_k}{k^{1/2}} \right) \tag{3}$$

where $c_k \to \infty$. Then

$$\lim_{N=\infty} \left| \underset{t}{\mathsf{E}} \left\{ \sum_{k=1}^{N} \left(\cos 2\pi n_k (t - \vartheta_k) < \omega \cdot N^{1/2} \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2}\, du . \tag{4}$$

It seems likely that the Theorem remains true if it is not assumed that the n_k are integers. On the other hand if $n_{k+1}/n_k \to 1$ is an arbitrary sequence of integers it is easy to construct examples which show that (1) is not enough

[1] R. SALEM and A. ZYGMUND: "On lacunary trigonometric series I. and II.", *Proc. Math. Acad. Sci. USA* **33** (1947) 333–338 and **34** (1948) 54–62.

For the history of the problem see M. KAC: "Probability methods in analysis and number theory". *Bull. Amer. Math. Soc.* **55** (1949) 641–665.

for the truth of (2). It is possible that (3) and

$$\lim_{N=\infty}\left(\sum{}' a_k^2+b_k^2\right)^{1/2}\left(\sum_{k=1}^{N}(a_k^2+b_k^2)\right)^{-1/2}=0$$

where in $\sum' \frac{1}{2} n_N < n_k \leqq n_N$ suffices for the truth of our Theorem. But I can not at present decide this question and in this paper only consider the case $a_k = b_k = 1$.

I can show that Theorem 1. is best possible in the following sense: To every constant c there exists a sequence n_k for which $n_{k+1} > n_k\left(1+\frac{c}{k^{1/2}}\right)$ but (4) is not true. To see this let u_k tend to infinity sufficiently fast. Put

$$n_{k^2+l} = n_k + lc_1\left[\frac{n_k}{k}\right],\quad 1 \leqq l \leqq 2k+1 .$$

Clearly $n_{r+1} > n_r\left(1+\frac{c}{r^{1/2}}\right)$ if c_1 is sufficiently large and it is not difficult to see that (4) can not be satisfied. We do not give the details.

Further I can prove the following

Theorem 2. *Let $n_1 < n_2 < \ldots$ be an infinite sequence of integers for which for every $\varepsilon > 0$ there exists a $k_0 = k_0(\varepsilon)$ so that for every $k > k_0$*

$$n_{k+1} > n_k + n_{k-[\varepsilon k^{1/2}]}\,. \tag{5}$$

Then (4) *holds.*

It is not difficult to construct sequences for which (3) holds but (5) does not hold and sequences for which (5) holds and (3) not, or Theorems 1 and 2 are incomparable. (3) seems to be easier to verify, thus Theorem 1 is probably more useful. We will not give the proof of Theorem 2 since it is similar to that of Theorem 1.

To simplify the computations we will work out the proof of Theorem 1 only for a cosine series, the proof of the general case follows the same lines.

Theorem 1'. *Let $n_1 < n_2 < \ldots$ be an infinite sequence of integers satisfying* (3). *Then*

$$\lim_{N=\infty}\left|\underset{t}{\mathsf{E}}\left\{\sum_{k=1}^{N}\cos 2\pi n_k t < \omega\left(\frac{N}{2}\right)^{1/2}\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\omega} e^{-u^2/2}\,du\,. \tag{4'}$$

A well known theorem of Chebyshev implies that to prove Theorem 1′ it will suffice to show that for every l, $1 \leqq l < \infty$

$$\lim_{N=\infty} I_N^{(l)} = \lim_{N=\infty}\int_0^1\left(\frac{\sum\limits_{k=1}^{N}\cos 2\pi n_k t}{\left(\frac{N}{2}\right)^{1/2}}\right)^l dt = (2\pi)^{-1/2}\int_{-\infty}^{\infty} x^l e^{-x^2/2}\,dx = \begin{cases} 0 & \text{if } l \text{ is odd,} \\ \dfrac{l!}{2^{l/2}\left(\frac{l}{2}\right)!} & \text{if } l \text{ is even.}\end{cases} \tag{6}$$

It is easy to see that ($\varepsilon_1 = \pm 1$, $1 \leq i \leq l$)

$$(7) \quad \int_0^1 \prod_{i=1}^{l} \cos 2\pi n_i t = \frac{1}{2^l} \int_0^1 \sum_{\varepsilon_1, \ldots, \varepsilon_l} \cos\left(2\pi \sum_{i=1}^{l} \varepsilon_i n_i t\right) dt = \frac{h(n_1, \ldots, n_l)}{2^l}$$

where $h(n_1, \ldots, n_l)$ denotes the number of solutions of $\sum_{i=1}^{l} \varepsilon_i n_i = 0$. From (7) we have

$$(8) \qquad \left(\frac{N}{2}\right)^{l/2} I_N^{(l)} = \frac{1}{2^l} \sum h(n_{i_1}, \ldots, n_{i_l})$$

where $i_1, \ldots, i_l$ runs through all the l-tuples formed from the integers $1 \leq r \leq N$ (where order counts). Clearly $\Sigma\, h(n_{i_1}, \ldots, n_{i_l})$ equals the number of solutions of

$$(9) \qquad \sum_{i=1}^{l} \varepsilon_i n_{r_i} = 0, \quad 1 \leq r_i \leq N \quad \text{(order counts here too).}$$

Thus to estimate $I_n^{(l)}$ we only have to estimate the number of solutions of (9). Assume first l even $l = 2s$. Then (9) has trivial solutions such that among the terms in (9) each n_r occurs the same number of times with a positive as with a negative sign. The number of these trivial solutions clearly equals

$$(10) \qquad (1 + o(1)) \frac{l!}{\left(\frac{l}{2}\right)!} N^{l/2}.$$

Now we prove the following

Lemma 1. *Let $\{n_k\}$ be a sequence of integers satisfying* (3). *Denote by $g_l(A, N)$ the number of solutions of*

$$(11) \qquad \sum_{i=1}^{l} \varepsilon_i n_{r_i} = A, \qquad 1 \leq r_l \leq \ldots \leq r_1 \leq N$$

where the trivial solutions are excluded.

Then

$$(12) \qquad \max_A g_l(A, N) = o(N^{l/2}).$$

(The trivial solutions can only occur if $A = 0$ and l is even).

From Lemma 1, (8) and (10) it follows that

$$\lim_{N=\infty} I_N^{(l)} = \begin{cases} 0 & \text{if } l \text{ is odd} \\ \dfrac{l!}{2^{l/2}\left(\frac{l}{2}\right)!} & \text{if } l \text{ is even} \end{cases}$$

which implies Theorem 1. Thus to complete our proof it will suffice to prove Lemma 1, and in fact Lemma 1 is the only new and difficult part of our paper.

First we show that the Lemma holds for $l = 1$ and $l = 2$. For $l = 1$ the Lemma is trivial, the number of solutions of (11) is at most one for $l = 1$. Now we need

Lemma 2. *The number of* n_i *satisfying* $(k \to \infty)$

$$n_k x^{-1} < n_i < n_k x$$

is $o(k^{1/2} \log x) + o(1) + o((\log x)^2)$.

The Lemma follows immediately from (3). (The term $o(1)$ is needed only for small x and the term $o((\log x)^2)$ only for very large x.)

If $\pm n_{r_1} \pm n_{r_2} = A (n_{r_1} > n_{r_2})$ we must have (by (3))

$$\left|\frac{A}{2}\right| \leqq n_{r_1} \leqq |A| N. \tag{13}$$

From (13) and Lemma 2 we obtain that the number of solutions of (11) for $l = 2$ is $o(N^{1/2} \log N) = o(N)$ uniformly in A which proves Lemma 1 for $l = 2$.

Now we use induction with respect to l. Assume that (12) holds for all $l' < l$, we shall then prove that (12) holds for l too. We assume now $l \geqq 3$ and distinguish four cases.

In case I.

$$\frac{1}{N} n_{r_i} \leqq n_{r_{i+1}} \leqq n_{r_i} \tag{14}$$

holds for all $1 \leqq i \leqq l - 1$.

Put $(1 \leqq s \leqq l - 1)$

$$2^n \leqq \max n_{r_i}/n_{r_{i+1}} = n_{r_s}/n_{r_{s+1}} < 2^{n+1}. \tag{15}$$

Clearly $0 \leqq n \leqq \log N/\log 2$. Evidently there are at most N choices for n_{r_1}. Let $i < l - 1$. If $n_{r_1}, \ldots, n_{r_i}$ have already been determined then by (15) and Lemma 2 there are at most $o(N^{1/2} n)$ choices for $n_{r_{i+1}}$. Now we show that for n_{r_s} there at most are $o(N^{1/2}/2^n) + o(1) = o\left(\frac{N^{1/2}}{2^{n/4}}\right)$ choices (if $n_{r_1}, \ldots, n_{r_{s-1}}$ has already been chosen). To see this observe that from (15) we have

$$\left|\sum_{i=s+1}^{l} \varepsilon_i n_{r_i}\right| < \frac{l \cdot n_{r_s}}{2^n}. \tag{16}$$

Thus from (11) and (16)

$$A - \sum_{i=1}^{s-1} \varepsilon_i n_{\ i} = \varepsilon_s n_{r_s} + \frac{\theta l}{2^n} n_{r_s}, \ |\theta| < 1. \tag{17}$$

(17) implies that n_{r_s} must lie in an interval (α, β) with $\alpha < \beta < \alpha\left(1 + \frac{cl}{2^n}\right)$. Thus from Lemma 2 there are at most $o\left(\frac{N^{1/2}}{2^n}\right) + o(1) = o\left(\frac{N^{1/2}}{2^{n/4}}\right)$ choices for n_{r_s} as stated. Finally if $n_{r_1}, \ldots, n_{r_{l-1}}$ has already been determined there are at most 2^{l-1} choices for n_{r_l} (i. e. $\sum_{i=1}^{l-1} \varepsilon_i n_{r_i}$ can be chosen in 2^{l-1} ways). Thus the

total number of choices for $n_{r_1}, \ldots, n_{r_l}$ satisfying (15) is at most

$$cN\left(o(N^{1/2}\,n)\right)^{l-3} o\left(\frac{N^{1/2}}{2^{n/4}}\right) = o(N^{l/2})\,\frac{n^{l-3}}{2^{n/4}}\,. \tag{18}$$

From (18) we evidently obtain that the number of solutions of (11) in case I is

$$o(N^{l/2}) \sum_{n=0}^{\infty} \frac{n^{l-3}}{2^{n/4}} = o(N^{l/2})\,.$$

In case II (14) holds for $i < j, j \geqq 3$ and for $i = j \leqq l - 1$

$$n_{r_{j+1}} < \frac{1}{N}\, n_{r_j}\,. \tag{19}$$

We show that if $n_{r_1}, \ldots, n_{r_{j-1}}$ has already been determined, then there are only a bounded number of choices of n_{r_j}. To see this observe that by (19)

$$\left(\sum_{i=j+1}^{l} \varepsilon_i\, n_{r_i}\right) < \frac{l}{N}\, n_{r_j}.$$

Thus from (11)

$$A - \sum_{i=1}^{j-1} \varepsilon_i\, n_{r_i} = \varepsilon_j\, n_{r_j} + \theta\, \frac{l n_{r_j}}{N}\,, \quad |\theta| < 1 \tag{20}$$

or n_{r_j} must lie in an interval (α, β) with $\alpha < \beta < \alpha\left(1 + \dfrac{cl}{N}\right)$. Thus by Lemma 2 there are only a bounded number of choices for n_{r_j}.

Put

$$2^n \leqq \max_{1 \leq i \leq j-1} n_{r_i}/n_{r_{i+1}} = n_{r_s}/n_{r_{s+1}} < 2^{n+1}\,. \tag{15'}$$

As in case I. there are at most $o(N^{1/2}/2^{n/4})$ choices for n_{r_s}, $o(N^{1/2}n)$ choices for n_{r_i}, $1 < i < j$, $i \neq s$ and at most N choices for n_{r_1}. Thus we see as in case I that for $n_{r_1}, \ldots, n_{r_j}$ there are at most $o(N^{j/2})$ choices. If $n_{r_1}, \ldots, n_{r_j}$ are already chosen there are 2^j choices for $\sum_{i=1}^{j} \varepsilon_i\, n_{r_i}$. Hence there are only $2^j o(N^{j/2}) = o(N^{j/2})$ choices for $\sum_{i=1}^{j} \varepsilon_i\, n_{r_i}$. By our induction hypothesis there are $o(N^{(l-j)/2})$ solutions of

$$A - \sum_{i=1}^{j} \varepsilon_i\, n_{r_i} = \sum_{i=j+1}^{l} \varepsilon_i\, n_{r_i} \tag{21}$$

in $n_{r_{j+1}}, \ldots, n_{r_l}$. Thus finally there are $o(N^{l/2})$ solutions of (11) in case II.

In case III (14) holds for $i = 1$, but

$$n_{r_3} < \frac{1}{N}\, n_{r_2}\,.$$

The same proof as in case II shows that if n_{r_1} has already been chosen there are only a bounded number of choices for n_{r_2}. Thus since there are at most N choices for n_{r_1} there are at most cn choices for $\varepsilon_1\, n_{r_1} + \varepsilon_2\, n_{r_2}$. Hence arguing as in (21) we see that by our induction hypothesis the number of solutions of (11) is $o(N^{l/2})$ in case III too.

In case IV $n_{r_2} < \frac{1}{N} n_{r_1}$ i. e. (14) never holds. We see by the same argument as in (20) that there are only a bounded number of choices for n_{r_1} and therefore again arguing as in (21) we obtain by our induction hypothesis that in case IV (11) has $o(N^{l/2})$ solutions.

Thus combining the four cases we obtain that the number of solutions of (11) is $o(N^{l/2})$ uniformly in A, or (12) — and therefore Lemma 1 is proved. Hence the proof of Theorem 1 is complete.

Let $f(k) \to \infty$ monotonically. It is easy to see that

$$n_k = [e^{k^{1/2} f(k)}] \tag{22}$$

satisfies (3), hence Theorem 1 holds for the sequence (22).

It is not difficult to see that Lemma 1 is best possible in some sense, namely if (3) is replaced by

$$n_{k+1} > n_k\left(1 + \frac{c}{k^{1/2}}\right) \qquad c \text{ independent of } k$$

then (12) in general will not hold. On the other hand (12) may very well hold for special sequences which do not satisfy (3). In particular I would guess that (12) and therefore Theorem 1 will hold if $n_k = [e^{k^\alpha}]$ for every $\alpha > 0$. I cannot even prove this for $\alpha = {}^1/_2$.

(Received August 25, 1961.)

О ЛАКУНАРНЫХ ТРИГОНОМЕТРИЧЕСКИХ РЯДАХ

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Резюме

В работе доказывается следующая теорема: пусть $n < n_2 < \ldots$ последовательность натуральных чисел для которых

$$n_{k+1} > n_k\left(1 + \frac{c_k}{\sqrt{k}}\right) \qquad (k = 1, 2, \ldots),$$

где

$$\lim_{k\to\infty} c_k = +\infty .$$

Пусть $S_N(t) = \sum_{k=1}^{N} \cos 2\pi n_k(t - \vartheta_k)$ где вещественные числа ϑ_k произвольные. Пусть $\underset{t}{\mathsf{E}}\{\quad\}$ обозначает множество тех чисел t в интервале $0 \leq t \leq 1$ для которых условие в скобках выполняется, и пусть $|\underset{t}{\mathsf{E}}|$ — мера Lebesgue-а множества $\underset{t}{\mathsf{E}}$. Тогда имеем для всех ω $(-\infty < \omega < \infty)$

$$\lim_{N\to\infty} |\underset{t}{\mathsf{E}}\{S_N(t) < \omega\sqrt{N}\}| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-u^2/2}\, du .$$